# Aggregation and truncation of reversible Markov Chains modulo state renaming

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**Abstract.** The theory of time-reversibility has been widely used to derive the expressions of the invariant measures and, consequently, of the equilibrium distributions for a large class of Markov chains which found applications in optimisation problems, computer science, physics, and bioinformatics. One of the key-properties of reversible models is that the truncation of a reversible Markov chain is still reversible. In this work we consider a more general notion of reversibility, i.e., the reversibility modulo state renaming, called $\rho$-reversibility, and show that some of the properties of reversible chains cannot be straightforwardly extended to $\rho$-reversible ones. Among these properties, we show that in general the truncation of the state space of a $\rho$-reversible chain is not $\rho$-reversible. Hence, we derive further conditions that allow the formulation of the well-known properties of reversible chains for $\rho$-reversible Markov chains. Finally, we study the properties of the state aggregation in $\rho$-reversible chains and prove that there always exists a state aggregation that associates a $\rho$-reversible process with a reversible one.

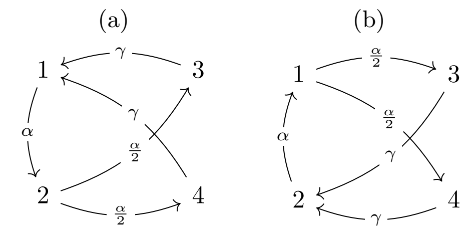
## 1 Introduction

Reversibility of Markov chains at discrete or continuous time has been extensively studied in [13,25]. Given a stationary Markov chain $X(t)$ we say that it is reversible if for all $t_1, t_2, \dots, t_n, \tau$, $(X(t_1), \dots, X(t_n))$ has the same equilibrium distribution as $(\rho(X)(\tau - t_1), \dots, \rho(X)(\tau - t_n))$ where $t_1, \dots, t_n, \tau$ belongs to the time domain, i.e., $\mathbb{Z}$ for discrete time Markov chains (DTMCs) and $\mathbb{R}$ for continuous time Markov chains (CTMCs). Reversibility is a key-property for studying the stationary behaviour of Markov chains and there are several examples of models with underlying reversible processes such as the *loss networks* [14] which found applications for studying telecommunication systems, models of wireless networks [5] just to mention a non exhaustive list of applications. In many practical cases, reversibility allows for the derivation of an exact analysis of the stationary behaviour of the model without resorting to simulation, approximate decompositions (see e.g., [6,3]) or limit-based analysis (see e.g., [4,7]).

However, the largest application field of reversible Markov chains is in queueing theory. Queueing theory is the foundation of many works in operation research (see, e.g., [23,16,8] just to mention some recent works) and some of them are based on reversible models or their variation [13,1,12,24,2].

Markov chain reversibility is a special case of a more general notion of reversibility that we call $\rho$-reversibility. A $\rho$-reversible chain $X(t)$ is stochastically indistinguishable from $X(\tau - t)$ modulo a state renaming which is a bijective function $\rho$ from the chain's state space $\mathcal{S}$ to itself. An example of such a chain is shown in Figure 1 where we can easily see that the forward CTMC (Figure 1-(a)) is not reversible since a simple necessary structural condition for reversibility is that whenever there is a transition from state $s$ to state $s'$ there is also its inverse from $s'$ to $s$. Figure 1-(b) shows the transition diagrams of $X(\tau - t)$ and we can observe that it is stochastically indistinguishable from $X(t)$ modulo the renaming of states $\rho(1) = 2, \rho(2) = 1, \rho(3) = 4$ and $\rho(4) = 3$.

Fig. 1: A simple $\rho$-reversible CTMC: (a) Forward process, (b) Reversed process.




In this case function $\rho$ is an involution. In the literature of stochastic processes when $\rho$ is an involution the notion of $\rho$-reversibility is known as *dynamic reversibility* and has been studied in [13,25]. It is worth of notice that the concept of $\rho$-reversibility is more general than that of dynamic reversibility, i.e., there exist Markov processes which are $\rho$-reversible but there does not exist any involution for which they are also dynamically reversible [18,21].

Reversible Markov chains enjoy some important properties that can be readily formulated also for $\rho$-reversible chains. Specifically, in both cases one may decide if a chain is reversible/$\rho$-reversible by inspection of a base of minimal cycles of the chain and the computation of a non-trivial invariant measure can be done by performing only multiplications and using the detailed balance equations [13,25,19,21]. However, other important properties that hold for reversible Markov chains do not straightforwardly hold for the $\rho$-reversible ones. Specifically, if $\mathcal{S}$ is the state space of a reversible CTMC, $\mathcal{A} \subset \mathcal{S}$ and if the graph of $\mathcal{A}$ is irreducible, then also the chain whose state space is $\mathcal{A}$ and the transitions are only those of the original one for the states in $\mathcal{A}$ is reversible. We say in this case that the resulting process is *truncated* to the set $\mathcal{A}$. A similar result holds if the transition rates from set $\mathcal{S} \setminus \mathcal{A}$ to $\mathcal{A}$ are changed by the same multiplicative factor. In this paper we prove that, in general, these results do not hold for $\rho$-reversible and dynamically reversible chains, but they require some further conditions that are trivially satisfied in the case of reversible chains. It is worth to observe that, to the best of our knowledge, this is the first work that studies the truncation properties for Markov chains that are reversible modulo state renaming, including those that are know to be dynamically reversible. In fact, in [13,25] the authors consider only the truncation of reversible processes. We also investigate the definition of the aggregated process for reversible and $\rho$-reversible chains and prove that they also are reversible or $\rho$-reversible.

The paper is structured as follows. Section 2 illustrates the preliminary notions and the notation which are necessary to keep the paper self-contained. In Sections 3 and 4 we prove the new results about process aggregation and truncation, respectively. Finally, Section 5 concludes the paper.

## 2 Preliminaries

Let us consider a Markov chain $X(t)$ defined on the state space $\mathcal{S}$. For the sake of brevity we study the continuous time case, i.e., $t \in \mathbb{R}$. Given a stationary CTMC $X(t)$ the process $X(\tau - t)$, denoted by $X^R(t)$, is still a stationary CTMC [13] and the equilibrium state probability $\pi$ for $X(t)$ is the same of that of $X^R(t)$.

In [19] the notion of reversibility for CTMC has been generalized to a notion of reversibility under state renaming named *$\rho$-reversibility*.

Formally, a *renaming* $\rho$ over the state space of a Markov chain is a bijection from $\mathcal{S}$ to itself. For a Markov chain $X(t)$ with state space $\mathcal{S}$ we denote by $\rho(X)(t)$ the same process where the state names are changed according to $\rho$.

The notion of $\rho$-reversibility is defined as follows.

**Definition 1.** ($\rho$-reversibility) [19,21] *Let $X(t)$ be a stationary CTMC with state space $\mathcal{S}$ and $\rho$ be a renaming on $\mathcal{S}$. $X(t)$ is said to be $\rho$-*reversible *if for all $t_1, t_2, \ldots, t_n, \tau \in \mathbb{R}$, $(X(t_1), \ldots, X(t_n))$ has the same equilibrium distribution as $(\rho(X)(\tau - t_1), \ldots, \rho(X)(\tau - t_n))$. Moreover, if $\rho$ is the identity we say that $X(t)$ is* reversible *whereas if $\rho$ is a non-trivial involution, i.e., $\forall s \in \mathcal{S}$ $\rho(\rho(s)) = s$ but $\rho$ is not the identity, then $X(t)$ is said to be* dynamically reversible *[13].*

Notice that from Definition 1 and the fact that $X(t)$ and $X^R(t)$ have the same equilibrium state distribution, it follows that:

$$\pi(s) = \pi(\rho(s)) \quad \text{for all } s \in \mathcal{S}\,.$$

It is important to observe that a CTMC may be $\rho_1$-reversible and $\rho_2$-reversible for some $\rho_1 \neq \rho_2$. In [18] we prove that the extension of dynamic reversibility to $\rho$-reversibility is non-trivial since there exist CTMCs such that they have a function $\rho$ for which they are $\rho$-reversible but there does not exist any involution that makes them dynamically reversible.

The following proposition, proved in [19], gives necessary and sufficient conditions for a CTMC to be $\rho$-reversible given a certain $\rho$.

**Proposition 1.** ($\rho$-detailed balance equations*)* *Let $X(t)$ be an ergodic CTMC with state space $\mathcal{S}$ and infinitesimal generator matrix $\mathbf{Q}$. Let $\rho$ be a renaming on $\mathcal{S}$. $X(t)$ is $\rho$-*reversible *if and only if there exists a collection of positive numbers*

*$\pi(s)$, $s \in \mathcal{S}$, summing to unity that satisfy the following system of* $\rho$-detailed balance equations*:*

$$\pi(s)q(s,s') = \pi(\rho(s'))q(\rho(s'),\rho(s)) \quad \textit{for all } s,s' \in \mathcal{S}\,, \tag{1}$$

*where $q(s,s')$ denotes the transition rate from state $s$ to $s'$, with $s \neq s'$. If such a solution $\boldsymbol{\pi}$ exists then it is the equilibrium distribution of both $X(t)$ and $\rho(X^R)(t)$ and $\pi(s) = \pi(\rho(s))$ for all $s \in \mathcal{S}$.*

If the equilibrium distribution of $X(t)$ is known, the following corollary gives a straightforward way to decide if $X(t)$ is $\rho$-reversible given a certain $\rho$.

**Corollary 1.** *Let $X(t)$ be an ergodic CTMC with state space $\mathcal{S}$, infinitesimal generator matrix $\mathbf{Q}$ and equilibrium distribution $\boldsymbol{\pi}$. Let $\rho$ be a renaming on the state space $\mathcal{S}$. If the transition rates of $X(t)$ satisfy the following system of equations:*

$$\pi(s)q(s,s') = \pi(s')q(\rho(s'),\rho(s)) \quad \textit{for all } s,s' \in \mathcal{S}$$

*then $X(t)$ is* $\rho$-reversible.

The previous methods to decide the property of $\rho$ reversibility are based on the computation or the knowledge of the equilibrium distribution. In contrast, Kolmogorov's critera are purely structural, i.e., they depend only on the structure of the underlying transition graph and on the transition rates and do not require the solution of a linear system of equations.

**Proposition 2.** *Let $X(t)$ be an ergodic CTMC with state space $\mathcal{S}$ and infinitesimal generator matrix $\mathbf{Q}$, and $\rho$ be a renaming on $\mathcal{S}$. $X(t)$ is* $\rho$-reversible *if and only if for every finite sequence $s_1, s_2, \dots s_n \in \mathcal{S}$,*

$$\begin{aligned} &q(s_1,s_2)\ \cdots\ q(s_{n-1},s_n)\,q(s_n,s_1) = \\ &\qquad\qquad q(\rho(s_1),\rho(s_n))\,q(\rho(s_n),\rho(s_{n-1}))\ \cdots\ q(\rho(s_2),\rho(s_1)) \end{aligned}$$

*and $q(s) = q(\rho(s))$ for every state $s \in \mathcal{S}$.*

The equilibrium distribution of a $\rho$-reversible CTMC can be computed as stated in Proposition 3. Notice that Proposition 3 gives a numerically stable method to compute a non-trivial invariant measure of the process since for each state it requires the computation only of products.

**Proposition 3.** *Let $X(t)$ be an ergodic CTMC with state space $\mathcal{S}$ and infinitesimal generator matrix $\mathbf{Q}$, $\rho$ be a renaming on $\mathcal{S}$, and $s_0, s_1, s_2, \dots s_n = s \in S$ be a finite sequence of states. If $X(t)$ is* $\rho$-reversible *then for all $s \in \mathcal{S}$,*

$$\pi(s) = C_{s_0} \prod_{k=1}^{n} \frac{q(\rho(s_{k-1}),\rho(s_k))}{q(s_k,s_{k-1})} \tag{2}$$

*where $s_0 \in \mathcal{S}$ is an arbitrary reference state and $C_{s_0} \in \mathbb{R}^+$.*

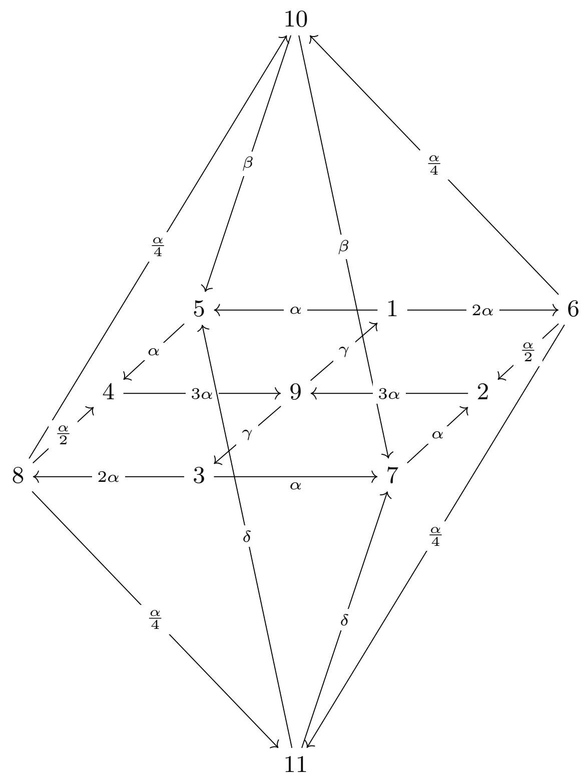



Fig. 2: A $\rho$-reversible CTMC.

Recall that a permutation $\rho$ on a set $S$ admits a unique decomposition into *cycles* of different states:

$$(s, \rho(s), \rho(\rho(s)), \ldots, \rho^n(s) \equiv s)\,.$$

The set of states in a cycle form an orbit. Then, every permutation can be decomposed into a collection of cycles on disjoint orbits.

*Example 1.* If we consider the CTMC depicted in Figure 2 we can prove that it is $\rho$-reversible where $\rho$ is described by the following orbits: $(1, 2, 3, 4)$; $(5, 6, 7, 8)$; $(9)$; $(10)$; $(11)$.

Now, we review an aggregation technique for CTMCs that preserve the equilibrium distribution, i.e., the equilibrium probability of the macro state is given by the sum of the equilibrium probability of its elements in the original, non aggregated, process. More formally, let $\sim$ be an equivalence relation over the state space $\mathcal{S}$ of a CTMC $X(t)$. In general, the process obtained by the observation of the macro state jump process is *not* a Markov process (for instance the residence time in an aggregated state is not exponentially distributed) unless we have a lumping [15]. However, we can still define CTMC $\widetilde{X}(t)$ corresponding to a certain aggregation $\sim$ as follows: the state space is the set of the equivalence classes $\mathcal{S}/\sim$ and its infinitesimal generator matrix $\widetilde{\mathbf{Q}}$ can be derived from the following general aggregation equation for any $S_i, S_j \in \mathcal{S}/\sim$ ,

$$\widetilde{q}(S_i, S_j) = \frac{\sum_{s' \in S_i} \pi(s') \sum_{s \in S_j} q(s', s)}{\sum_{s' \in S_i} \pi(s')} \tag{3}$$

The following proposition shows that the equilibrium distribution of the aggregated process is such that the equilibrium probability of each macro-state is the sum of the equilibrium probabilities of the states in the original process forming it.

**Proposition 4.** *Let $X(t)$ be an ergodic CTMC with state space $\mathcal{S}$ and $\sim$ be an equivalence relation over $\mathcal{S}$. Let $\widetilde{X}(t)$ be the aggregated process with respect to $\sim$. Let $\boldsymbol{\pi}$ and $\widetilde{\boldsymbol{\pi}}$ be the equilibrium distributions of $X(t)$ and $\widetilde{X}(t)$, respectively. Then for all $S \in \mathcal{S}/\sim$,*

$$\widetilde{\pi}(S) = \sum_{s \in S} \pi(s).$$

## 3 Aggregation of $\boldsymbol{\rho}$-reversible processes

Aggregation is a technique for reducing the state space of a model and hence for deriving some quantitative measures more efficiently. Unfortunately, for general processes, an aggregation of states that respects the equilibrium distributions (i.e., the equilibrium probability of a macro state is given by the sum of the equilibrium probabilities of the states that it aggregates) is as hard to compute as the computation of the model equilibrium distribution as shown by Equation (3). Strong lumping [15] is a structural approach to state aggregation, i.e., the definition of the aggregated chain does not require the knowledge of its equilibrium distribution. In this section we will show that also the class of $\rho$-reversible CTMCs can be aggregated in a process whose transition rates can be obtained without the knowledge of the equilibrium distribution and hence can be performed efficiently. Before stating the results on the aggregation in $\rho$-reversible (and hence also reversible) CTMCs, we need to introduce a definition of compatibility of an aggregation with a renaming $\rho$. Intuitively, we say that an aggregation $\sim$ respects renaming $\rho$ if its equivalence classes are either singletons or if they contain more states then they must cluster together all the states of the corresponding orbits.

**Definition 2.** *An aggregation $\sim$ respects a renaming $\rho$ on $\mathcal{S}$ if for each $S \in \mathcal{S}/\sim$ at least one of the following conditions is satisfied:*

– $|S| = 1$, *or*
– $s \in S$ *implies* $\rho(s) \in S$.

We stress on the fact that Definition 2 does not require that the state partitions correspond to the orbits of $\rho$, but it states that if we aggregate two states, then all the states in their orbits must belong to the same partition. However, states that are not aggregated do not need to satisfy this conditions.

*Example 2.* Consider the CTMC with states space $\mathcal{S} = \{s_1, \dots s_8\}$ and let the orbit of $\rho$ be $(s_1, s_2)$, $(s_3, s_4)$, $(s_5, \dots, s_8)$, then the following partitions of states respects $\rho$:

– $S_1 = \{s_1, s_2\}$, $S_2 = \{s_3, \dots, s_8\}$
– $S_1 = \{s_1\}$, $S_2 = \{s_2\}$, $S_3 = \{s_3, s_4\}$, $S_4 = \{s_5, \dots, s_8\}$
– $S_i = \{s_i\}$ (the trivial partition)

Theorem 1 states that an aggregation $\sim$ of a $\rho$-reversible chain $X(t)$ is $\widetilde{\rho}$-reversible for a certain renaming $\widetilde{\rho}$ if $\sim$ respects $\rho$.

**Theorem 1.** *Let $X(t)$ be a $\rho$-reversible CTMC and let $\sim$ be an aggregation that respects $\rho$ according to Definition 2. Then, Markov chain $\widetilde{X}(t)$ is $\widetilde{\rho}$-reversible where $\widetilde{\rho}$ is defined as follows:*

$$\widetilde{\rho}(S_i) = \begin{cases} S_j & \text{if } S_i = \{s\} \wedge S_j = \{\rho(s)\}\,, \\ S_i & \text{if } |S_i| > 1\,. \end{cases} \tag{4}$$

Let us analyse some consequences of Theorem 1. Let $X(t)$ be a $\rho$-reversible CTMC with state space $\mathcal{S}$ and $\sim$ be the equivalence relation over $\mathcal{S}$ such that $s_1 \sim s_2$ if and only if $s_1$ and $s_2$ belongs to the same orbit with respect to the permutation $\rho$. Then clearly $\sim$ respects $\rho$ according to Definition 2, $\widetilde{\rho}$ is the identity on $\mathcal{S}/\sim$ and $\mathcal{S}/\sim$ denotes the set of all orbits induced by $\rho$ in $\mathcal{S}$. In this case we say that $\sim$ is the equivalence relation induced by $\rho$ in $\mathcal{S}$.

**Corollary 2.** *Let $X(t)$ be a $\rho$-reversible CTMC with state space $\mathcal{S}$ and infinitesimal generator matrix $\mathbf{Q}$. Let $\sim$ be the equivalence relation over $\mathcal{S}$ induced by $\rho$. Then $\widetilde{X}(t)$ is reversible.*

*Proof.* The proof follows from Theorem 1 and the observation that $\widetilde{\rho}$ is the identity (see Definition 1).

For this type of aggregation the transition rates of the aggregated process can be calculated without the computation of the equilibrium state distribution $\pi$.

**Proposition 5.** *Let $X(t)$ be a $\rho$-reversible CTMC with state space $\mathcal{S}$ and infinitesimal generator matrix $\mathbf{Q}$. Let $\sim$ be the equivalence relation over $\mathcal{S}$ induced by $\rho$. Then, the infinitesimal generator matrix $\widetilde{\mathbf{Q}}$ of $\widetilde{X}(t)$ is defined as:*

$$\widetilde{q}(S_i, S_j) = \frac{\sum_{s' \in S_i} \sum_{s \in S_j} q(s', s)}{|S_i|} \tag{5}$$

*where $|S_i|$ denotes the cardinality of the orbit $S_i$.*

The next corollary follows immediately from Theorem 1 and states that any aggregation of a reversible chain is still reversible.

**Corollary 3.** *Let $X(t)$ be a reversible CTMC, then for any aggregation $\sim$ on its state space $\mathcal{S}$ we have that $\widetilde{X}(t)$ is still reversible.*

*Proof.* Observe that if $X(t)$ is reversible then $\rho$ is the identity and hence any aggregation $\sim$ respects $\rho$. The proof follows by observing that by definition also $\widetilde{\rho}$ is the identity and hence $\widetilde{X}(t)$ is reversible. □

*Example 3.* Let us aggregate the $\rho$-reversible process of Figure 2 with respect to relation $\sim$ induced by the orbits of the CTMC. Then, by Proposition 5 we can straightforwardly derive the aggregated process of Figure 3. It is easy to observe that the resulting CTMC is reversible.

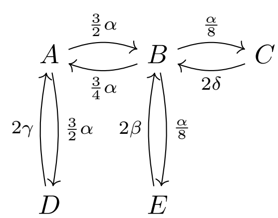



Fig. 3: Aggregation according to the orbits of the CTMC shown in Figure 2.

## 4 Truncation of $\rho$-reversible processes

The truncation of a reversible CTMC is a very useful technique to study models in which some agents compete for a set of resources. For instance, consider a reversible chain that models $N$ agents performing a set of operations some of which require a certain resource whose availability is $M < N$. In order to study the equilibrium properties of the model, we may assume that the resource is always available for all the agents and prove the reversibility of the underlying process, then we have to exclude the transitions that would take the model to states in which more than $M$ resources are used simultaneously. In [13, Lemma 1.9, Corollary 1.10] the author proves that if the original process is reversible then also the truncated one is reversible.

In this section we study the same problem with $\rho$-reversible processes. The main result that we derive is that, in general, the truncation of a $\rho$-reversible process is not $\rho$-reversible. In fact, in order to prove the analogue result of Lemma 1.9 and Corollary 1.10 of [13] we require that the truncation respects the orbits of $\rho$, i.e., each orbit is either entirely truncated or kept.

A reversible CTMC may be altered by changing the transition rates in such a way that the equilibrium distribution is not changed. As observed in [13] if a reversible CTMC $X(t)$ has transition rate $q(s,u) > 0$ and $q(u,s) > 0$, then also the CTMC $X'(t)$ whose transition rates are the same of $X(t)$ with the exception of $q'(s,u) = cq(s,u)$ and $q'(u,s) = cq(u,s)$ for $c > 0$ is still reversible. The result follows immediately from Proposition 1 assuming $\rho$ to be the identity. We notice that this result is in general not applicable to $\rho$-reversible CTMCs since the modification of $q(s,u)$ to $cq(s,u)$ changes the residence time of state $i$ and the definition of $\rho$-reversibility requires that all the states in the orbit of $s$ must have the same residence time.

*Example 4.* Let us consider the model of Figure 1-(a) and let us write the $\rho$-detailed balance equation associated with the transition from state 1 to state 2:

$$\pi(1)q(1,2) = \pi(\rho(2))q(\rho(2),\rho(1))\,.$$

Notice that, since $\rho(1) = 2$ and $\rho(2) = 1$ we have that $q(1,2) = q(\rho(2),\rho(1))$, hence the detailed balance equation is satisfied even if we set $q'(1,2) = c\alpha$, for $c > 0$ and $c \neq 1$. Nevertheless, the CTMC $X'(t)$ is *not* $\rho$-reversible since the residence time in state 1 has mean $(c\alpha)^{-1}$ while in state 2 has mean $\alpha^{-1}$.

The following Lemma is the version of Lemma 1.9 in [13] for $\rho$-reversible CTMCs.

**Lemma 1.** *Let $X(t)$ be a $\rho$-reversible CTMC with state space $\mathcal{S}$ and let $\sim$ be an equivalence relation that induces only two non-empty equivalence classes $\mathcal{A} \subset \mathcal{S}$ and $\mathcal{S} \setminus \mathcal{A}$. Then, if $\sim$ respects $\rho$ we have that for any positive constant $c \in \mathbb{R}$ the chain $X'(t)$ whose transition rates $q'(s,u)$ are defined as follows:*

$$q'(s,u) = \begin{cases} cq(s,u) & \text{if } s \in \mathcal{A} \wedge u \in \mathcal{S} \setminus \mathcal{A} \\ q(s,u) & \text{otherwise} \end{cases}$$

*is still $\rho$-reversible if the residence time of the states in $X'(t)$ are identically distributed for all the states belonging to the same orbit. Moreover, if $X'(t)$ is $\rho$-reversible, then the equilibrium distribution $\pi'(s)$ for $X'(t)$ is:*

$$\pi'(s) = \begin{cases} B\pi(s) & \text{if } s \in \mathcal{A} \\ Bc\pi(s) & \text{if } s \in \mathcal{S} \setminus \mathcal{A} \end{cases},$$

*where $B$ is a normalising constant.*

The following corollary follows from Lemma 1 where $c = 0$ and is the analogue of Corollary 1.10 in [13].

**Corollary 4.** *Let $X(t)$ be a $\rho$-reversible CTMC with state space $\mathcal{S}$ and let $\sim$ be an equivalence relation that induces only two non-empty equivalence classes $\mathcal{A} \subset \mathcal{S}$ and $\mathcal{S} \setminus \mathcal{A}$. Let $\sim$ respect $\rho$, and define the chain $X'(t)$ on the state space $\mathcal{A}$ with transition rates:*

$$q'(s,u) = \begin{cases} q(s,u) & \text{if } s, u \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases}.$$

*Then if $X'(t)$ is irreducible and the residence time of every state $s \in A$ is the same of every other state $u$ in the same orbit of $s$, we have that $X'(t)$ is $\rho$-reversible. In this case the equilibrium probabilities of $s \in \mathcal{A}$ are:*

$$\pi(s) = \frac{\pi(s)}{\sum_{u \in \mathcal{S}} \pi(u)} .$$

*Proof.* The proof follows straightforwardly from Lemma 1. □

*Example 5.* Let us consider a manufacturing system where $K$ independent machines produce parts of a product that will be assembled once all the $K$ components are available. Let us assume that the time required to produce one component from a machine is modelled by an independent and exponentially distributed random variable with rate $\mu$. The components wait for being assembled in $K$ *join queues*. This is usually reffered as a kitting process. In [17,20] the authors proved that join queue lengths tend to grow infinitely due to the variance of the component production time. Moreover, assuming that the assembly operation is instantaneous, the underlying CTMC $X(t)$ can be studied by means of a dynamically reversible process. It is sufficient to encode the state as a vector $\mathbf{n} = (n_1, \ldots, n_K)$ of integer components that represent the difference in the number of pieces produced by the $k$-th machine and its neighbour $k^+$ where

$$k^+ = \begin{cases} k+1 & \text{if } k < K \\ 1 & \text{if } k = K . \end{cases}$$

The state space of the model is $\mathcal{S} = \{\mathbf{n} : \sum_{k=1}^{K} n_k = 0 \wedge n_k \in \mathbb{Z}\}$. The expected queue length becomes finite [20] if we can modulate the rates of the component production machines as follows:

$$\mu(n_k) = \begin{cases} \frac{\mu}{n_k+1} & \text{if } n_k \geq 0 \\ \mu & \text{otherwise} \end{cases} .$$

Such a CTMC is dynamically reversible and hence $\rho$-reversible according to the following renaming function:

$$\rho(\mathbf{n}) = \rho(n_1, \ldots, n_K) = (n_K, \ldots, n_1) = \mathbf{n}^R ,$$

and the equilibrium distribution is given by the expression [20]:

$$\pi(\mathbf{n}) = \frac{1}{G_K} \frac{1}{\prod_{i=1}^{K} (n_i \delta_{n_i>0})!} , \tag{6}$$

where $\delta_{n_i>0} = 1$ if $n_i$ is positive, 0 otherwise and $G_K$ is a normalising constant.

Let us assume that we want to change the model such that we impose that the difference between the number of components given by production like $k$ and $k^+$ is smaller or equal to $T$, i.e. $n_k \leq T$ for all $k = 1, \ldots, K$. The machine that saturates its join queue according to this condition is stopped and will restart working when its neighbour will complete a job. This means that under the immediate assembly time assumption, the maximum join-queue length that we can observe is $(K-1)\cdot T$. Clearly, the CTMC $X'(t)$ underlying such a model is a truncation of the original one, where $\mathcal{A} = \{\mathbf{n} \in \mathcal{S}, n_k \leq T$ for all $k = 1, \dots, K\}$. To prove that $X'(t)$ is still $\rho$-reversible, we use Corollary 4 and we have to show that:

- The partition respects $\rho$,
- The residence time of $\mathbf{n} \in \mathcal{A}$ and $\mathbf{n}^R$ have the same mean in $X'(t)$.

The first point is easy to prove since if $\mathbf{n} \in \mathcal{A}$ then also $\mathbf{n}^R \in \mathcal{A}$ and vice versa. The second one is trivial since the sum of the arrival rates of the components in $\mathbf{n}$ and $\mathbf{n}^R$ are the same. Therefore, Equation (6) is an invariant measure for $X'(t)$.

## 5 Conclusion

In this paper we have studied the aggregation and truncation properties of Markov chains which are reversible modulo a renaming of states. In physics (see e.g., [10,9,11]) and computer science (e.g., [22,20]) we can find numerous applications of this theory in the formulation known as *dynamic reversibility*. By the notion of $\rho$-reversibility, we generalised this definition to arbitrary state renaming functions and showed that the extension is non-trivial, i.e., there are Markov chains which are not dynamically reversible but are $\rho$-reversible for some $\rho$ which is not an involution. In this paper we have established an important link between $\rho$-reversibility and the well-known notion of Kelly's reversibility by showing that a certain aggregation of a $\rho$-reversible chain originates a reversible chain. Although this aggregation is *not* a strong lumping in the sense of Kemeny and Snell work [15], we still have that the aggregated process can be constructed without the computation of the equilibrium distribution of the original chain. Finally, we have revised the well-know results about the truncation of reversible processes in the context of $\rho$-reversibility and have shown some results that generalise them. Specifically, while the truncation of a reversible chain is always reversible (provided that the irreducibility of the transition graph is maintained) we need some further conditions in order to prove that the truncation of a $\rho$-reversible chain is also $\rho$-reversible. These conditions are always trivially satisfied for reversible chains.

## References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## A Proofs of the results

### Proof of Theorem 1

*Proof.* By Proposition 1 and Definition 1, to prove that $\widetilde{X}(t)$ is $\widetilde{\rho}$-reversible it is sufficient to show that for all $S_i, S_j \in \mathcal{S}/\sim$ with $i \neq j$,

$$\widetilde{\pi}(S_i)\widetilde{q}(S_i, S_j) = \widetilde{\pi}(\widetilde{\rho}(S_j))\widetilde{q}(\widetilde{\rho}(S_j), \widetilde{\rho}(S_i)) .$$

By Equation (3) and Proposition 4, this is equivalent to:

$$\left(\sum_{s\in S_i} \pi(s)\right) \frac{\sum_{s\in S_i} \pi(s) \sum_{s'\in S_j} q(s,s')}{\sum_{s\in S_i} \pi(s)} = \left(\sum_{s'\in\widetilde{\rho}(S_j)} \pi(s')\right) \frac{\sum_{s'\in\widetilde{\rho}(S_j)} \pi(s') \sum_{s\in\widetilde{\rho}(S_i)} q(s',s)}{\sum_{s'\in\widetilde{\rho}(S_j)} \pi(s')} ,$$

which can be written as:

$$\sum_{s\in S_i} \sum_{s'\in S_j} \pi(s)q(s,s') = \sum_{s'\in\widetilde{\rho}(S_j)} \sum_{s\in\widetilde{\rho}(S_i)} \pi(s')q(s',s) . \tag{7}$$

We now proceed by considering four cases.

1. Assume that $S_i = \{s\}$ and $S_j = \{s'\}$, then Equation (7) becomes:

   $$\pi(s)q(s,s') = \pi(\rho(s'))q(\rho(s'), \rho(s)) ,$$

   where we have used the definition of $\widetilde{\rho}$ for singletons. This is true since by hypothesis $X(t)$ is $\rho$-reversible and hence satisfies the $\rho$-detailed balance equation.
2. Assume $S_i = \{s\}$ and $|S_j| > 1$, and recall that $\widetilde{\rho}(S_j) = S_j$ by definition. Then Equation (7) can be rewritten as:

   $$\sum_{s'\in S_j} \pi(s)q(s,s') = \sum_{s'\in S_j} \pi(s')q(s', \rho(s)) .$$

   Since $\sim$ respects $\rho$ we have that $\rho$ restricted to the elements of $S_j$ is still a bijection and hence we can write:

   $$\sum_{s'\in S_j} \pi(s)q(s,s') = \sum_{s'\in S_j} \pi(\rho(s'))q(\rho(s'), \rho(s)) ,$$

   which is true by the hypothesis of $\rho$-reversibility of $X(t)$.
3. Assume $|S_i| > 1$ and hence $\widetilde{\rho}(S_i) = S_i$ and $S_j = \{s'\}$, then Equation (7) can be written as:

   $$\sum_{s\in S_i} \pi(s)q(s,s') = \sum_{s\in S_i} \pi(\rho(s'))q(\rho(s'), s) .$$

Since $\rho$ restricted to the elements of $S_i$ is a bijection, then we have:

$$\sum_{s \in S_i} \pi(s) q(s, s') = \sum_{s \in S_i} \pi(\rho(s')) q(\rho(s'), \rho(s)) \,,$$

which is an identity.

4. Assume $|S_i| > 1$ and $|S_j| > 1$, and hence $\widetilde{\rho}(S_i) = S_i$ and $\widetilde{\rho}(S_j) = S_j$. Then we can rewrite Equation (7) as:

$$\sum_{s \in S_i} \sum_{s' \in S_j} \pi(s) q(s, s') = \sum_{s \in S_i} \sum_{s' \in S_j} \pi(s') q(s', s) \,.$$

Since $\rho$ restricted to $S_i$ and to $S_j$ is still a bijection because $\sim$ respects $\rho$, we can rewrite the previous equation as:

$$\sum_{s \in S_i} \sum_{s' \in S_j} \pi(s) q(s, s') = \sum_{s \in S_i} \sum_{s' \in S_j} \pi(\rho(s')) q(\rho(s'), \rho(s)) \,.$$

which is true by hypothesis. □

## Proof or Proposition 5

*Proof.* By the general aggregation Equation (3), for any $S_i, S_j \in \mathcal{S}/\sim$ ,

$$\widetilde{q}(S_i, S_j) = \frac{\sum_{s' \in S_i} \pi(s') \sum_{s \in S_j} q(s', s)}{\sum_{s' \in S_i} \pi(s')} \,, \tag{8}$$

Since $X(t)$ is $\rho$-reversible and each $S_i \in S/\sim$ is an orbit for $\rho$, it holds that $\pi(s) = \pi(s')$ for all $s, s' \in S_i$. Let us denote by $\pi(S_i)$ the equilibrium probability of each $s$ belonging to the orbit $S_i$. Hence, $\sum_{s' \in S_i} \pi(s') = |S_i| \pi(S_i)$ and Equation (8) can be written

$$\widetilde{q}(S_i, S_j) = \pi(S_i) \frac{\sum_{s' \in S_i} \sum_{s \in S_j} q(s', s)}{|S_i| \pi(S_i)} \tag{9}$$

proving the statement. □

## Proof of Lemma 1

*Proof.* To prove the lemma we use Proposition 1. In fact, let us consider two states $s, u \in \mathcal{A}$, then the corresponding $\rho$-detailed balance equation is $B\pi(s)q(s,u) = B\pi(\rho(u))q(\rho(u), \rho(s))$ since we have by assumption that the partition respects $\rho$ and hence also $\rho(t), \rho(s) \in \mathcal{A}$. This equation is satisfied because $X(t)$ is $\rho$-reversible. If $s, u \in \mathcal{S} \setminus \mathcal{A}$ the corresponding detailed balance equation is $Bc\pi(s)q(s,u) = Bc\pi(\rho(u))q(\rho(u), \rho(s))$ that is also satisfied for the same reasons. Let us consider $s \in \mathcal{A}$ and $u \in \mathcal{S} \setminus \mathcal{A}$, then we have that the transition rates are modified and hence $B\pi(s)\left(cq(s,u)\right) = Bc\pi(\rho(u))q(\rho(u), \rho(s))$ which is an identity since $\sim$ respects $\rho$. Finally, we have to consider the case of $s \in \mathcal{S} \setminus \mathcal{A}$ and $u \in \mathcal{A}$. The corresponding detailed balance equation is $Bc\pi(s)q(s,u) = B\pi(\rho(u))\left(cq(\rho(u), \rho(s))\right)$ which is satisfied by hypothesis. The fact that the residence times in the states belonging to the same orbits of $\rho$ in $X'(t)$ are identically distributed is an assumption of the lemma. □