# Automated Verification of Behavioural Properties of Prolog Programs

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**Abstract.** Program verification is a crucial issue in the field of program development, compilation and debugging. In this paper, we present an analyser for Prolog which aims at verifying whether the execution of a program behaves according to a given specification (behavioural assumptions). The analyser is based on the methodology of abstract interpretation. A novel notion of abstract sequence is introduced, that includes an over-approximatimation of successful inputs (this is useful to detect mutual exclusion of clauses), and expresses size relation information between successful inputs and the corresponding outputs, together with cardinality information in terms of input argument sizes.

**Keywords**: Program Verification, Static Analysis, Logic Programming, Prolog.

## 1 Introduction

Declarative languages have received great attention in the last years, as they allow the programmer to focus on the description of the problem to be solved while ignoring low level implementation details. Nevertheless, the implementation of declarative languages remains a delicate issue. Very often, languages include additional "impure" features which are intended to improve the efficiency of the programs but do not respect their declarative nature. This is what happens in logic programming with Prolog, where a number of these "impure" features arise, e.g., the incomplete (depth-first) search rule, the non logical negation by failure, the non-logical test predicates like `var`, and the cut.
Many static analysis techniques have been proposed in the literature to improve on this situation. Some analyses aim at optimizing programs automatically, relieving the programmer from using impure control features [8, 18]. Other analyses are designed to verify that a non declarative implementation of a program does behave according to its declarative meaning [6, 9]. These analyses may also be useful to transform a first (declaratively but not operationally correct) version of a program into a both declaratively and operationally correct program.

In this paper, we describe an analyser for Prolog programs which aims at verifying whether the concrete execution of a program behaves according to a

given specification consisting of a set of behavioural assumptions. The analyser is based on the methodology of abstract interpretation [5, 14]: it can be seen as an online approximation of (sets of) concrete program executions. However, instead of performing a fixpoint computation, the analyser makes use of declarations on allowed program executions (*behaviours*) provided by the user. Therefore, the emphasis here is not on the analysis of substitution properties, like modes, sharing and types, that can be automatically inherited from previous works [14], but on the automatic verification of assumptions like the number of solutions, or the size relations between input and output arguments.
Our analyser is built upon the notion of *abstract sequence* [2, 3, 12]. Abstract sequences describe pairs $\langle \theta, S \rangle$, where $\theta$ is a substitution and $S$ is the sequence of answer substitutions resulting from executing a program (a procedure, a clause, etc.) with input substitution $\theta$. In this paper, we revisit this notion so that

- the new notion includes an over-approximatimation of successful inputs: this is useful to detect mutual exclusion of clauses,
- it allows to express size relation information between successful inputs and the corresponding outputs,
- it allows to express cardinality information in terms of input argument sizes.

Basically, the new notion of abstract sequence is more "relational", since it may relate the number of solutions and the size of output terms to the size of input terms in full generality. For instance, it may relate the input and output sizes of the same [4] term without requiring any invariance under instantiation.

This paper presents the main features of our analyser which uses abstract sequences for computing non trivial information on size relations and cardinality. The interested reader may found more details and correctness proofs in [10]. The practical implementation of the analyser (which is still under progress) is based on the generic system *GAIA* [14] and on the polyhedron library described in [19] to manipulate size information.

The paper is organized as follows. Section 2 provides an overview of the analyser. Section 3 recalls some basic concepts. Section 4 illustrates our domain of abstract sequences. Section 5 describes the analyser. Section 6 discusses the implementation of two abstract operations. Section 7 concludes the paper.

## 2 Overview of the Analyser

In this section, we introduce the main functionalities of the analyser by discussing a simple example. Consider the Prolog procedure `select/3` depicted below.

```
select(X, L, LS):- L=[H|T], H=X, LS=T, list(T).
select(X, L, LS):- L=[H|T], LS=[H|TS], select(X, T, TS).
```

Declaratively, it defines a relation between three terms[5] X, L, and LS that holds

[4] i.e., bound to the same program variable.

[5] We use roman letters to denote the values to which program variables are instantiated. Syntactic objects are denoted by typewriter characters.

if and only if the terms L and LS are lists and LS is obtained by removing one occurrence of X from L. Our analyser is not aimed at verifying this (informal) declarative specification. Instead, it checks a number of operational properties which ensure that the program execution of this program actually computes the specified relation, provided that the procedure is "declaratively" correct (and its operational specification guarantees such correctness). We assume one particular and reasonable class of input calls, i.e., calls such that X and LS are *distinct* variables and L is any ground term (not necessarily a list). For this class of input calls, the user has to provide a description of the expected behaviour of the procedure by means of an abstract sequence $B$ and a size expression $se$. The abstract sequence $B$ is a tuple $\langle \beta_{in}, \beta_{ref}, \beta_{out}, E_{ref_out}, E_{sol} \rangle$, where

1. $\beta_{in}$ is an abstract substitution describing the above class of input calls;
2. $\beta_{ref}$ is an abstract substitution describing an over approximation of the successful input calls, i.e., those that produce at least one solution: in our example, $\beta_{ref}$ states that L is a non empty ground list;
3. $\beta_{out}$ is an abstract substitution describing an over approximation of the set of outputs corresponding to the successful calls: in our example, $\beta_{out}$ states that X is a ground term and LS is a ground list;
4. $E_{ref_out}$ describes a relation between the size of the terms of a successful call and the size of the terms returned by the call: in our example, $E_{ref_out}$ states that the input length of L is equal to the output length of LS plus 1;
5. $E_{sol}$ describes a relation between the size of the terms occurring in a successful call and the number of solutions returned by the call: in our example, $E_{sol}$ states that the number of solutions is equal to the input length of L.

The size expression $se$ is a positive integer expression over the formal parameters of the procedure denoting the size of the corresponding input terms; this expression must decrease strictly through recursive calls. In our example, $se$ is equal to `L` representing the input length of L, denoted by $\|L\|$.

Starting from $B$ and $se$, the analyser computes a number of abstract sequences, one for every prefix of the body of every clause, one for every clause, and, finally, one for the complete procedure. In our example, the analyser computes an abstract sequence $B_1$ for the first clause, expressing that for the specified class of input calls, the first clause succeeds if and only if L is a non empty list, and it succeeds exactly once. The derivation of this information is possible because the analyser is able to detect that the unification `L=[H|T]` succeeds if and only if L is of the form $[t_1|t_2]$ (not necessarily a list) and that both unifications `H=X` and `LS=T` surely succeed since X and LS are free and do not share. $B_1$ is obtained by combining this information with the abstract sequence that describes the behaviour of the procedure `list/1`. The latter states that, for ground calls, the literal succeeds only for lists, and it succeeds exactly once.

The second clause of `select/3` is treated similarly. Only the recursive call deserves a special treatment. First, the analyser infers that the recursive call will be executed at most once and, in fact, exactly once when L is of the form $[t_1|t_2]$. It also infers that X and TS are distinct variables and that T is ground and

strictly smaller than L with respect to the norm $\|\cdot\|$. Thus, it can be assumed by induction that the recursive call satisfies the conditions provided by the user through the abstract sequence $B$. The analyser deduces that the recursive call succeeds only if T is a *non-empty* list and that it returns a number of solutions equal to the length of T. It also infers that X is ground and that TS is a ground list whose size is the same as the size of T minus 1. Putting all pieces together, the analyser computes the abstract sequence $B_2$ for the second clause, which states that the second clause succeeds only for a list L of at least two elements and that the output size of LS is equal to the size of L minus 1, i.e., $\|L\| - 1$; moreover, the number of solution is also equal to $\|L\| - 1$.

The last step for the analyser is to combine the abstract sequences $B_1$ and $B_2$ to get a new abstract sequence $B_{out}$ describing the behaviour of the whole procedure. Once again, a careful analysis is necessary to get the most precise result. When L is a list of at least two elements, the first clause succeeds once and the second one succeeds $\|L\| - 1$ times. Thus, the procedure succeeds $\|L\|$ times. Otherwise, when the length of L is equal to 1, the second clause fails and the first one succeeds exactly once. Thus, in both cases the procedure succeeds $\|L\|$ times. Hence, putting the abstract sequences $B_1$ and $B_2$ together, the analyser is able to reconstruct exactly the information provided by the user, which is automatically verified to be correct.

## 3 Preliminaries

The reader is assumed to be familiar with the basic concepts of logic programming and abstract interpretation [5, 17]. We denote by $\mathcal{T}$ the set of all terms, and for any set of indices $I$, we denote by $\mathcal{T}^I$ the set of all tuples of terms $\langle t_i \rangle_{i \in I}$. A size measure (or norm) is a function $\|\cdot\| : \mathcal{T} \rightarrow \mathbf{N}$. Here, we refer to the list-length measure defined for any term $t$ by $\|t\| = 1 + \|t_2\|$ if $t$ is of the form $[t_1|t_2]$ and $\|t\| = 0$ otherwise. The *disjoint union* of two (possibly non disjoint) sets $A$ and $B$ is an arbitrarily set $A + B$ in which the elements of $A$ (resp. $B$) can be identified. Formally, $A + B$ is equipped with two injections functions $in_A$ and $in_B$ such that: for any set $C$ and for any pair of functions $f_A : A \rightarrow C$ and $f_B : B \rightarrow C$, there exists a unique function $f : A + B \rightarrow C$ with $f_A = f \circ in_A$ and $f_B = f \circ in_B$ (the symbol $\circ$ is the usual function composition). We denote the function $f$ by $f_A + f_B$. For any set $V$, we denote by $\mathbf{Exp}_V$ the set of all integer linear expressions with variables in $V$. An element $se \in \mathbf{Exp}_{\{X_1,\ldots,X_m\}}$ can also be seen as a function from $\mathbf{N}^m$ to $\mathbf{N}$. The value of $se(\langle n_1, \ldots, n_m \rangle)$ is obtained by evaluating the expression $se$ where each $X_i$ is replaced by $n_i$.

Programs are assumed to be normalized as follows. A *(normalized) program* $P$ is a non empty set of procedures $pr$. A procedure is a non empty sequence of clauses $c$. Each clause has the form $h{:}-g$ where the head $h$ is of the form $p(X_1, \ldots, X_n)$ and $p$ is a predicate symbol of arity $n$, whereas the body $g$ is a possibly empty sequence of literals. A literal $l$ is either a built-in of the form $X_{i_1} = X_{i_2}$, or a built-in of the form $X_{i_1} = f(X_{i_2}, \ldots, X_{i_n})$ where $f$ is a functor of arity $n-1$, or an atom $p(X_{i_1}, \ldots, X_{i_n})$. The variables occurring in a literal are

all distinct; all clauses of a procedure have exactly the same head. We denote by $\mathcal{P}$ the set of all predicate symbols occurring in the program $P$. Variables used in the clauses are called *program variables* and are denoted by $X_1, \ldots, X_i, \ldots$.

A *(program) substitution* $\theta$ is a finite set $\{X_{i_1}/t_1, \ldots, X_{i_n}/t_n\}$ where variables $X_{i_1}, \ldots, X_{i_n}$ are distinct program variables and the $t_i$'s are terms. The domain of $\theta$, denoted by $dom(\theta)$, is the set of variables $\{X_{i_1}, \ldots, X_{i_n}\}$. Variables occurring in $t_1, \ldots, t_n$ are taken from the set of *standard variables* which is disjoint from the set of program variables. A *standard substitution* $\sigma$ is a substitution in the usual sense which only uses standard variables. The application of a standard substitution $\sigma$ to a program substitution $\theta = \{X_{i_1}/t_1, \ldots, X_{i_n}/t_n\}$ is the program substitution $\theta\sigma = \{X_{i_1}/t_1\sigma, \ldots, X_{i_n}/t_n\sigma\}$. We say that $\theta_1$ is *more general* than $\theta_2$, noted $\theta_2 \leq \theta_1$, iff there exists $\sigma$ such that $\theta_2 = \theta_1\sigma$. $mgu(t_1, t_2)$ denotes the set of standard substitutions that are a most general unifier of $t_1$ and $t_2$. The *restriction* of $\theta$ to a set of variables $D$, denoted by $\theta_{/D}$, is such that $dom(\theta_{/D}) = D$ and $X_i\theta = X_i(\theta_{/D})$, for all $X_i \in D$. A *(program) substitution sequence* $S$ is a *finite* sequence $< \theta_1, \ldots, \theta_n >$ of (program) substitutions with the same domain. We denote by $< >$ the empty sequence and by $Subst(S)$ the set of all substitutions in $S$. The symbol :: denotes the sequence concatenation.

Our analyser refers to the concrete semantics of Prolog programs presented in [13], which has been proven equivalent to Prolog operational semantics in [11]. The concrete semantics for a program $P$ is a total function from the set of pairs $\langle\theta, p\rangle$, where $p \in \mathcal{P}$ has arity $n$ and $dom(\theta) = \{X_1, \ldots, X_n\}$, to the set of substitution sequences. The fact that $\langle\theta, p\rangle$ is mapped to the sequence $S$ is denoted by $\langle\theta, p\rangle \longmapsto S$. Here, this means that the execution of $p(X_1, \ldots, X_n)\theta$ terminates and produces the (finite, possibly empty) sequence of answer substitutions $S$.

## 4 Abstract Domains

In this section, we describe the abstract objects used by the analyser, namely, *abstract substitutions*, *abstract sequences* and *behaviours*.

*Abstract Substitutions.* Our domain of *abstract substitutions* is an instantiation of the generic domain $\mathtt{Pat}(\Re)$ [4]. Here, we give only an informal presentation of it and we refer the reader to our previous papers [14, 4] for more details.
An abstract substitution $\beta$ with domain $\{X_1, \ldots, X_n\}$ describes a set of program substitutions with the same domain giving information not only about the terms to which $X_1, \ldots, X_n$ are bound, but also about some subterms of them. The terms described in $\beta$ are denoted by indices from a set $I$. We say that $\beta$ is an abstract substitution over $I$. The following properties of terms are captured: the *pattern*, which specifies the main functor of a term as well as the subterms that are its arguments; the *mode* (e.g., **ground**, **var**); the *type* (e.g., **list**); and the *possible sharing* with other subterms. We donote by $Cc(\beta)$ the set of all substitutions described by $\beta$, by $\bot$ the abstract substitution describing the empty set, i.e., $Cc(\bot) = \emptyset$, and by $\mathtt{DECOMP}(\theta, \beta)$ the set of all tuples of terms $\langle t_i\rangle_{i\in I}$ respecting the term properties described in $\beta$ and such that $\theta = \{X_1/t_{i_1}, \ldots, X_n/t_{i_n}\}$ (if $i_1, \ldots, i_n$ denote the terms bound to $X_1, \ldots, X_n$, respectively).

As an example, consider the abstract substitutions $\beta_{in}$, $\beta_{ref}$, $\beta_{out}$ informally described in Section 2. They can be represented as follows. The terms bound to the formal parameters X, L and LS are denoted by indices 1, 2 and 3, respectively, whereas the subterms of L in $\beta_{ref}$ and $\beta_{out}$ are denoted by indices 4 and 5, in

$\beta_{in}$ : $\texttt{X}_1$/var(1), $\texttt{X}_2$/ground(2), $\texttt{X}_3$/var(3), noshare(1, 3)
$\beta_{ref}$ : $\texttt{X}_1$/var(1), $\texttt{X}_2$/[ground(4)|ground_list(5)](2), $\texttt{X}_3$/var(3), noshare(1, 3)
$\beta_{out}$ : $\texttt{X}_1$/ground(1), $\texttt{X}_2$/[ground(4)|ground_list(5)](2), $\texttt{X}_3$/ground_list(3)

*Abstract Sequences.* We formally describe here our domain of *abstract sequences* which is substantially more elaborate than the similar notion used in [12, 13]. An abstract sequence $B$ is a tuple $\langle \beta_{in}, \beta_{ref}, \beta_{out}, E_{ref_out}, E_{sol} \rangle$. The first element $\beta_{in}$ is the input abstract substitution; $\beta_{ref}$ is a refinement of $\beta_{in}$ approximating the set of concrete substitutions in $Cc(\beta_{in})$ that surely succeeds (i.e., whose execution produces at least one result); $\beta_{out}$ approximates output information about variable instantiation. $E_{ref_out}$ represents size relations between the output and the input arguments (we refer to $\beta_{ref}$ for the input) whereas $E_{sol}$ expresses the number of solutions in terms of the input argument sizes. The size components $E_{ref_out}$ and $E_{sol}$ are abstract objects representing tuples of natural numbers.

In this paper, we assume that a size component $E$ over a set of indices $I$ is a system of linear equations and inequations over $\mathbf{Exp}_I$. It represents the set of all tuples of natural numbers $\langle n_i \rangle_{i \in I} \in \mathbf{N}^I$ which are solutions of $E$. We denote by $Cc(E)$ this set. In order to distinguish indices of $I$, considered as variables, from integer coefficient and constants, when writing elements of $\mathbf{Exp}_I$, we wrap up each element $i$ of $I$ into the symbol $\mathbf{sz}(i)$. If $f$ is a function from one set of indices to another one, such that $f(i) = i'$ and $f(j) = j'$, the expression $\texttt{sz}(f(i)) = \texttt{sz}(f(j)) + \texttt{1}$ stands for the syntactical equation $\texttt{sz}(i') = \texttt{sz}(j') + \texttt{1}$.

In the following definition, the symbol *sol* denotes a special index representing the number of substitutions belonging to the approximated sequences.

**Definition 1** (Abstract Sequence). An *abstract sequence* $B$ is either $\perp$ or a tuple of the form $\langle \beta_{in}, \beta_{ref}, \beta_{out}, E_{ref_out}, E_{sol} \rangle$ where

- $\beta_{in}$ is an abstract substitution over $I_{in}$;
- $\beta_{ref}$ is an abstract substitution over $I_{ref}$ with $dom(\beta_{ref}) = dom(\beta_{in})$ and $Cc(\beta_{ref}) \subseteq Cc(\beta_{in})$;
- $\beta_{out}$ is an abstract substitution over $I_{out}$ with $dom(\beta_{out}) \supseteq dom(\beta_{in})$;
- $E_{ref_out}$ is a size component over $I_{ref} + I_{out}$;
- $E_{sol}$ is a size component over $I_{ref} + \{sol\}$;
- for all $\theta' \in Cc(\beta_{out})$, $\exists \theta \in Cc(\beta_{ref})$ such that $\theta'_{/dom(\beta_{ref})} \leq \theta$.

The abstract sequence $B$ represents the set of all pairs $\langle \theta, S \rangle$, noted $Cc(B)$, such that $\theta \in Cc(\beta_{in})$, $S$ is a sequence of substitutions with $Subst(S) \subseteq Cc(\beta_{out})$ and

- if $S \neq < >$ then $\theta \in Cc(\beta_{ref})$;
- $\forall \theta' \in Subst(S)$, if $\langle t_i \rangle_{i \in I_{ref}} \in \texttt{DECOMP}(\theta, \beta_{ref})$ and $\langle s_i \rangle_{i \in I_{out}} \in \texttt{DECOMP}(\theta', \beta_{out})$ then $\langle \|t_i\| \rangle_{i \in I_{ref}} + \langle \|s_i\| \rangle_{i \in I_{out}} \in Cc(E_{ref_out})$;

- if $\langle t_i \rangle_{i \in I_{ref}} \in \mathtt{DECOMP}(\theta, \beta_{ref})$ then $\langle \|t_i\| \rangle_{i \in I_{ref}} + \{sol \mapsto |S|\} \in Cc(E_{sol})$.

Consider the abstract substitutions $\beta_{in}, \beta_{ref}$, and $\beta_{out}$ described above where $I_{ref} = I_{out} = \{\mathtt{1}, \mathtt{2}, \mathtt{3}, \mathtt{4}, \mathtt{5}\}$. Let $in_{ref} : I_{ref} \rightarrow I_{ref} + I_{out}$, $in_{out} : I_{out} \rightarrow I_{ref} + I_{out}$ and $in_{sol} : I_{ref} \rightarrow I_{ref} + \{sol\}$ be injection functions. The behaviour for the procedure `select/3` described in Section 2 can be expressed in terms of the abstract sequence $B = \langle \beta_{in}, \beta_{ref}, \beta_{out}, E_{ref_out}, E_{sol} \rangle$ where $E_{ref_out} = \{\mathtt{sz}(in_{ref}(\mathtt{2})) = \mathtt{sz}(in_{out}(\mathtt{3})) + \mathtt{1}\}$ and $E_{sol} = \{sol = \mathtt{sz}(in_{sol}(\mathtt{2}))\}$.

*Behaviours.* A behaviour for a procedure is a formalization of its behavioural properties provided by the user. Formally, a *behaviour* $Beh_p$ for a procedure $p/n$ is a finite set of pairs $\{\langle B_1, se_1 \rangle, \ldots, \langle B_m, se_m \rangle\}$ where for all $k \in \{1 \ldots, m\}$, $B_k = \langle \beta^k_{in}, \beta^k_{ref}, \beta^k_{out}, E^k_{ref_out}, E^k_{sol} \rangle$ is an abstract sequence with $dom(\beta^k_{in}) = dom(\beta^k_{ref}) = dom(\beta^k_{out}) = \{X_1, \ldots, X_n\}$, and $se_k$ is a positive linear expression from $\mathbf{Exp}_{\{X_1,\ldots,X_n\}}$. As an example, the behaviour for `select/3` described in Section 2 is simply $\{\langle B, \mathtt{X_2} \rangle\}$ where $B$ is the abstract sequence defined above.
In the following, we assume that a set of behaviours $SBeh$ for a program $P$ contains exactly one behaviour $Beh_p$ for each procedure name $p \in \mathcal{P}$.

**Definition 2** (Consistency). We say that a set of behaviours $SBeh$ for a program $P$ is *consistent with respect to the concrete semantics of* $P$ iff for all $p \in \mathcal{P}$ and $\langle B, se \rangle \in Beh_p$, the execution of the procedure $p$ called with a substitution $\theta$ described by the input of $B$ terminates and $\langle \theta, p \rangle \mapsto S$ implies $\langle \theta, S \rangle \in Cc(B)$.

## 5 The Analyser

The analyser follows the standard top-down verification technique: for a given program, it analyses each procedure; for each procedure, it analyses each clause; for each clause, it analyses each atom such that if an atom is a procedure call, then it looks up the behaviour to infer information about its execution. The algorithm of the analyser is depicted in the Appendix. We specify here its main operations. To simplify the presentation, we assume that programs contain no mutually recursive procedures. We discuss this point below.

The analysis of a program $P$ with a set of behaviours $SBeh$ returns a boolean value *success*. If *success* is true, then the program satisfies the set of behaviours $SBeh$ and, in particular, every procedure call (allowed by $SBeh$) terminates. Otherwise, if *success* is false then the analyser is not able to infer whether the program is correct with respect to the set of behaviours $SBeh$.
The analyser computes the glb-closure of the set of behaviours $SBeh$ through the function `MAKE_SAT`. This is useful when analysing an atom which is a procedure call: in that case, a look-up to such a set is performed. Formally, $\mathtt{MAKE_SAT}(SBeh)$ returns a family $sat = \langle sat_p \rangle_{p \in \mathcal{P}}$ of sets of abstract sequences such that for all $p \in \mathcal{P}$, $sat_p$ is the smallest set containing $\{B \mid \exists se : \langle B, se \rangle \in Beh_p\}$ which is closed under greatest lower bound. The results of the analysis of clauses in a same procedure are "concatenated" through the operation `CONC` (see Section 6).

The analysis of a clause $c \equiv p(X_1, \ldots, X_n) \; :- \; l_1, \ldots, l_s$. with respect to $\langle B, se\rangle \in Beh_p$ consists in the following steps:

1. extending the input substitution $\beta_{in}$ of $B$ to an abstract sequence $B_0$ on all the variables in the clause through the operation `EXTC`;
2. computing $B_k$ from $B_{k-1}$ and $l_k$ ($k \in \{1, \ldots, s\}$);
3. restricting $B_s$ to the variables in the head of $c$ through the operation `RESTRC`.

Each $B_k$ is computed from $B_{k-1}$ and $l_k$ by

1. restricting the domain of the output abstract substitution $\beta_{out}$ of $B_{k-1}$ to the variables $X_{i_1}, \ldots, X_{i_n}$ of $l_k$ and renaming them into $X_1, \ldots, X_n$ through the operation `RESTRG`;
2. executing the literal $l_k$ with $\beta^k_{inter}$ which returns an abstract sequence $B^k_{aux}$;
3. propagating this result on $B_{k-1}$ by computing $B_k = \texttt{EXTGS}(l_k, B_{k-1}, B^k_{aux})$.

The execution of $l_k$ with $\beta^k_{inter}$ depends on the form of $l_k$.

1. If $l_k$ is a built-in of the form $X_{i_1} = X_{i_2}$ then $B^k_{aux} = \texttt{UNIF_VAR}(\beta^k_{inter})$ (see Section 6).
2. If $l_k$ is of the form $X_{i_1} = f(X_{i_2}, \ldots, X_{i_n})$ then $B^k_{aux} = \texttt{UNIF_FUNC}(\beta^k_{inter}, f)$. This operation is defined similarly to the previous one.
3. If $l_k$ is a non-recursive call $q(X_{i_1}, \ldots, X_{i_m})$ (i.e., $q \neq p$) then the analyser looks at *sat*, the glb-closed set of behaviours, to find an abstract sequence general enough to give information about this call.
4. If $l_k$ is a recursive call $p(X_{i_1}, \ldots, X_{i_n})$ then the analyser checks whether the size of the arguments decreases through operation $\texttt{CHECK_TERM}(l_k, B_{k-1}, se)$, i.e., it checks whether for all $\langle \theta, S\rangle \in Cc(B_{k-1})$ and for all $\theta' \in Subst(S)$, $se(\langle \|X_{i_1}\theta'\|, \ldots, \|X_{i_n}\theta'\|\rangle) < se(\langle \|X_1\theta\|, \ldots, \|X_n\theta\|\rangle)$.

*Mutual Recursion.* Mutual recursion is treated by extending the termination test to all mutual recursive procedures (above, such a test is applied only to recursive procedures). Mutual recursive procedures are found out by a first-stage analysis which returns all pairs $(\langle p, B_p, se_p\rangle, \langle q, B_q, se_q\rangle)$ with $\langle B_p, se_p\rangle \in Beh_p$ and $\langle B_q, se_q\rangle \in Beh_q$, describing possibly mutual recursive calls.

**Theorem 3** (Correctness [10]). *Let $P$ be a program and SBeh be a set of behaviours for $P$. The analyser called with $P$ and SBeh as inputs returns a boolean value* success *as output such that if* success *is true, then SBeh is consistent with respect to the concrete semantics of the program $P$.*

*Proof.* (sketch) In order to prove the theorem, we introduce the notion of *procedure call allowed by SBeh* which is a tuple $t = \langle \theta, B, se, p\rangle$ such that $p \in \mathcal{P}$, $\langle B, se\rangle \in Beh_p$ and $\theta$ is described by the input abstract substitution of $B$. We also introduce a *well-founded relation* on the set of all the allowed procedure calls as follows: for any allowed procedure calls $t = \langle \theta, B, se, p\rangle$ and $t' = \langle \theta', B', se', p'\rangle$, $t < t'$ iff either the procedure $p'$ is used in the definition of $p$ or $t'$ is a (mutually) recursive call that may be reached during the execution of $t$. In the second case,

we also require that the "size" of $\theta'$ is (strictly) smaller than the "size" of $\theta$, i.e., $se_{p'}(\langle \|X_{i_1}\theta'\|, \ldots, \|X_{i_m}\theta'\| \rangle) < se_p(\langle \|X_1\theta\|, \ldots, \|X_n\theta\| \rangle)$.

The proof is done by induction on the ordering on allowed procedure calls: we assume that *SBeh* correctly describes the executions of all procedure calls $t'$ such that $t' < t$ and that these executions terminate and we prove that the execution of $t$ terminates and is correcly described by *SBeh*. □

## 6 Abstract Operations

In this section we describe the implementation of two main operations, namely **UNIF_VAR** and **CONC**. For a complete description of the operations used by the analyser and the corresponding correctness proofs, the reader is referred to [10].

*Unification Operation.* The unification operation **UNIF_VAR** is used for executing built-ins of the form $X_i = X_j$ with an input abstract substitution $\beta$. It returns an abstract sequence $B' = \langle \beta'_{in}, \beta'_{ref}, \beta'_{out}, E'_{ref_out}, E'_{sol} \rangle$. The principle of the implementation is the following: first, we (re)use the operation **UNIF_VAR**$_{old}$ [14] to compute the abstract result of the execution of $X_i = X_j$ called with $\beta$; then we refine $\beta$ to the set of $\theta \in Cc(\beta)$ for which the unification succeeds through operation **REF**$_{ref}$; finally, we derive constraints between the input and argument sizes as well as constraints on the number of solutions. Below, we state the specifications of the operations **UNIF_VAR**$_{old}$ and **REF**$_{ref}$ whereas we detail the implementation of **UNIF_VAR**.

**UNIF_VAR**$_{old}(\beta) = \langle \beta_{out}, ss, sf, tr, U \rangle$. This operation is similar [6] to the one defined in [14]. Given an abstract substitution $\beta$ with $dom(\beta) = \{X_1, X_2\}$, it returns an abstract substitution $\beta_{out}$ describing the unification of $X_1\theta$ and $X_2\theta$ for all $\theta \in Cc(\beta)$; two boolean values *ss* and *sf* specifying whether sure success or sure failure can be inferred at the abstract level, a so-called *structural mapping* *tr* between the indices of $\beta$ and the indices of $\beta_{out}$ representing corresponding terms before and after the unification, and a set of indices $U$ representing the set of terms in $\theta$ whose norm is not affected by the instantiation.

**REF**$_{ref}(\beta_1, \beta_2, tr) = \langle \beta', tr' \rangle$. This operation takes as inputs two abstract substitutions $\beta_1$ and $\beta_2$ and a structural mapping *tr* between the indices of $\beta_1$ and $\beta_2$. It refines the abstract substitution $\beta_1$ by keeping substitutions in $Cc(\beta_1)$ that have at least an instance in $Cc(\beta_2)$. It returns an abstract substitution $\beta'$ and a structural mapping $tr'$ between the indices of $\beta'$ and $\beta_2$ such that $\beta'$ is at least as precise as $\beta_1$ and $\theta_k \in Cc(\beta_k)$ $(k = 1, 2)$ with $\theta_2 \leq \theta_1$ implies $\theta_1 \in Cc(\beta')$.

**UNIF_VAR**$(\beta) = B'$. Let $\beta$ be an abstract substitution with $dom(\beta) = \{X_1, X_2\}$ and $\langle \beta_{out}, tr, ss, sf, U \rangle =$ **UNIF_VAR**$_{old}(\beta)$. $B'$ is defined as follows.

[6] Actually, the signature in [14] of this operation is **UNIF_VAR**$(\beta) = \beta'$, as there was no need there to export sure success/failure information. Adapting that definition to our purposes is straightforward (see [13]). That's why we call it simply **UNIF_VAR**$_{old}$.

$$
\begin{array}{lll}
\beta'_{in} & = \beta & \\
\beta'_{out} & = \beta_{out} & \\
\langle \beta'_{ref}, tr_{ref_out} \rangle & = \langle \beta'_{in}, tr \rangle & \text{if } ss \\
 & \langle \bot, undef \rangle & \text{if } sf \\
 & \mathtt{REF}_{ref}(\beta'_{in}, \beta'_{out}, tr) & \text{if } \neg ss \text{ and } \neg sf \\
E'_{ref_out} & = \bot & \text{if } sf \\
 & \{ \mathtt{sz}(in_{ref}(i)) = \mathtt{sz}(in_{out}(tr_{ref_out}(i))) : & \\
 & i \in tr_{in_ref}(U) \} & \text{otherwise} \\
E'_{sol} & = \{ sol = 1 \} & \text{if } ss \\
 & \bot & \text{if } sf \\
 & \{ 0 \leq sol, sol \leq 1 \} & \text{if } \neg ss \text{ and } \neg sf.
\end{array}
$$

where $tr_{in_ref}$ is a canonical inclusion, and the following commutative diagram is satisfied by $tr_{in_ref}$, $tr_{ref_out}$ and the injection functions $in_{ref}$ and $in_{out}$.

$$
U \subseteq I = I'_{in} \xrightarrow{tr_{in_ref}} I'_{ref} \xrightarrow{tr_{ref_out}} I'_{out}, \qquad I'_{ref} \xrightarrow{in_{ref}} I'_{ref} + I'_{out} \xleftarrow{in_{out}} I'_{out}
$$

The accuracy of this operation may be improved in practice by using a reexecution strategy [15]: we may repeatedly apply $\mathtt{UNIF_VAR}_{old}$ and $\mathtt{REF}_{ref}$ to $\beta'_{ref}$ until sure success or sure failure is inferred or $\beta'_{ref}$ stabilizes.

*Concatenation Operation.* The concatenation operation `CONC` is the counterpart for abstract sequences of the operation `UNION`, used in [14], which simply collects information provided by two abstract substitutions into a single one. In fact, `CONC` differs from `UNION` only for the computation of the number of solutions to a procedure which is the sum of the numbers of solutions of its clauses, not an "upper bound" of them. To obtain a good precision, we detect mutual exclusion of clauses [2, 13] by computing the greatest lower bound of the $\beta_{ref}$ component of the two abstract sequences. If it is $\bot$, then the clauses are exclusive: in this case, we only collect the numbers of solutions of the two clauses. Otherwise, we compute the sum of the numbers of solutions for the greatest lower bound only. The implementation of `CONC` uses special operations, namely $tr^{>}(E)$ and $tr^{<}(E)$, to manipulate size components (see [16]). If $E$ is a size component over a set of indices $I$ and $tr : I \rightarrow I'$ is a (possibly partial) function, then $tr^{>}(E)$ returns $E'$ over $I'$ such that $(n_i)_{i \in I} \in Cc(E)$ and $n_i = n'_{tr(i)}$ $(\forall i \in dom(tr))$ imply $(n'_i)_{i \in I'} \in Cc(E')$. Analogously, if $E$ is as above and $tr : I' \rightarrow I$, then $tr^{<}(E)$ returns $E'$ over $I'$ such that $(n_i)_{i \in I} \in Cc(E)$ and $n_{tr(i)} = n'_i$ $(\forall i \in dom(tr))$ imply $(n'_i)_{i \in I'} \in Cc(E')$. The following auxiliary operations are also used.

$\mathtt{LUB}(\beta_1, \beta_2)$ returns a triplet $\langle \beta', tr_1, tr_2 \rangle$ where $\beta' = \beta_1 \sqcup \beta_2$ and $tr_k$ are two structural mappings between $\beta'$ and $\beta_k$, i.e., $tr_k : I' \rightarrow I_k$ $(k = 1, 2)$.

$\mathtt{EXT_LUB}(\beta_1, \beta_2)$ is an extension of the previous operation returning an additional boolean value $st$ (standing for "strict union") such that $st = true$ implies that $Cc(\beta') = Cc(\beta_1) \cup Cc(\beta_2)$.

$\mathtt{GLB}(\beta_1, \beta_2)$ returns the triplet $\langle \beta', tr_1, tr_2 \rangle$ where $\beta' = \beta_1 \sqcap \beta_2$ and $tr_k$ are two structural mappings between $\beta_k$ and $\beta'$, i.e., $tr_k : I_k \rightarrow I'$ $(k = 1, 2)$.

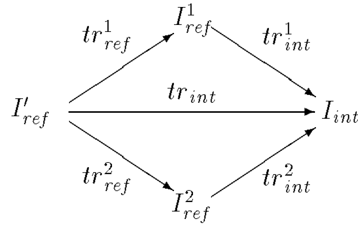
$\mathtt{SUM}_{sol}(E_1, E_2)$ returns a size component $E'$ satisfying the following relation: if $E_k$ $(k = 1, 2)$ are two size components over $I + \{sol\}$ then $E'$ is a size component over over $I + \{sol\}$ such that $(n_i^k)_{i \in I + \{sol\}} \in Cc(E_k)$ $(k = 1, 2)$, $n_i^1 = n_i^2 = n_i$ $(i \in I)$ and $n_{sol} = n_{sol}^1 + n_{sol}^2$ imply $(n_i)_{i \in I + \{sol\}} \in Cc(E')$.

Now we are in position to describe the implementation of $\mathtt{CONC}$. Let $B_k = \langle \beta_{in}, \beta_{ref}^k, \beta_{out}^k, E_{ref_out}^k, E_{sol}^k \rangle$ $(k = 1, 2)$ be two abstract sequences. $\mathtt{CONC}(B_1, B_2)$ returns an abstract sequence $B'$ such that $\langle \theta, S_1 \rangle \in Cc(B_1)$ and $\langle \theta, S_2 \rangle \in Cc(B_2)$ imply $\langle \theta, S_1 :: S_2 \rangle \in Cc(B')$. $B'$ can be implemented as follows[7].

$$\begin{aligned}
\beta'_{in} &= \beta_{in} \\
\langle \beta'_{ref}, tr_{ref}^1, tr_{ref}^2, st \rangle &= \mathtt{EXT_LUB}(\beta_{ref}^1, \beta_{ref}^2) \\
\langle \beta'_{out}, tr_{out}^1, tr_{out}^2 \rangle &= \mathtt{LUB}(\beta_{out}^1, \beta_{out}^2) \\
E'_{ref_out} &= (tr_{ref}^1 + tr_{out}^1)^{<}(E_{ref_out}^1) \sqcup (tr_{ref}^2 + tr_{out}^2)^{<}(E_{ref_out}^2)
\end{aligned}$$

$$E'_{sol} = \begin{cases} (tr_{ref}^1 + \{sol \mapsto sol\})^{<}(E_{sol}^1) \sqcup \\ (tr_{ref}^2 + \{sol \mapsto sol\})^{<}(E_{sol}^2) \sqcup & \text{if } st \\ (tr_{int} + \{sol \mapsto sol\})^{<}(\mathtt{SUM}_{sol}(\overline{E}_{sol}^1, \overline{E}_{sol}^2)) \\ \\ (tr_{ref}^1 + \{sol \mapsto sol\})^{<}(E_{sol}^1) \sqcup \\ (tr_{ref}^2 + \{sol \mapsto sol\})^{<}(E_{sol}^2) \sqcup \\ (tr_{int} + \{sol \mapsto sol\})^{<}(\mathtt{SUM}_{sol}(\overline{E}_{sol}^1, \overline{E}_{sol}^2)) \sqcup & \text{if } \neg st. \\ tr_{sol}^{>}(\{sol = 0\}) \end{cases}$$

where $\langle \beta_{int}, tr_{int}^1, tr_{int}^2 \rangle = \mathtt{GLB}(\beta_{ref}^1, \beta_{ref}^2)$, $\overline{E}_{sol}^k = (tr_{int}^k + \{sol \mapsto sol\})^{>}(E_{sol}^k)$ $(k = 1, 2)$, $tr_{sol} : \{sol\} \rightarrow I'_{ref} + \{sol\}$ is the canonical injection and the structural mappings $tr_{ref}^k$, $tr_{int}^k$, $tr_{int}$ satisfy the commutative diagram below.




[7] The least upper bound operator $\sqcup$ between (in)equation systems is implemented as convex union (see [19]).

## 7 Conclusion

The analyser presented in this paper may apply to any kind of Prolog program (without dynamic predicates such as `assert` and `retract`). To cite some of its applications, it could be integrated in a programming environment to check correctness of Prolog programs and/or to derive efficient Prolog programs from purely logic descriptions. In particular, it may be integrated in the FOLON environment [6, 9] which was developed for supporting the automatable aspects of Deville's methodology for logic program construction [7]. Our system could support the automatable aspects of other works on verification, e.g., [1]. Moreover, since the information provided by the user is certified by the system, it can be used by a compiler to optimize the object code. Finally, since it may verify precise relations between the size of the arguments and the number of solutions to a procedure, it can be used as a basis for an automatic complexity analysis. This is the main topic of future work.

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## A The Algorithm of the Analyser

This appendix contains the implementation of the three main procedures of our analyser, namely analyse_program, analyse_procedure and analyse_clause.

```
PROCEDURE analyze_program(P, SBeh) =
  success := true
  sat := MAKE_SAT(SBeh)
  for all p ∈ 𝒫,  for all ⟨B, se⟩ ∈ Beh_p
     success := success ∧ analyse_procedure(p, B, se)
  return success.
```

```
PROCEDURE analyze_procedure(p, B, se) =
  for k := 1 to r do
     ⟨success_k, B_k⟩ := analyze_clause(c_k, B, se)
  if there exists k ∈ {1, ..., r} such that ¬success_k,
     then  success := false
     else  B_out := CONC(B_1, ..., B_r)
           success := (B_out ≤ B)
  return success.
```

```
PROCEDURE analyze_clause(c, B) =
  β_in := input(B)
  B_0 := EXTC(c, β_in)
  for k := 1 to s do
     β^k_inter := RESTRG(l_k, B_{k-1})
     if l_k ≡ X_{i_1} = X_{i_2} then B^k_aux := UNIF_VAR(β^k_inter)
     if l_k ≡ X_{i_1} = f(X_{i_2}, ..., X_{i_m}) then B^k_aux := UNIF_FUNC(β^k_inter, f)
     if l_k ≡ q(X_{i_1}, ..., X_{i_m}) and q ≠ p then
               ⟨B^k_aux, success_k⟩ := LOOK_UP(β^k_inter, q, sat)
     if l_k ≡ p(X_{i_1}, ..., X_{i_m}) then
               ⟨B^k_aux, success'_k⟩ := LOOK_UP(β^k_inter, q, sat)
               success_k := success'_k ∧ CHECK_TERM(l_k, B_{k-1}, se)
     B_k := EXTGS(l_k, B_{k-1}, B^k_aux)
  if there exists k such that
        either l_k ≡ q(X_{i_1}, ..., X_{i_m}) ∧ ¬success_k
        or l_k ≡ p(X_{i_1}, ..., X_{i_n}) ∧ (¬success_k ∨ β^k_inter ≰ β_in)
     then success = false
     else success = true and B_out = RESTRC(c, B_s)
  return⟨success, B_out⟩.
```

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