

Algebra and Topology in Lambda Calculus

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Lambda calculus

- Lambda calculus was introduced around 1930 by A. Church as part of an investigation in the formal foundations of mathematics and logic.
- Why is lambda calculus important?
 - Full computational formalism
 - Functional programming languages are based on λ -calculus

- What is lambda calculus?

- A theory of functions
- The name of a function contains a description of the function as a program
- Untyped world: every element in lambda calculus is contemporaneously
 - * Function
 - * A possible argument for a function
 - * A possible result of the application of a function to an argument
- No Partiality: every function can be applied to any other function including itself

Lambda terms

- λ -notation:
Expression: $a + 2$ Function: $f(a) = a + 2$ $\lambda_a(a + 2)$
- Algebraic similarity type Σ :
 - Nullary operators: $a, b, c, \dots \in A$ (formal parameters)
 - Binary operator: \bullet (application)
 - Unary operators: $\lambda_a (a \in A)$ (λ -abstractions)
- A **λ -term** is a ground Σ -term (no algebraic variable x, y, z, \dots)

$\lambda_a(a)$ YES $\lambda_a(a) \cdot x$ NO

- $a =$ generic function
- $M \cdot N =$ function M applied to argument N
- $\lambda_a(M) =$ function of a whose body is expression M

How to compute (informally)

- Bound and free parameters: $\lambda_a(a \cdot b)$

- α -conversion: $\lambda_a(a \cdot b) = \lambda_c(c \cdot b)$
The name of a bound parameter does not matter

- β -conversion: $\lambda_a(a) \cdot b = b$
 $\lambda_a(a \cdot a) \cdot \lambda_a(a \cdot a) = \lambda_a(a \cdot a) \cdot \lambda_a(a \cdot a) = \dots$

- $\alpha\beta$ -conversion:
 $(\lambda_a(\lambda_b(a)) \cdot b) \cdot a = \lambda_b(b) \cdot a = a$ NO!
 $(\lambda_a(\lambda_b(a)) \cdot b) \cdot a = (\lambda_a(\lambda_c(a)) \cdot b) \cdot a = \lambda_c(b) \cdot a = b$

- Non-extensionality:
 $M \neq \lambda_a(M \cdot a)$ (a not free in M)
 $\lambda_a(M \cdot a) \cdot N = M \cdot N$

- Every λ -term has a fixpoint: $(\forall M)(\exists P) M \cdot P = P$

The classic λ -calculus

- The λ -term algebra is the absolutely free Σ -algebra over an empty set of generators:

$$\Lambda = (\Lambda, \cdot, \lambda_a, a)_{a \in A}$$

The object of study of λ -calculus is any congruence on Λ (called λ -theory) including α - and β -conversion:

– β -conversion:

$$\lambda_a(M) \cdot N = M[N/a]$$

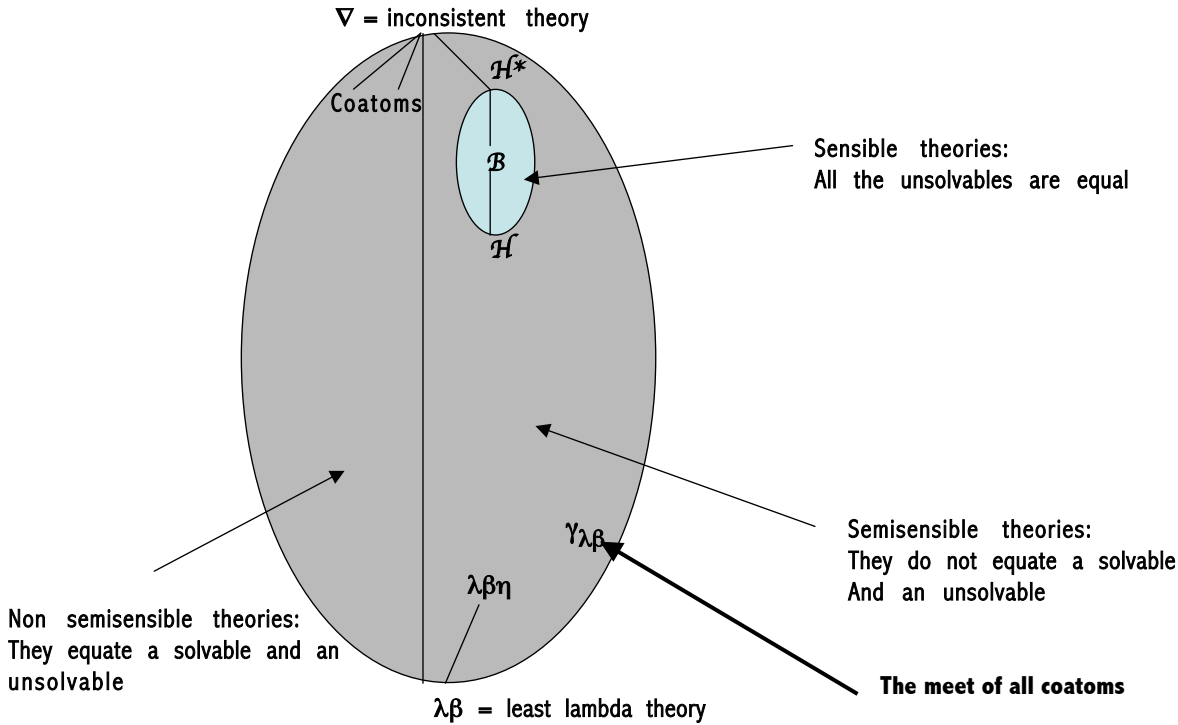
– α -conversion:

$$\lambda_a(M) = \lambda_b(M[b/a]) \quad (b \text{ not free in } M)$$

- The lattice of λ -theories \equiv The congruence lattice of $\Lambda/\lambda\beta$
($\lambda\beta$ is the least congruence on Λ including α - and β -conversion)

The Lattice of λ -Theories

A continuum of λ -theories and coatoms.



The algebraic lambda calculus

- The classic λ -calculus is not interesting from the algebraic viewpoint
- Variety generated by the term algebra $\Lambda/\lambda\beta$
- Pigozzi-S. (1993) introduced the variety of **lambda abstraction algebras** (LAAs, for brevity)

The algebraic lambda calculus

Variety LAA of lambda abstraction algebras is axiomatized by:

$$(\beta_1) \lambda_a(a) \cdot x = x$$

$$(\beta_2) \lambda_a(b) \cdot x = b \quad (b \neq a)$$

$$(\beta_3) \lambda_a(x) \cdot a = x$$

$$(\beta_4) \lambda_a(\lambda_a(x)) \cdot y = \lambda_a(x)$$

$$(\beta_5) \lambda_a(x \cdot y) \cdot z = (\lambda_a(x) \cdot z) \cdot (\lambda_a(y) \cdot z)$$

$$(\beta_6) \lambda_b(y) \cdot c = y \Rightarrow \lambda_a(\lambda_b(x)) \cdot y = \lambda_b(\lambda_a(x) \cdot y) \quad (c \neq b, a \neq b)$$

$$(\alpha) \lambda_b(x) \cdot c = x \Rightarrow \lambda_a(x) = \lambda_b(\lambda_a(x) \cdot b) \quad (a \neq b)$$

$\lambda_a(x) \cdot b = x$ ($a \neq b$) means “ x does not depend on parameter b ”

The algebraic lambda calculus

- Examples of LAAs:
 - The term algebra Λ/T of a λ -theory T
 - Algebras of functions obtained by coordinatizing the models of lambda calculus
- Representation Theorems (1993-1999)

Theorem 1 (Goldblatt-S. 1999) *Every LAA is representable as an algebra of functions obtained by coordinatizing a suitable model of lambda calculus.*

Universal algebra at work

Why is important to study the subvarieties of LAA?

- λ -theory $T \Leftrightarrow$ Variety generated by term algebra Λ/T .
- λ -calculus problem = Problem of existence of a suitable subvariety of LAA.

Theorem 2 (S. 2000) *The lattice of the equational theories of LAAs is isomorphic to the lattice of λ -theories.*

Corollary 1 *Every variety of LAAs is generated by the term algebra Λ/T of a suitable lambda theory T . In particular,*

$$\text{LAA} = \mathcal{V}(\Lambda/\lambda\beta)$$

Universal algebra at work

BUT!

Theorem 3 (Lusin-S. 2004) *Every congruence identity holding in LAA is trivial.*

Are there interesting algebraic properties consistent with lambda calculus?

Theorem 4 (Berline-S. 2006) *There exists a congruence distributive variety of LAAs.*

The algebraic counterpart of an open problem

- Church (around 1930): Lambda calculus
- Scott (1969): First model
- Meyer-Scott (around 1980): There exists a first-order axiomatization of what is a model of λ -calculus.

\mathcal{D} model \Rightarrow $\text{Th}(\mathcal{D}) = \{M = N : M \text{ and } N \text{ have the same interpretation}\}$

We can construct an LAA expansion $\lambda(\mathcal{D})$ of \mathcal{D} (algebra of functions) in such a way that

$$\Lambda/\text{Th}(\mathcal{D}) \hookrightarrow \lambda(\mathcal{D}).$$

- Scott Semantics (1969-2007) A Scott model \mathcal{D} is a Scott topological space (no: Hausdorff; yes: poset with bottom) and two Scott continuous maps

$$i : \mathcal{D} \rightarrow [\mathcal{D} \rightarrow \mathcal{D}]; \quad j : [\mathcal{D} \rightarrow \mathcal{D}] \rightarrow \mathcal{D}; \quad i \circ j = id_{[\mathcal{D} \rightarrow \mathcal{D}]}$$

- Does the logical part of a λ -calculus a model (BA, HA, Lattice with λ -calculus extra operations) influence the λ -theory of the model?

The algebraic counterpart of an open problem

Definition 1 A lambda theory T is **order-incomplete** if $T \neq Eq(\mathcal{D})$, for every model \mathcal{D} such that its LAA expansion $\lambda(\mathcal{D})$ admits a non-trivial compatible (with respect to application and λ -abstractions) partial ordering.

Open Question (Selinger 1996): Does it exist an order-incomplete lambda theory?

What we know:

Theorem 5 (S. 2001) There exists a lambda theory T such that $T \neq Eq(\mathcal{D})$ for every partially ordered model \mathcal{D} with bottom element.

Fact: Order-incompleteness = Existence of an n -permutable variety of LAAs

Theorem 6 (Plotkin-Selinger-Simpson 1996) There exists no 2-permutable and 3-permutable variety of LAAs.

Open Question: Is n -permutability ($n \geq 4$) consistent with lambda calculus?

Boolean algebras for λ -calculus

- Let \mathbf{A} be an algebra. There exists a bijective correspondence between:
Pairs of compl. factor congr. \Leftrightarrow Decomposition op. \Leftrightarrow Factorizations
- A decomposition operation is an algebra homomorphism $f : A \times A \rightarrow A$ such that $f(x, x) = x$ and $f(f(x, y), z) = f(x, z) = f(x, f(y, z))$.
- Let $\mathbf{t} \equiv \lambda_a(\lambda_b(a))$ and $\mathbf{f} \equiv \lambda_a(\lambda_b(b))$. The least reflexive compatible relation equating

$$\mathbf{t} = \mathbf{f}$$

is trivial.

Boolean algebras for LAAs

- **A is a ring**
- **BA of idempotent elements of A**
- **A idempotent $\iff a \cdot a = a$**
- **a idempotent $\implies A = A_a \times A_{-a}$**
- **Boolean operations:**
 - **Meet:** $a \wedge b = a \cdot b$;
 - **Complement:** $-a = 1 - a$
 - **Bottom:** 0
 - **Top:** 1
- **$A \twoheadrightarrow \prod_{i \in I} A_i$, embedding**
 $I =$ Boolean space of maximal ideals of the BA of idempotent elements
 $A_i = A/\text{congruence determined by the maximal ideal } i$
 A_i directly indecomposable
- **A is an LAA**
- **BA of central elements of A**
- **$e \in A$ is **central** \iff the function $f: A \times A \rightarrow A$ defined by $f(x,y) = (e \cdot x) \cdot y$ is a decomposition operation.**
- **$t \equiv \lambda_a(\lambda_b(a))$; $f \equiv \lambda_a(\lambda_b(b))$.**
- **Boolean operations on central elements:**
 - **Meet:** $d \wedge e = (d \cdot t) \cdot e$
 - **Complement:** $-d = (d \cdot f) \cdot t$
 - **Bottom and Top:** $0 = t$; $1 = f$
- **e central $\implies A = A/\theta(1,e) \times A/\theta(1,-e)$**
- **$A \twoheadrightarrow \prod_{i \in I} A_i$, embedding**
 $I =$ Boolean space of maximal ideals of the BA of central elements
 $A_i = A/\text{congruence determined by the maximal ideal } i$
 A_i directly indecomposable
- **The Representation Theorem:** Every LAA is isomorphic to a weak Boolean product of directly indecomposable LAAs.
- **Proof.** By using a theorem by Vaggione.

The lattice λT of λ -theories

Conjecture: Every nontrivial lattice identity fails in λT

- (S. 2000) λT is isomorphic to the lattice of equational theories of LAA's.
- (Lampe 1986) λT satisfies the Zipper condition:

$$\bigvee \{b : a \wedge b = c\} = 1 \Rightarrow a = c.$$

- (S. 2001) λT is not modular.
- (Berline-S. 2006) (\exists λ -theory T) the interval $[T, \nabla]$ is distributive.
- (Statman 2001) The meet of all coatoms is $\neq \lambda\beta$.
- (Visser 1980)
 - Every countable poset embeds into λT by an order-preserving map.
 - Every interval $[T, S]$ with T, S r.e. has a continuum of elements.

- (S. 2006) $(\forall n)(\exists T_n)$ such that the interval sublattice $[T_n, \nabla]$ is isomorphic to the finite Boolean lattice with 2^n elements.
- (Diercks-Erné-Reinhold 1994) There exists no λ -theory T such that the interval sublattice $[T, \nabla]$ is isomorphic to an infinite Boolean lattice.