A Note on Absolutely Unorderable Combinatory Algebras

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Abstract

Plotkin [16] has conjectured that there exists an *absolutely unorderable* combinatory algebra, namely a combinatory algebra which cannot be embedded in another combinatory algebra admitting a non-trivial compatible partial order. In this paper we prove that a wide class of combinatory algebras admits extensions with a non-trivial compatible partial order.

Introduction

Although the axioms of the lambda calculus are all in the form of equations, the lambda calculus is not a true equational theory since the variable-binding properties of lambda abstraction prevent variables in lambda calculus from operating as real algebraic variables. However, there have been several attempts to reformulate the lambda calculus as a purely algebraic theory. The earliest, and best known, algebraic models are the combinatory algebras of Curry and Schönfinkel (see [6], [23]). Combinatory algebras have a simple purely equational characterization. Curry also specified (by a considerably less natural set of axioms) a purely equational subclass of combinatory algebras, the λ-*algebras* (see Barendregt [1, Def. 5.2.5]), that he viewed as algebraic models of the lambda calculus. Although lambda calculus has been the subject of research by logicians since the early 1930's, its model theory developed only much later, following the pioneering model construction made by Dana Scott. The notion of an *environment model* (the name is due to Meyer [12]) originated with Hindley and Longo [8]. They are functional domains where λ-terms can be properly interpreted. Meyer describes them as "the natural, most general formulation of what might be meant by mathematical models of the untyped lambda calculus". The main result in [12] is a completeness theorem demonstrating that every lambda theory is the theory associated with some environment model. The drawback of environment models is that they are higher-order structures. However, there exists an intrinsic characterization (up to isomorphism) of environment models as a special class of λ-algebras called *lambda models* ([1, Def. 5.2.5]). They were first axiomatized by Meyer [12] and independently by Scott [24]; the axiomatization, while elegant, is not equational. It turns out however that the variety of λ-algebras is generated by the lambda models.

In [13, 14] Pigozzi and Salibra have introduced *lambda abstraction algebras* (LAA's) which constitute a purely algebraic theory of the untyped lambda calculus alternative to Curry's highly combinatorial models. Combinatory algebras (CA's) and lambda abstraction algebras are both defined by universally quantified equations and thus form varieties in the universal algebraic sense. There are important differences however that result in theories of very different character. Functional application is taken as a fundamental operation in both CA's and LAA's. Lambda (i.e., functional) abstraction is also fundamental in LAA's but in CA's is defined in terms of the combinators $\mathbf{k}$ and $\mathbf{s}$. A more important difference is connected with the role variables play in the lambda calculus as place holders. In a lambda abstraction algebra this is also abstracted. It takes the form of a system of fundamental elements (nullary operations) of the algebra. This is a crucial feature of LAA's that has no direct analogue in CA's. One important consequence of the abstraction of variables is the abstraction of term-for-variable substitution in lambda abstraction algebras. Among the seven axioms characterizing LAA's, the first six constitute a recursive definition of the abstract substitution operator; they express precisely the metamathematical content of β-conversion. The last axiom is an algebraic translation of α-conversion.

The theory of lambda abstraction algebras can be regarded as axiomatizing the equations that hold between contexts of the lambda calculus, as opposed to lambda terms with free variables. We recall from Barendregt ([1, Def. 14.4.1]) that a context is a λ-term with some 'holes' in it. The essential feature of a context is that a free variable in a λ-term may become bound when we substitute it for a 'hole' within the context. So, Barendregt's 'holes' play the role of algebraic variables, and the contexts are the algebraic terms in the similarity type of lambda abstraction algebras. In [22] Salibra and Goldblatt have shown that the explicit finite equational axiomatization for the variety of lambda abstraction algebras provides also an explicit axiomatization of the equations between contexts valid in every lambda theory, where a lambda theory satisfies an identity between contexts if all the instances of the identity fall within the lambda theory.

In [21] Salibra has shown that the lattice of the subvarieties of lambda abstraction algebras is dually isomorphic to the lattice of lambda theories of the lambda calculus; for every variety of lambda abstraction algebras there exists exactly one lambda theory whose term algebra generates the variety. For example, the variety generated by the term algebra of the minimal lambda theory $\lambda\beta$ is the variety lambda abstraction algebra of all lambda abstraction algebras, so that an identity between contexts is true in every lambda theory if and only if it is true in the minimal lambda theory. These results prove useful in the lambda calculus as a way for applying the methods of universal algebra: we can study the properties of a lambda theory by means of the variety of lambda abstraction algebras generated by its term algebra.

In this paper we study the problem of the ordinability of combinatory algebras and lambda abstraction algebras. There are, in fact, evident computational reasons to consider some ordered algebraic structures as models of lambda calculus. However not every model is *orderable*, that is, it admits a non-trivial compatible partial order. For example, Selinger [25] has recently proven that the term model of the minimal lambda theory $\lambda\beta$ does not admit a non-trivial compatible partial order, while Salibra [17] has shown that there exists a lambda theory whose term model generates a quasi-variety (i.e., a class of algebras axiomatized by equational implications) of unorderable combinatory algebras.

Plotkin has conjectured [16] that there exists an *absolutely unorderable* combinatory algebra, that is, a combinatory algebra which cannot be embedded in an orderable one. Selinger [25] has given an algebraic characterization of an absolutely unorderable $\mathcal{V}$-algebra in an algebraic variety $\mathcal{V}$. He shows that a $\mathcal{V}$-algebra is absolutely unorderable if, and only if, it has a family of so-called *generalized Mal'cev operators*. The main problem is to prove whether it is consistent to add such Mal'cev operators to the lambda calculus. In general this is an open problem.

In this paper we prove that there exists a wide class of combinatory algebras (the term algebras of the, so-called, "generic" lambda theories) which are not absolutely unorderable, i.e. they admit extensions with a non-trivial compatible partial order.

The present paper is organized as follows. In Section 1 we review the basic definitions of the lambda calculus and summarize definitions and results concerning the theory of lambda abstraction algebras that will be needed in the subsequent part of the paper. In particular, we recall the formal definition of a lambda abstraction algebra, the notion of dimension set (an abstraction of a variable occurring free in a term of the lambda calculus) and the definition of locally finite lambda abstraction algebra.

Section 2 is devoted to unorderable algebras. We prove a proposition (Prop. 2.4) that states when a locally finite lambda abstraction algebra is unorderable.

The main result of the paper is presented in the third section, where we deal with the problem of absolutely unorderability. In this section we introduce the new concept of a *σ-semisensible* lambda theory (with σ a *semimorphism* according to Def. 3.5). We prove that, when σ is the map that associates with a lambda term its Böhm tree, then a lambda theory is σ-semisensible if, and only if, it is semisensible in the usual sense. Finally, we define *generic lambda theories* (Def. 3.10) and we prove that the term algebra of a generic lambda theory is not absolutely unorderable.

1 Lambda Abstraction Algebras: Basic Notions and Notation

To keep this article self-contained, we summarize some definitions and results that we need in the subsequent part of the paper. Our main references will be [14] and [15] for lambda abstraction algebras, Barendregt's book [1] for lambda calculus, and

Grätzer's book [7] for universal algebra.

1.1 Algebras

In this Section we recall some algebraic notions that will be used in the following.

Let ω be the set of natural numbers. A *similarity type* of algebras is an ω-indexed family $\mathcal{F} = (\mathcal{F}_n : n \in \omega)$ of function symbols. If $f \in \mathcal{F}_n$, we say that n is the *arity* of the function symbol f. An *algebra* $\mathbf{A}$ of type $\mathcal{F}$ is an ordered pair (A, F), where A is a nonempty set and $F = (f^{\mathbf{A}} : A^n \to A \mid f \in \mathcal{F}_n, n \in \omega)$ is an $\mathcal{F}$-indexed family of finitary operations. The set A is called the *universe* (or *underlying set*) of $\mathbf{A}$.

An algebra $\mathbf{B}$ of type $\mathcal{F}$ is a *subalgebra* of an algebra $\mathbf{A}$ of the same type if $B \subseteq A$ and $f^{\mathbf{B}}(b_1, \dots, b_n) = f^{\mathbf{A}}(b_1, \dots, b_n)$ for all $f \in \mathcal{F}$ and all $b_1, \dots, b_n \in B$. A subset X of an algebra $\mathbf{A}$ *generates* the algebra if $\mathbf{A}$ is the least subalgebra including X.

Given an algebra $\mathbf{A}$ of type $\mathcal{F}$, a binary relation $\theta \subseteq A \times A$ is *compatible* if for all $f \in \mathcal{F}_n$ and all $a_i, b_i \in A$, we have

$$a_1 \theta b_1, ..., a_n \theta b_n \Rightarrow \ f(a_1, \dots, a_n) \theta f(b_1, \dots, b_n).$$

A *congruence* θ of $\mathbf{A}$ is a compatible equivalence relation. A congruence θ is *trivial* if either $\theta = 1_{\mathbf{A}}$ or $\theta = 0_{\mathbf{A}}$, where $1_{\mathbf{A}}$ and $0_{\mathbf{A}}$ denote, respectively, $A \times A$ and $\{(a, a) \mid a \in A\}$.

An algebra $\mathbf{A}$ of type $\mathcal{F}$ is a *reduct* of an algebra $\mathbf{B}$ of type $\mathcal{G}$ if $\mathbf{A}$ and $\mathbf{B}$ have the same universe, $\mathcal{F}_n$ is a subset of $\mathcal{G}_n$ for all n, and $f^{\mathbf{A}} = f^{\mathbf{B}}$ for all operation symbols $f \in \mathcal{F}$.

A nonempty class K of algebras of type $\mathcal{F}$ is called a *variety* if it is closed under subalgebras, homomorphic images and direct products. By Birkhoff's theorem (see [7]) a class of algebras is a variety if, and only if, it is an equational class (that is, it is axiomatized by a set of equations). A variety K of algebras is generated by an algebra $\mathbf{A} \in K$ if every equation satisfied by $\mathbf{A}$ is also satisfied by every algebra in K.

Let K be a class of algebras of type $\mathcal{F}$, $\mathbf{U}$ be an algebra of the same type and X be a subset of U. We say that $\mathbf{U}$ has the *universal mapping property for K over X* iff for every $\mathbf{A} \in K$ and for every mapping $g : X \to A$, there is a homomorphism $f : \mathbf{U} \to \mathbf{A}$ that extends g (i.e., $f(x) = g(x)$ for every $x \in X$). We say that $\mathbf{U}$ is *free in K over X* iff $\mathbf{U} \in K$, $\mathbf{U}$ is generated by X and $\mathbf{U}$ has the universal mapping property for K over X. If $\mathbf{U}$ is free in K over X, then X is called a *free generating set for* $\mathbf{U}$, and $\mathbf{U}$ is said to be *freely generated by* X.

Let K be a variety of algebras of type $\mathcal{F}$ and $\mathbf{C} \in K$. We denote by $\mathbf{C}[X]$ the *free extension of* $\mathbf{C}$ *by a set* X in the variety K. $\mathbf{C}[X]$ is an expansion of $\mathbf{C}$ (i.e., $\mathbf{C}$ is a subalgebra of $\mathbf{C}[X]$) defined up to isomorphism by the following universal mapping properties (see [7]): ($C[X]$ is the universe of $\mathbf{C}[X]$.) (1) $X \subseteq C[X]$; (2) $\mathbf{C}[X] \in K$; (3) for every homomorphism $h : \mathbf{C} \to \mathbf{A}$ from $\mathbf{C}$ into an algebra $\mathbf{A} \in K$ and every mapping $g : X \to A$ there exists a unique homomorphism $f : \mathbf{C}[X] \to \mathbf{A}$ extending both h and g.

1.2 Lambda calculus

The untyped lambda calculus was introduced by Church (Church [3, 4]) as a foundation for logic. Although the appearance of paradoxes caused the program to fail, a consistent part of the theory turned out to be successful as a theory of "functions as rules" (formalized as terms of the lambda calculus) that stresses the computational process of going from argument to value. Every object is at the same time a function and an argument; in particular a function can be applied to itself, contrary to the usual notion of function in set theory. The two primitive notions of the lambda calculus are *application*, the operation of applying a function to an argument (expressed as juxtaposition of terms), and *lambda* (*functional*) *abstraction*, the process of forming a function from the "rule" that defines it.

The set $\Lambda_I(C)$ of ordinary terms of lambda calculus over an infinite set I of variables and a set C of constants is constructed as usual [1]:

1. every variable $x \in I$ and every constant $c \in C$ is a λ-term;
2. if M and N are λ-terms, then so are (MN) and $(\lambda x.M)$ for each variable $x \in I$.

We will write Λ_I for $\Lambda_I(\emptyset)$, the set of λ-terms without constants.

The symbol $\equiv$ denotes syntactic equality.

The following are some well-known λ-terms:

$$\mathbf{i} \equiv \lambda x.x; \quad \mathbf{s} \equiv \lambda xyz.xz(yz); \quad \mathbf{k} \equiv \lambda xy.x; \quad \mathbf{1} \equiv \lambda xy.xy.$$

An occurrence of a variable x in a λ-term is *bound* if it lies within the scope of a lambda abstraction λx; otherwise it is *free*. $FV(M)$ is the set of free variables of a λ-term M. A λ-term without free variables is said to be *closed*. $\Lambda_I^0(C)$ is the set of closed λ-terms of $\Lambda_I(C)$. A λ-term N is *free for* x in M if no free occurrence of x in M lies within the scope of a lambda abstraction with respect to a variable that occurs free in N. $M[N/x]$ is the result of substituting N for all free occurrences of x in M subject to the usual provisos about renaming bound variables in M to avoid capture of free variables in N. The above proviso is empty if N is free for x in M.

The axioms of the $\lambda\beta$-calculus are as follows: M and N are arbitrary λ-terms and x, y variables.

(α) $\lambda x.M = \lambda y.M[y/x]$ for any variable y that does not occur free in M;

(β) $(\lambda x.M)N = M[N/x]$ for any N free for x in M.

(β)-conversion expresses the way of calculating a function $(\lambda x.M)$ on an argument N, while (α)-conversion says that the name of bound variables does not matter. The rules for deriving equations from instances of (α) and (β) are the usual ones from equational calculus asserting that equality is a congruence for application and abstraction.

Let $M \in \Lambda_I(C)$. M *is* a *β-normal form* (β-nf) if M does not contain any subterm of the form $(\lambda x.P)Q$. A λ-term M *has* a β-nf if there exists a λ-term N such that

$M =_\beta N$ and N is a β-nf. A λ-term M is a *$\beta\eta$-normal form* if it is a β-normal form and it does not contain any subterm of the form $\lambda x.Nx$ ($x \notin FV(N)$).

A λ-term $M \in \Lambda_I^0(C)$ is *solvable* if there exist an integer n and $N_1, \dots, N_n \in \Lambda_I(C)$ such that $MN_1 \dots N_n = \mathbf{i}$. $M \in \Lambda_I(C)$ is *solvable* if the closure of M, that is $\lambda x_1 \dots x_n.M$ with $\{x_1 \dots x_n\} = FV(M)$, is solvable. $M \in \Lambda_I(C)$ is *unsolvable* if it is not solvable. A λ-term M is solvable if, and only if, it has a *head normal form*, that is, $M =_\beta \lambda x_1 \dots x_n.yM_1 \dots M_k$ for some $n, k \geq 0$ and λ-terms $M_1, \dots, M_k$.

We can give a graphic representation of a λ-term M in function of its property to be either solvable or unsolvable. In this way we obtain the Böhm tree $BT(M)$ of M as a finite or infinite labelled tree. If M is unsolvable, then $BT(M) = \bot$, that is, $BT(M)$ is constituted by a unique node labelled by $\bot$. If M is solvable and $M =_\beta \lambda x_1 \dots x_n.yM_1 \dots M_m$, then we have:

$$BT(M) = \lambda x_1 \dots x_n.y$$

$$BT(M_1) \dots\dots BT(M_m)$$

A compatible λ-relation $\mathcal{T}$ is any set of equations between λ-terms that is closed under the following rules, for all λ-terms M, N and P:

- $M = N \in \mathcal{T} \Longrightarrow \lambda x.M = \lambda x.N \in \mathcal{T}$;
- $M = N \in \mathcal{T} \Longrightarrow MP = NP \in \mathcal{T}$;
- $M = N \in \mathcal{T} \Longrightarrow PM = PN \in \mathcal{T}$.

We will write on occasion $\mathcal{T} \vdash M = N$ for $M = N \in \mathcal{T}$.

A *lambda theory* $\mathcal{T}$ is any compatible λ-relation which is an equivalence relation and includes (α) and (β) conversion. The set of all lambda theories is naturally equipped with a structure of complete lattice with meet defined as set theoretical intersection. The join of two lambda theories $\mathcal{T}$ and $\mathcal{S}$ is the least equivalence relation including $\mathcal{T} \cup \mathcal{S}$. The least lambda theory including a set $\mathcal{W}$ of equations will be denoted by $\mathcal{W}^+$. A lambda theory $\mathcal{T}$ is *consistent* if there exist two lambda terms M and N such that $\mathcal{T} \not\vdash M = N$.

The definition of lambda theory used here is different from the standard definition. Usually, one defines a lambda theory to be a set of equations between closed λ-terms in the language without constants (see [1, Def. 4.1.1]). Of course, every lambda theory $\mathcal{T}$ in our sense is determined by its restriction to closed λ-terms: for every sequence $x_1 \cdots x_n$ of variables including all the free variables of M and N, $\mathcal{T} \vdash M = N$ if, and only if, $\mathcal{T} \vdash \lambda x_1 \cdots x_n.M = \lambda x_1 \cdots x_n.N$.

$\lambda\beta$ is the least lambda theory, while $\lambda\eta$ is the least extensional lambda theory (axiomatized by $\mathbf{i} = \mathbf{1}$). $\mathcal{H}$ is the lambda theory generated by equating all the unsolvable λ-term (i.e., $\mathcal{H} = \mathcal{H}_0^+$ where $\mathcal{H}_0 = \{M = N \mid M, N \text{ closed and unsolvable}\}$), while $\mathcal{H}^*$ is the unique maximal consistent extension of $\mathcal{H}$ (see [1]). $\mathcal{H}\eta$ is the least extensional lambda theory which includes $\mathcal{H}$. $\mathcal{B}$ is the lambda theory generated by equating two λ-terms if they have the same Böhm tree (i.e., $\mathcal{B} = \{M = N \mid BT(M) = BT(N)\}$)

A lambda theory $\mathcal{T}$ is *sensible* if $\mathcal{H} \subseteq \mathcal{T}$. $\mathcal{T}$ is *semisensible* if $\mathcal{T}$ does not equate a solvable and an unsolvable.

1.3 Models of Lambda Calculus

Combinatory logic is a formalism for writing expressions which denote functions. Combinators are designed to perform the same tasks as λ-terms, but without using bound variables. Schönfinkel and Curry discovered that a formal system of combinators, having the same expressive power of the lambda calculus, can be based on only two primitive combinators.

An algebra $\mathbf{C} = (C, \cdot, \mathbf{k}, \mathbf{s})$, where $\cdot$ is a binary operation and $\mathbf{k}, \mathbf{s}$ are constants, is called a *combinatory algebra* (see [6]) if it satisfies the following identities (as usual the symbol $\cdot$ is omitted, and association is to the left):

$$\mathbf{k}xy = x; \quad \mathbf{s}xyz = xz(yz).$$

$\mathbf{k}$ and $\mathbf{s}$ are called the *basic combinators*. In the equational language of combinatory algebras the derived combinators $\mathbf{i}$ and $\mathbf{1}$ are defined as follows: $\mathbf{i} := \mathbf{skk}$ and $\mathbf{1} := \mathbf{s}(\mathbf{ki})$. Hence, every combinatory algebra satisfies the identities $\mathbf{i}x = x$ and $\mathbf{1}xy = xy$.

A function $f : C \to C$ is *representable* in $\mathbf{C}$ if there exists an element $c \in C$ such that $cz = f(z)$ for all $z \in C$. If this last condition is satisfied, we say that c represents f in $\mathbf{C}$. Two elements $c, d \in C$ are called *extensionally equal* if they represent the same function in $\mathbf{C}$. For example, the elements c and $\mathbf{1}c$ are extensionally equal for every $c \in C$.

The class of the models of lambda calculus has an intrinsic characterization as an elementary subclass of combinatory algebras called λ-*models* [1, Def. 5.2.7]. They were first axiomatized by Meyer [12] and independently by Scott [24]; the axiomatization, while elegant, is not equational. We now define the notion of a λ-model.

Let $\mathbf{C}$ be a combinatory algebra and let $\bar{c}$ be a new symbol for each $c \in C$. Extend the language of the lambda calculus by adjoining $\bar{c}$ as a new constant symbol for each $c \in C$. Let $\Lambda_I^o(C)$ be the set of closed λ-terms with constants from C. The interpretation of terms in $\Lambda_I^o(C)$ with elements of C can be defined by induction as follows (for all $M, N \in \Lambda_I^o(C)$ and $c \in C$):

$$|\bar{c}|_{\mathbf{C}} = c; \; |(MN)|_{\mathbf{C}} = |M|_{\mathbf{C}}|N|_{\mathbf{C}}; \; |\lambda x.M|_{\mathbf{C}} = \mathbf{1}m,$$

where m is any element of C representing the following function $f : C \to C$:

$$f(c) = |M[x := \bar{c}]|_{\mathbf{C}}, \quad \text{for all } c \in C. \tag{1}$$

The drawback of the previous definition is that, if $\mathbf{C}$ is an arbitrary combinatory algebra, it may happen that the function f is not representable. The axioms of an elementary subclass of combinatory algebras, called λ-*models* or models of the lambda calculus, were expressly chosen to make coherent the previous definition of interpretation (see [12], [24], [1, Def. 5.2.7]).

The *Meyer-Scott axiom* is the most important axiom in the definition of a λ-model. In the first-order language of combinatory algebras it takes the following form:

$$\forall x \forall y (\forall z (xz = yz) \Rightarrow \mathbf{1}x = \mathbf{1}y)$$

and it makes the combinator $\mathbf{1}$ an inner choice operator. Indeed, given any c, the element $\mathbf{1}c$ is in the same equivalence class as c w.r.t. extensional equality; and, by Meyer-Scott axiom, $\mathbf{1}c = \mathbf{1}d$ for every d extensionally equal to c. Thus, the set $Y = \{c : cz = f(z) \text{ for all } z \in C\}$ of elements representing the function f defined in (1) admits $\mathbf{1}m$ as a canonical representative and this does not depend on the choice of $m \in Y$.

For every λ-model $\mathbf{C}$, the lambda theory generated by the set $\{M = N : M, N \in \Lambda_I^o,\ |M|_{\mathbf{C}} = |N|_{\mathbf{C}}\}$ is called *the equational theory of* $\mathbf{C}$.

1.4 Lambda abstraction algebras

Let I be a nonempty set. The similarity type of *lambda abstraction algebras of dimension* I is constituted by a binary operation symbol "$\cdot$" formalizing application, a unary operation symbol "λx" for every $x \in I$, and a constant symbol (i.e., nullary operation symbol) "x" for every $x \in I$. The elements of I are the variables of lambda calculus although in their algebraic transformation they no longer play the role of variables in the usual sense. In the remaining part of the paper we will refer to them as *λ-variables*. The actual variables of the similarity type of lambda abstraction algebras are referred to as *context variables* and denoted by the Greek letters ξ, ν, and μ possibly with subscripts.

The terms of the similarity type of lambda abstraction algebras are called *λ-contexts*. They are constructed in the usual way: every λ-variable x and context variable ξ is a λ-context; if t and s are λ-contexts, then so are $t \cdot s$ and $\lambda x(t)$.

Because of their similarity to the terms of the lambda calculus we use the standard notational conventions of the latter. The application operation symbol "$\cdot$" is normally omitted, and the application of t and s is written as juxtaposition ts. When parentheses are omitted, association to the left is assumed. The left parenthesis delimiting the scope of a lambda abstraction is replaced with a period and the right parenthesis is omitted. For example, $\lambda x(ts)$ is written $\lambda x.ts$. Successive λ-abstractions $\lambda x \lambda y \lambda z \ldots$ are written $\lambda xyz \ldots$.

An occurrence of a λ-variable x in a λ-context is *bound* if it falls within the scope of the operation symbol λx; otherwise it is *free*. The *free λ-variables* of a λ-context are the λ-variables that have at least one free occurrence. A λ-context without free λ-variables is said to be *closed*. Note that λ-contexts without any context variables coincide with ordinary terms of the lambda calculus without constants.

Our notion of a λ-context coincides with the notion of *context* defined in Barendregt ([1, Def.14.4.1]); our context variables correspond to Barendregt's notion of a 'hole'. The main difference between Barendregt's notation and our's is that 'holes' are denoted here by Greek letters $\xi, \mu, \ldots$, while in Barendregt's book by $[\,], [\,]_1, \ldots$.

The essential feature of a λ-context is that a free λ-variable in a λ-term may become bound when we substitute it for a 'hole' within the context. For example, if $t(\xi) = \lambda x.x(\lambda y.\xi)$ is a λ-context, in Barendregt's notation: $t([\,]) = \lambda x.x(\lambda y.[\,])$, and $M = xy$ is a λ-term, then $t(M) = \lambda x.x(\lambda y.xy)$.

A lambda theory has a natural algebraic interpretation. Let $\mathcal{T}$ be a lambda theory over the language $\Lambda_I(C)$ and let $\mathbf{\Lambda}_I(C)$ be the following algebra in the similarity type of lambda abstraction algebras (of dimension I):

$$\mathbf{\Lambda}_I(C) := \langle \Lambda_I(C), \cdot^{\mathbf{\Lambda}_I(C)}, \lambda x^{\mathbf{\Lambda}_I(C)}, x^{\mathbf{\Lambda}_I(C)} \rangle_{x \in I} \tag{2}$$

where for $M, N \in \Lambda_I(C)$

$$M \cdot^{\mathbf{\Lambda}_I(C)} N = (MN); \quad \lambda x^{\mathbf{\Lambda}_I(C)}(M) = (\lambda x.M); \quad x^{\mathbf{\Lambda}_I(C)} = x.$$

We will write $\mathbf{\Lambda}_I$ for $\mathbf{\Lambda}_I(\emptyset)$. The lambda theory $\mathcal{T}$ is a congruence (i.e., a compatible equivalence relation) on $\mathbf{\Lambda}_I(C)$. We denote by $\mathbf{\Lambda}_I^{\mathcal{T}}(C)$ the quotient of $\mathbf{\Lambda}_I(C)$ by $\mathcal{T}$ and call it the *term algebra* of the lambda theory $\mathcal{T}$. We denote by $M_{\mathcal{T}}$ the equivalence class of the λ-term M (i.e. $M_{\mathcal{T}} = \{N \in \Lambda_I(C) \mid \mathcal{T} \vdash M = N\}$ and call it *$\mathcal{T}$-block* of the λ-term M.

We say that $\mathcal{T}$ satisfies an identity between contexts $t(\xi_1, \dots, \xi_n) = u(\xi_1, \dots, \xi_n)$ if the term algebra $\mathbf{\Lambda}_I^{\mathcal{T}}(C)$ of $\mathcal{T}$ satisfies it; i.e., if all the instances of the above identity, obtained by substituting λ-terms for context variables in it, fall within the lambda theory: $\mathcal{T} \vdash t(M_1, \cdots, M_n) = u(M_1, \cdots, M_n)$, for all λ-terms $M_1, \cdots, M_n \in \Lambda_I(C)$. For example, every lambda theory satisfies the identity $(\lambda x.x)\xi = \xi$ because $\lambda\beta \vdash (\lambda x.x)M = M$ for every λ-term M.

Lambda abstraction algebras are meant to axiomatize those identities between contexts that are valid for the lambda calculus.

We now give the formal definition of a lambda abstraction algebra (see [14, 15, 20, 21, 22, 10]).

Definition 1.1 *By a* **lambda abstraction algebra of dimension** I *we mean an algebraic structure of the form:*

$$\mathbf{A} := \langle A,\ \cdot^{\mathbf{A}},\ \lambda x^{\mathbf{A}},\ x^{\mathbf{A}} \rangle_{x \in I}$$

satisfying the following identities between λ-contexts, for all $x, y, z \in I$:

(β_1) $(\lambda x.x)\xi = \xi$;

(β_2) $(\lambda x.y)\xi = y, \quad x \neq y$;

(β_3) $(\lambda x.\xi)x = \xi$;

(β_4) $(\lambda xx.\xi)\mu = \lambda x.\xi$;

(β_5) $(\lambda x.\xi\mu)\nu = (\lambda x.\xi)\nu((\lambda x.\mu)\nu)$;

(β_6) $(\lambda xy.\mu)((\lambda y.\xi)z) = \lambda y.(\lambda x.\mu)((\lambda y.\xi)z), \quad x \neq y, z \neq y$;

(α) $\lambda x.(\lambda y.\xi)z = \lambda y.(\lambda x.(\lambda y.\xi)z)y, \quad z \neq y$.

I is called the **dimension set** *of* $\mathbf{A}$. $\cdot^{\mathbf{A}}$ *is called* **application** *and* $\lambda x^{\mathbf{A}}$ *is called* **λ-abstraction** *with respect to* x.

The class of lambda abstraction algebras of dimension I is denoted by LAA_I and the class of all lambda abstraction algebras of any dimension by LAA. We also use LAA_I as shorthand for the phrase "lambda abstraction algebra of dimension I", and similar for LAA. An LAA_I is *infinite dimensional* if I is infinite.

LAA_I is a variety (= equational class) for every dimension set I, and therefore it is closed under the formation of subalgebras, homomorphic (in particular isomorphic) images, and Cartesian products.

In [15, 21, 22] it is shown that the term algebra of every lambda theory is a lambda abstraction algebra and that every variety of lambda abstraction algebras is generated by the term algebra of a suitable lambda theory over the language Λ_I (with an empty set of constants). In particular, the term algebra of the least lambda theory $\lambda\beta$ generates the variety LAA_I. Hence the explicit finite equational axiomatization for the variety of lambda abstraction algebras provides also an explicit axiomatization of the identities between contexts satisfied by the term algebra of the least lambda theory $\lambda\beta$. The variety of LAA_I's generated by the term algebra $\mathbf{\Lambda}_I^{\mathcal{T}}$ will be denoted by $\mathsf{LAA}_I^{\mathcal{T}}$. Note that the lattice of all lambda theories is naturally isomorphic to the lattice of all congruences of the term algebra $\mathbf{\Lambda}_I^{\lambda\beta}$.

We would like to explicitly mention at this point that the equational theory axiomatized by $(\beta_1) - (\beta_6)$ and (α) is a conservative extension of lambda beta-calculus: for any two λ-terms M and N, the identity $M = N$ between λ-terms is a logical consequence of $(\beta_1) - (\beta_6)$ and (α) (in symbols, $LAA_I \models M = N$) if, and only if, $M = N$ is derivable in the lambda beta-calculus. This can be immediately inferred from the fact, stated previously, that LAA_I is generated as a variety by the term algebra of the lambda theory $\lambda\beta$.

We note here one very useful immediate consequence of the axioms $(\beta_1) - (\beta_6)$ and (α): in any LAA_I $\mathbf{A}$ the functions λx are always one-one, i.e., for all $x \in I$,

$$\lambda x.a = \lambda x.b \ \text{ iff } \ a = b, \quad \text{for all} \ \ a, b \in A.$$

In fact, if $\lambda x.a = \lambda x.b$, then by (β_3), $a = (\lambda x.a)x = (\lambda x.b)x = b$.

An LAA with only one element is said to be *trivial*. It is interesting that any nontrivial LAA_I $\mathbf{A}$ of positive dimension is infinite, since the one-one map λx is not onto. To see this, assume by way of contradiction that x is in the range of λx; then $x = \lambda x.b$ for some element $b \in A$. Since $\mathbf{A}$ is nontrivial, there exists an element $a \in A$ such that $a \neq x$. Then a contradiction results from (β_1) and (β_4):

$$a = (\lambda x.x)a = (\lambda x.\lambda x.b)a = \lambda x.b = x.$$

Definition 1.2 ([14, Def. 1.3]) *Let* $\mathbf{A}$ *be an* LAA_I. *Let* $a \in A$ *and* $x \in I$. *a is said to be* **algebraically dependent on** x (**over** $\mathbf{A}$) *if* $(\lambda x.a)z \neq a$ *for some* $z \in I$*; otherwise a is* **algebraically independent of** x (**over** $\mathbf{A}$). *The set of all $x \in I$ such that a is algebraically dependent on x over* $\mathbf{A}$ *is called the* **dimension set** *of a and is denoted by* $\mathbf{\Delta}a$*; thus:*

$$\mathbf{\Delta}a = \{x \in I : (\lambda x.a)z \neq a \textit{ for some } z \in I\}.$$

a is **finite** (**infinite**) **dimensional** *if* $\mathbf{\Delta} a$ *is finite (infinite). An element* a *is called* **zero-dimensional** *if* $\mathbf{\Delta} a = \emptyset$. *We denote the set of zero-dimensional elements by* $\mathrm{Zd}\,\mathbf{A}$.

For example, if $a = xy$ then a is algebraically dependent on x because $(\lambda x.xy)z = zy \neq xy$ for every $z \in I \setminus \{x, y\}$.

Proposition 1.3 ([14, Prop. 1.7]) *Let* $\mathbf{A} \in \mathsf{LAA}_I$, $a, b \in A$, *and* $x \in I$.

1. $\mathbf{\Delta}(ab) \subseteq \mathbf{\Delta} a \cup \mathbf{\Delta} b$.
2. $\mathbf{\Delta}(\lambda x.a) = \mathbf{\Delta} a \setminus \{x\}$.
3. $\mathbf{\Delta} x \subseteq \{x\}$, *with equality holding if* $\mathbf{A}$ *is nontrivial.*

Proposition 1.4 ([21, Prop. 4]) *Let* $\mathbf{A} \in \mathsf{LAA}_I$, $a \in A$, *and* $x \in I$. *The following are equivalent.*

(i) $(\lambda x.a)z = a$ *for some* $z \in I$;

(ii) $(\lambda x.a)z = a$ *for all* $z \in I$ *(i.e.,* $x \notin \mathbf{\Delta} a$*)*;

(iii) $(\lambda x.a)b = a$ *for all* $b \in A$.

If M is a λ-term without constants and $\mathbf{A}$ is an LAA_I, then $M^{\mathbf{A}}$ will denote the value of M in $\mathbf{A}$ when each λ-variable x occurring in M is interpreted as $x^{\mathbf{A}}$. By Prop. 1.3 the dimension set of $M^{\mathbf{A}}$ is a subset of the set of free λ-variables of M.

Suitable reducts of arbitrary LAA's turn out to be combinatory algebras. Let $\mathbf{A}$ be an LAA_I. By the *combinatory reduct* of $\mathbf{A}$ we mean the algebra

$$\mathrm{Cr}\,\mathbf{A} = \langle A,\ \cdot^{\mathbf{A}},\ \mathbf{k}^{\mathbf{A}},\ \mathbf{s}^{\mathbf{A}} \rangle,$$

where

$$\mathbf{k}^{\mathbf{A}} = (\lambda xy.x)^{\mathbf{A}} \quad \text{and} \quad \mathbf{s}^{\mathbf{A}} = (\lambda xyz.xz(yz))^{\mathbf{A}}.$$

$\mathrm{Cr}\,\mathbf{A}$ is a combinatory algebra [22]. A subalgebra of the combinatory reduct of an LAA_I $\mathbf{A}$ (i.e., a subset of $\mathbf{A}$ containing $\mathbf{k}^{\mathbf{A}}$ and $\mathbf{s}^{\mathbf{A}}$ and closed under $\cdot^{\mathbf{A}}$) is called a *combinatory subreduct* of $\mathbf{A}$. The *zero-dimensional subreduct* of $\mathbf{A}$ is the combinatory subreduct

$$\mathbf{Zd}\,\mathbf{A} = \langle \mathrm{Zd}\,\mathbf{A}, \cdot^{\mathbf{A}}, \mathbf{k}^{\mathbf{A}}, \mathbf{s}^{\mathbf{A}} \rangle,$$

where $\mathrm{Zd}\,\mathbf{A} = \{\, a \in A : \mathbf{\Delta} a = \emptyset \,\}$, the set of zero-dimensional elements of $\mathbf{A}$.

The *open term model* of a lambda theory $\mathcal{T}$, as defined in Barendregt's book [1], is the combinatory reduct $\mathrm{Cr}\,\mathbf{\Lambda}_I^{\mathcal{T}}(C)$ of the term algebra $\mathbf{\Lambda}_I^{\mathcal{T}}(C)$, while the *closed term model* of $\mathcal{T}$ is its zero-dimensional subreduct $\mathbf{Zd}\,\mathbf{\Lambda}_I^{\mathcal{T}}(C)$.

1.5 Locally finite LAA's

There is a strong connection between the lambda theories and the subclass of LAA's whose elements are finite dimensional.

Definition 1.5 ([14, Def. 2.1]) *A lambda abstraction algebra* $\mathbf{A}$ *is* **locally finite** *if it is of infinite dimension (i.e., I is infinite) and every* $a \in A$ *is of finite dimension (i.e.,* $|\mathbf{\Delta} a| < \aleph_0$*).*

The class of locally finite LAA_I's is denoted by LFA_I, which is also used as shorthand for the phrase "locally finite lambda abstraction algebra of dimension I".

For every infinite I the term algebra $\mathbf{\Lambda}_I^{\mathcal{T}}$ of a lambda theory $\mathcal{T}$ is locally finite. This is a direct consequence of the trivial fact that every λ-term is a finite string of symbols and hence contains only finitely many λ-variables.

The following result characterizes those congruences on the algebra $\mathbf{\Lambda}_I(C)$ (defined in Section 1.4) that are lambda theories.

Lemma 1.6 ([21, Lemma 8]) *Let* I *be an infinite set. A congruence* θ *on* $\mathbf{\Lambda}_I(C)$ *is a lambda theory over the language* $\mathbf{\Lambda}_I(C)$ *if, and only if, the following two conditions are satisfied:*

1. *The quotient algebra* $\mathbf{\Lambda}_I(C)/\theta$ *is an* LAA_I*;*
2. $(\lambda x.c)y\ \theta\ c$ *for all* $c \in C$ *and all* $x, y \in I$*, i.e., the equivalence class* c/θ *of every element* $c \in C$ *is a zero-dimensional element of* $\mathbf{\Lambda}_I(C)/\theta$*.*

The following proposition is the algebraic analog of Prop. 1 and Prop. 3 in ([11, Ch. VII]).

Proposition 1.7 ([15, Prop. 2.4]) *Let* I *be an infinite set. An algebra* $\mathbf{A}$ *in the similarity type of lambda abstraction algebras of dimension* I *is (isomorphic to) the term algebra of a lambda theory if, and only if, it is an* LFA_I*.*

The following proposition provides an algebraic characterization of the term algebra of a lambda theory.

Proposition 1.8 *Let* I *be an infinite set and* $\mathsf{LAA}_I^{\mathcal{T}}$ *be the variety of lambda abstraction algebras generated by the term algebra* $\mathbf{\Lambda}_I^{\mathcal{T}}$ *of a lambda theory* $\mathcal{T}$*. Then* $\mathbf{\Lambda}_I^{\mathcal{T}}$ *is the free algebra over an empty set of generators in the variety* $\mathsf{LAA}_I^{\mathcal{T}}$*.*

Proof: The proof of this result in the hypothesis $\mathcal{T} = \lambda\beta$ can be found in [21, Prop. 10]. The proof can be easily generalized to an arbitrary lambda theory. □

2 Unorderable algebras

Although the λ-calculus was introduced in the early 1930's, its model theory developed only much later. Dana Scott discovered that models of lambda calculus can be constructed by a combination of order-theoretic and topological methods. Scott's methods have been widely studied and today one can choose from a wide array of model constructions that are based on Scott's principles. A reflexive object in the Cartesian closed category of complete partial orders and Scott-continuous function is a model of λ-calculus: a topological model. A topological model, through the interpretation function, induces a lambda theory. So it is natural to ask if all possible λ-theories are induced by a topological model. Honsell and Ronchi dalla Rocca [9] give a negative answer to this question: they exhibit a lambda theory which cannot be induced by a topological model (see also [17, 18, 19]). In [25] Selinger relaxes the above question by asking:

1. Can every model of the λ-calculus be partially ordered in a nontrivial way?
2. Can every model of the λ-calculus be embedded in one that admits a nontrivial partial order?

These two questions are treated in [25]. We quote only the main results, for details see [25].

An algebra is called *unorderable* if it does not admit a non-trivial partial order that is compatible with the algebraic structure. We know that unorderable algebras exist and Plotkin in [16] has recently constructed a *finitely separable* combinatory algebra, a property which implies unorderability. Selinger has proven that there exist "simple" unorderable combinatory algebras: the standard open and closed term models of the lambda theory $\lambda\beta$. Recall from Section 1.4 that the open and closed term models of $\lambda\beta$ are, respectively, the combinatory reduct and the zero-dimensional subreduct of the term algebra $\mathbf{\Lambda}_I^{\lambda\beta}(C)$.

Question (2) is more difficult to answer and the answer is still unknown. A combinatory algebra is *absolutely unorderable* (see Def. 3.1 below) if it cannot be embedded in an orderable one. Selinger gives an algebraic and interesting characterization of absolutely unorderable $\mathcal{V}$-algebras in any algebraic variety $\mathcal{V}$. He shows (see Thm. 3.4 below) that a $\mathcal{V}$-algebra is absolutely unorderable if, and only if, it has a family of, so called, *generalized Mal'cev operators*. Question (2) thereby reduces to the syntactic question whether it is consistent to add such Mal'cev operators to the λ-calculus. This is, in general, an open problem.

In this paper we prove that there exists a wide class of term algebras (associated with particular lambda theories) which can be embedded into orderable combinatory algebras. This includes the term algebra $\mathbf{\Lambda}_I^{\lambda\beta}$.

Definition 2.1 *Let ρ be a binary relation on a set A.*

1. *Two elements $a, b \in A$ are* **incomparable** *if neither $a\rho b$ nor $b\rho a$.*
2. *An element $a \in A$ is* **maximal** *if $a\rho b$ implies $a = b$.*

Proposition 2.2 *Let $\mathbf{A}$ be an LAA_I and let ρ be a non-trivial compatible binary relation on $\mathbf{A}$. Then the $\beta\eta$-normal forms are incomparable.*

Proof: Let M, N be two distinct $\beta\eta$-normal forms such that $M^{\mathbf{A}}\ \rho\ N^{\mathbf{A}}$. Then by Böhm theorem ([1, Section 10.3]) there exists a λ-context $t(\xi)$ such that $t(M) =_\beta \lambda xy.x$ and $t(N) =_\beta \lambda xy.y$. By the compatibility of ρ we have

$$\lambda xy.x = t(M)\ \rho\ t(N) = \lambda xy.y,$$

that implies

$$a = (\lambda xy.x)ab \ \rho \ (\lambda xy.y)ab = b$$

for all $a, b \in A$. It follows that ρ is a trivial relation and this contradicts the hypothesis. $\square$

Definition 2.3 *An algebra* $\mathbf{A}$ *is* **orderable** *if it admits a non-trivial compatible partial order; otherwise,* $\mathbf{A}$ *is said to be* **unorderable**.

We would like to characterize the unorderable LAA's. We have the following first result.

Proposition 2.4 *Let* $\mathbf{A}$ *an* LFA_I. $\mathbf{A}$ *is unorderable if, and only if, its zero-dimensional combinatory subreduct* $\mathbf{Zd\,A}$ *is unorderable.*

Proof: ($\Leftarrow$) We assume that $\mathbf{A}$ is orderable. Let $\leq$ be a non-trivial compatible partial order on $\mathbf{A}$ and let a, b be distinct elements of $\mathbf{A}$ with $a < b$. We recall from Def. 1.2 that

$$\mathbf{\Delta} a = \{x \in I \mid (\lambda x.a)z \neq a \text{ for some } z \in I\}.$$

Consider any finite set $\{x_1, \dots, x_n\}$ of λ-variables such that

$$\mathbf{\Delta} a \cup \mathbf{\Delta} b \subseteq \{x_1, \dots, x_n\}.$$

Then the elements $\lambda x_1 \dots x_n.a$ and $\lambda x_1 \dots x_n.b$ are zero-dimensional. By the injectivity of the lambda abstractions they are distinct elements of $\mathbf{Zd\,A}$ and by the compatibility of the partial order we have:

$$\lambda x_1 \dots x_n.a < \lambda x_1 \dots x_n.b.$$

So $\mathbf{Zd\,A}$ admits a nontrivial compatible partial order.
($\Rightarrow$) Assume a nontrivial partial order $\leq_0$ on $\mathbf{Zd\,A}$ which is compatible w.r.t. application. We now show that $\mathbf{A}$ is orderable, i.e., it admits a nontrivial partial order compatible w.r.t. application and λ-abstraction.

As a matter of notation, we write $e \subseteq_\omega I$ if e is a finite subset of I.

For every $e \subseteq_\omega I$, we define

$$A_e = \{a \in A \mid \Delta a \subseteq e\}.$$

By definition and by Prop. 1.3 the following properties hold:

(i) A_e is closed under the operations of application "$\cdot$" and λ-abstraction "λy" ($y \in I$);

(ii) $(A_e \mid e \subseteq_\omega I)$ is a directed family of subsets of A (because $A_e \subseteq A_{e \cup d}$ and $A_d \subseteq A_{e \cup d}$ for all $e, d \subseteq_\omega I$);

(iii) $A_\emptyset = \mathrm{Zd}\,\mathbf{A}$;

(iv) $A = \bigcup_{e \subseteq_\omega I} A_e$ (because $\mathbf{A}$ is locally finite).

We now define a family of partial orderings $\leq_e$ on A_e ($e \subseteq_\omega I$) which satisfy the following conditions:

(a) $\leq_e$ is compatible w.r.t. application and λ-abstraction λy ($y \in I$);

(b) If $d \subseteq e$ then $\leq_e$ extends $\leq_d$ (i.e., $a \leq_d b$ ($a, b \in A_d$) implies $a \leq_e b$).

The definition of $\leq_e$ is given by induction on the cardinality of e.

Base: Let $\leq_\emptyset$ be the partial order $\leq_0$ on $\mathbf{Zd}\,\mathbf{A}$ that we have by hypothesis. We have to check that $\leq_\emptyset$ is also compatible w.r.t. λ-abstraction. By applying axiom (β_6) and the compatibility of $\leq_\emptyset$ w.r.t. application, we have:

$$a \leq_\emptyset b \;\Rightarrow\; \lambda y.a = \mathbf{k}a \;\;\leq_\emptyset\;\; \mathbf{k}b = \lambda y.b, \quad \text{for all } a, b \in \mathrm{Zd}\,\mathbf{A}.$$

Inductive case: Let $e \neq \emptyset \subseteq_\omega I$. Assume by induction hypothesis that, for every proper subset d of e, we have a non-trivial partial order $\leq_d$ over A_d satisfying conditions (a)-(b). We define a partial order on A_e as follows, for all $a, b \in A_e$:

$$a \leq_e b \Leftrightarrow \text{ for all } x \in e,\ \lambda x.a \leq_{e-\{x\}} \lambda x.b. \tag{3}$$

The reflexivity and transitivity of the relation $\leq_e$ are trivial. The antisymmetry follows from the corresponding property of $\leq_{e-\{x\}}$ and from the injectivity of the lambda abstractions "λx".

In order to verify that $\leq_e$ also satisfies conditions (a)-(b), we divide the proof into Claims.

Claim 2.5 *The partial order $\leq_e$ extends $\leq_{e-\{x\}}$ for all $x \in e$, i.e.,*

$$a \leq_e b \Leftrightarrow a \leq_{e-\{x\}} b, \qquad \textit{for all } a, b \in A_{e-\{x\}}.$$

Assume $a \leq_e b$. Then by definition of $\leq_e$ we have that $\lambda x.a \leq_{e-\{x\}} \lambda x.b$. The elements $a, b \in A_{e-\{x\}}$ are independent of x, then by compatibility of $\leq_{e-\{x\}}$ and by Prop. 1.4 we obtain the conclusion $a = (\lambda x.a)a \leq_{e-\{x\}} (\lambda x.b)a = b$.
Assume now $a \leq_{e-\{x\}} b$. Then we have:

$$\begin{aligned} a \leq_{e-\{x\}} b \;\;&\Rightarrow\;\; \text{for all } y \in e - \{x\},\ \lambda y.a \leq_{e-\{x,y\}} \lambda y.b \quad (\text{by definition of } \leq_{e-\{x\}}) \\ &\Rightarrow\;\; \text{for all } y \in e - \{x\},\ \lambda y.a \leq_{e-\{y\}} \lambda y.b \quad (\text{by induction hypothesis (b)}). \end{aligned}$$

Moreover, by compatibility of $\leq_{e-\{x\}}$ we also have $\lambda x.a \leq_{e-\{x\}} \lambda x.b$. Then the conclusion $a \leq_e b$ follows from the definition of $\leq_e$.

Claim 2.6 *The partial order $\leq_e$ is compatible w.r.t. application.*

Consider $a_1, a_2, c \in A_e$ with $a_1 \leq_e a_2$. By (3) and by recalling that $\lambda x.ca_i = \mathbf{s}(\lambda x.c)(\lambda x.a_i)$ ($i = 1, 2$) for all $x \in e$, we have:

$$\lambda x.ca_1 \leq_{e-\{x\}} \lambda x.ca_2, \quad \text{for all } x \in e$$

and then

$$ca_1 \leq_e ca_2.$$

Claim 2.7 *The partial order $\leq_e$ is compatible w.r.t. all the λ-abstractions.*

It is an immediate consequence of the definition of the partial order $\leq_e$: if $a \leq_e b$ then $\lambda x.a \leq_{e-\{x\}} \lambda x.b$ for all $x \in e$. Since $\leq_e$ extends $\leq_{e-\{x\}}$ we get the conclusion $\lambda x.a \leq_e \lambda x.b$ for all $x \in e$.

The conclusion of the theorem is obtained by defining, for all $a, b \in A$, $a \leq b$ if and only if there exists $e \subseteq_\omega I$ such that $a \leq_e b$. □

Recall from [1] that the variety of λ-algebras is generated by the class of λ-models, so that it is the equational class of combinatory algebras which are more properly related to lambda calculus. The importance of Prop. 2.4 is immediate if we recall the following results:
(1) An algebra in the similarity type of LAA's is isomorphic to the term algebra of a lambda theory if, and only if, it is an LFA (see Prop. 1.7);
(2) A combinatory algebra is a λ-algebra if, and only if, it is the zero-dimensional combinatory subreduct of an LFA ([15, Cor. 3.1]).
So, the properties of unorderability in the variety of the λ-algebras and in the class of LFA's are equivalent.

By Prop. 2.4 and Selinger's unorderability result [25] we obtain that the term algebra $\mathbf{\Lambda}_I^{\lambda\beta}(C)$ of the lambda theory $\lambda\beta$ is unorderable.

Proposition 2.8 *$\mathbf{\Lambda}_I^{\lambda\beta}(C)$ is unorderable.*

Proof: We know from Prop. 1.7 that, for every infinite I, the term algebra $\mathbf{\Lambda}_I^{\lambda\beta}(C)$ is locally finite and from [25] that the zero-dimensional combinatory subreduct of $\mathbf{\Lambda}_I^{\lambda\beta}(C)$ (= the closed term model of $\lambda\beta$) is unorderable. So we can apply Prop. 2.4 to get the conclusion. □

3 Absolutely unorderable algebras

The term algebra $\mathbf{\Lambda}_I^{\lambda\beta}$ is unorderable by Prop. 2.8. However from the work of Di Gianantonio et al. [5] it follows that the open term model $\mathrm{Cr}\,\mathbf{\Lambda}_I^{\lambda\beta}$ of $\lambda\beta$ can be embedded in an orderable algebra. Plotkin has conjectured in [16] that there exists a combinatory algebra which is *absolutely unorderable* in the sense of the following definition.

Definition 3.1 *Let $\mathcal{V}$ be a class of algebras closed under isomorphism. An algebra $\mathbf{A} \in \mathcal{V}$ is* **absolutely unorderable** *(with respect to $\mathcal{V}$) if, for any embedding $\mathbf{A} \hookrightarrow \mathbf{B}$ of algebras in $\mathcal{V}$ (i.e. $\mathbf{B} \in \mathcal{V}$), $\mathbf{B}$ is unorderable.*

The following proposition is a consequence of the equivalence of the categories of λ-algebras and LFA_I's (see [15, Thm. 3.2]).

Proposition 3.2 *There exists an absolutely unorderable λ-algebra if, and only if, there exists an absolutely unorderable LFA_I.*

We now recall the definition of a polynomial operation because this concept will be useful later on.

Definition 3.3 *A polynomial operation on an* LAA_I $\mathbf{A}$ *is a function* $f : A^n \to A$ *for which there exists a* λ*-context* $t(\xi_1, \dots, \xi_n, \mu_1, \dots, \mu_k)$ *and elements* $b_1, \dots, b_k \in A$ *such that*

$$f(a_1, \dots, a_n) = t(a_1, \dots, a_n, b_1, \dots, b_k), \qquad \textit{for all } a_1, \dots, a_n \in A.$$

We explain how Selinger [25] has characterized absolutely unorderable algebras. Consider a variety $\mathcal{V}$ of algebras in the similarity type Σ. Let $\mathbf{A}$ be an algebra in $\mathcal{V}$ and let $\preceq$ be the smallest compatible preorder (i.e., reflexive and transitive binary relation) on $\mathbf{A}[x, y]$ such that $x \preceq y$, where $\mathbf{A}[x, y]$ is the free extension of $\mathbf{A}$ by the two indeterminates x and y (see Section 1.1). It is possible to prove ([25, Lemma 3.7]) that $\mathbf{A}$ is absolutely unorderable if, and only if, $y \preceq x$. Moreover, Selinger has shown that $\preceq$ is the transitive closure of the relation $\lhd$ on $\mathbf{A}[x, y]$ defined as follows: $a \lhd b$ if, and only if, there exists a unary polynomial function $f : \mathbf{A}[x, y] \longrightarrow \mathbf{A}[x, y]$ such that $f(x) = a$ and $f(y) = b$.

As a matter of notation, $\Sigma(A)$ denotes the similarity type Σ with an added constant c_a for each element a of A. In every extension of the algebra $\mathbf{A}$ the constant c_a is interpreted as a.

Selinger has shown the following syntactic characterization of absolute unorderability.

Theorem 3.4 ([25, Thm. 3.9]) *Let* $\mathcal{V}$ *be an algebraic variety. A* $\mathcal{V}$*-algebra* $\mathbf{A}$ *is absolutely unorderable if, and only if, for some* $n \geq 1$*, there exist* $\Sigma(A)$*-terms* $M_i(x_1, x_2, x_3)$ $(i = 1, \dots, n)$ *such that the following equations hold in the* $\Sigma(A)$*-algebra* $\mathbf{A}[x, y]$*:*

$$\begin{array}{rcl} y & = & M_1(y, x, x) \\ M_1(y, y, x) & = & M_2(y, x, x) \\ M_2(y, y, x) & = & M_3(y, x, x) \\ & \vdots & \\ M_n(y, y, x) & = & x. \end{array}$$

In this paper we prove that for a "generic" lambda theory $\mathcal{T}$ (see below for the definition of "generic" lambda theory) the term algebra $\mathbf{\Lambda}_I^{\mathcal{T}}$ is **not** absolutely unorderable.

We recall that a nonempty subset X of a partially ordered set $\mathcal{C} = (C, \leq)$ is *directed* if, for all $x, y \in X$, there exists $z \in X$ such that $x \leq z$ and $y \leq z$. $\mathcal{C}$ is a *complete partial order* (cpo, for short) if it admits least element $\bot$ and every directed subset X of C has least upper bound $\sqcup X \in C$.

Definition 3.5 *Let* $\mathcal{C} = \langle C; \leq \rangle$ *be a cpo and* Λ_I *be the set of* λ*-terms. We say that a map* $\sigma : \Lambda_I \to C$ *is a* **semimorphism** *if the following three conditions hold:*

1. $\lambda\beta \vdash M = N$ *implies* $\sigma(M) = \sigma(N)$.

2. *The preorder $\sqsubseteq_\sigma$ on Λ_I, defined by $M \sqsubseteq_\sigma N$ if, and only if, $\sigma(M) \leq \sigma(N)$, is compatible.*

3. *There exists a λ-term M such that $\sigma(M) = \bot$.*

Example 3.6 (Böhm trees)

For every λ-term M, we denote by $BT(M)$ the Böhm tree of M. BT is a semimorphism because of:

1. If $\lambda\beta \vdash M = N$, then $BT(M) = BT(N)$ ([1, Prop. 10.1.6]);
2. The preorder defined by $M \sqsubseteq_{BT} N$ if, and only if, $BT(M) \subseteq BT(N)$, is compatible ([1, Cor. 14.3.20]);
3. $BT(M) = \bot$ for all unsolvable λ-terms M.

We recall that a lambda theory $\mathcal{T}$ is semisensible if it does not equate a solvable and an unsolvable term. We now generalize the notion of a semisensible lambda theory.

Let $\mathcal{C} = \langle C; \leq \rangle$ be a cpo, $\sigma : \Lambda_I \to C$ be a semimorphism and $\mathcal{T}$ be a lambda theory over Λ_I. Since $\mathcal{T}$ is a congruence on $\mathbf{\Lambda}_I$, we define, for all $\mathcal{T}$-blocks (i.e., equivalence classes, see Section 1.4) a and b, the following relation:

$$a \sqsubseteq_\sigma b \iff (\exists M \in a)(\exists N \in b)\ M \sqsubseteq_\sigma N.$$

Since $\mathcal{T}$ is a congruence on $\mathbf{\Lambda}_I^{\mathcal{T}}$ and σ is a semimorphism, then the relation $\sqsubseteq_\sigma$ on $\mathcal{T}$-blocks is reflexive and compatible.

Now we are able to give the alternative definition of a semisensible lambda theory.

Definition 3.7 *Let σ be a semimorphism and $\mathcal{T}$ be a lambda theory. $\mathcal{T}$ is* **σ-semisensible** *if the relation $\sqsubseteq_\sigma$ on $\mathcal{T}$-blocks is non-trivial (i.e., it is not the case that $a \sqsubseteq_\sigma b$ for all a and b).*

The new definition of a semisensible lambda theory is equivalent to usual one when the semimorphism σ is exactly BT. In fact, in this case we prove the following result.

Proposition 3.8 *A lambda theory $\mathcal{T}$ is BT-semisensible if, and only if, it is semisensible in the usual sense, i.e., $\mathcal{T} \not\vdash M = N$ whenever M is solvable and N is unsolvable.*

Proof: ($\Rightarrow$) By the way of contradiction, assume that the lambda theory $\mathcal{T}$ is BT-semisensible but not semisensible, i.e., there exist a closed solvable λ-term M and a closed unsolvable λ-term N such that $\mathcal{T} \vdash M = N$. By the definition of solvability, there exist a natural number n and λ-terms $N_1, \dots, N_n$ such that $MN_1 \dots N_n = \mathbf{i}$. Then, for every λ-term P we have that $P = \mathbf{i}P = MN_1 \dots N_n P = NN_1 \dots N_n P$ and this last λ-term is unsolvable. So, for every $\mathcal{T}$-block a, there exists an unsolvable λ-term N such that $N \in a$. Since $BT(N) = \bot$ for all unsolvable λ-terms N and each $\mathcal{T}$-block contains an unsolvable, then from the definition $\sqsubseteq_{BT}$ it follows that $a \sqsubseteq_{BT} b$ for all $\mathcal{T}$-blocks a and b. This is a contradiction because we have supposed that $\mathcal{T}$ is BT-semisensible.

($\Leftarrow$) If $\mathcal{T}$ is semisensible, then there exists a $\mathcal{T}$-block a constituted by unsolvable λ-terms. This implies that $BT(M) = \bot$ for each $M \in a$. Let b be a $\mathcal{T}$-block of solvable λ-terms (a and b are distinct $\mathcal{T}$-blocks because $\mathcal{T}$ is semisensible). Since $BT(N) \neq \bot$ for every λ-term $N \in b$, then it holds:

$$BT(M) \leq BT(N) \qquad \text{for all } M \in a, \quad N \in b.$$

In conclusion $a \sqsubseteq_{BT} b$. On the other hand, we have:

$$(\forall M \in b)(\forall N \in a) \;\; BT(M) \not\leq BT(N),$$

that is $b \not\sqsubseteq_{BT} a$. So, the relation $\sqsubseteq_{BT}$ is non-trivial. □

Definition 3.9 *A λ-term M is called* **generic** *if $M = yx_1 \dots x_n$ for some distinct λ-variables $y, x_1, \dots, x_n \in I$.*

Definition 3.10 *A lambda theory $\mathcal{T}$ is called* **generic** *if there exists a cpo $\mathcal{C} = \langle C, \leq \rangle$ and a semimorphism $\sigma : \Lambda_I \to C$ such that $\mathcal{T}$ is σ-semisensible and, for every generic λ-term M, we have:*

1. *$\mathcal{T} \vdash M = P$ implies $\sigma(P)$ is a maximal element (w.r.t. the partial order on $\mathcal{C}$);*
2. *$\mathcal{T} \vdash M = P$ and $\sigma(P) = \sigma(N)$ implies $\mathcal{T} \vdash M = N$.*

We now give some examples of generic lambda theories.

Proposition 3.11 *Let $\mathcal{T}$ be a consistent lambda theory such that either $\mathcal{T} \subseteq \mathcal{B}$ or $\mathcal{B} \subseteq \mathcal{T}$, where $\mathcal{B}$ is the lambda theory associated with the Böhm trees. Then $\mathcal{T}$ is generic.*

Proof: By hypothesis $\mathcal{T}$ is semisensible, so that $\mathcal{T}$ is BT-semisensible by Prop. 3.8. It remains to show conditions (1) and (2) of Def. 3.10.

1. Let $\mathcal{T} \vdash M = P$ for a generic λ-term M. We distinguish two cases.

($\mathcal{T} \subseteq \mathcal{B}$) By hypothesis M and P have the same Böhm tree. Since M is a $\beta\eta$-normal form, its Böhm tree $BT(M) = BT(P)$ is a maximal element.

($\mathcal{B} \subseteq \mathcal{T}$) Since $\mathcal{T}$ is sensible and M is a $\beta\eta$-normal form, the Böhm tree of P is a (possible infinite) η-expansion of the Böhm tree of M (see [1, Thm. 10.2.31]). Every η-expansion of a $\beta\eta$-normal form is a maximal element in the cpo of Böhm trees since no node is labeled by $\bot$.

2. Let $\mathcal{T} \vdash M = P$ and $BT(P) = BT(N)$, where $M \equiv yx_1 \dots x_n$ is a generic λ-term. We distinguish two cases.

($\mathcal{T} \subseteq \mathcal{B}$) The Böhm tree of N is equal to the Böhm tree of the normal form M. This means that there exist a head reduction (see [1, Ch. 8]) from N to a λ-term of type $yQ_1 \dots Q_n$ and head reductions from Q_i to x_i for every $1 \leq i \leq n$. Then $\lambda\beta \vdash M = N$, so that we obtain the conclusion $\mathcal{T} \vdash M = N$.

($\mathcal{B} \subseteq \mathcal{T}$) From $BT(P) = BT(N)$ it follows that $\mathcal{B} \vdash P = N$. This implies $\mathcal{T} \vdash P = N$ from the hypothesis $\mathcal{B} \subseteq \mathcal{T}$. Then the conclusion follows from the transitivity rule of the equational calculus.

□

Now we prove the main result of the paper.

Theorem 3.12 *Let $\mathcal{T}$ be a generic lambda theory. Then the term algebra $\mathbf{\Lambda}_I^{\mathcal{T}}$ of $\mathcal{T}$ is not absolutely unorderable (w.r.t.* LAA_I*).*

Remark 3.13 The combinatory reduct of $\mathbf{\Lambda}_I^{\mathcal{T}}$ is a λ-algebra. As a consequence of the equivalence of the categories of λ-algebras and LFA_I's (see [15, Thm. 3.2]), by Thm. 3.12 we obtain that $\mathrm{Cr}\,\mathbf{\Lambda}_I^{\mathcal{T}}$ is not absolutely unorderable (w.r.t. the variety CA of combinatory algebras).

The proof is divided in Lemmata.
We recall (see Section 1.4) that $M_{\mathcal{T}}$ denotes the $\mathcal{T}$-block including the λ-term M.

Lemma 3.14 *Let $\mathcal{T}$ be a generic lambda theory. Then every $\mathcal{T}$-block a containing a generic λ-term M is maximal with respect to the relation $\sqsubseteq_\sigma$ (i.e., if there exists a $\mathcal{T}$-block b such that $a \sqsubseteq_\sigma b$ then $a = b$).*

Proof: Let a be a $\mathcal{T}$-block and let M be a generic λ-term such that $M \in a$. We suppose that there exists a $\mathcal{T}$-block b such that $a \sqsubseteq_\sigma b$. We prove that $a = b$.
From $a \sqsubseteq_\sigma b$ and from the fact that $\mathcal{T}$ is a σ-semisensible lambda theory it follows that there exist two λ-terms P and N such that $P \in a$, $N \in b$ and $P \sqsubseteq_\sigma N$, that is, $\sigma(P) \leq \sigma(N)$. Moreover, M and P belong to the same $\mathcal{T}$-block a and thus $\mathcal{T} \vdash M = P$ with M a generic λ-term. So, by Def. 3.10(1), $\sigma(P)$ is a maximal element with respect to the partial order on the cpo $\mathcal{C}$. We also know that $\sigma(P) \leq \sigma(N)$ and so (because $\sigma(P)$ is a maximal element) $\sigma(P) = \sigma(N)$. Then we can apply Def. 3.10(2) to $\mathcal{T} \vdash M = P$ and $\sigma(P) = \sigma(N)$. In conclusion, $\mathcal{T} \vdash M = N$ and a, b are the same $\mathcal{T}$-block. □

The following Lemma is a generalization of the Genericity Lemma of lambda calculus (see [1, Prop. 14.3.24]).

Lemma 3.15 *Let $\mathcal{T}$ be a generic lambda theory. Let M be a generic λ-term, P be a λ-term, and $t(\xi)$ be a λ-context. Then we have:*

$$\mathcal{T} \vdash t(P) = M \implies \mathcal{T} \vdash t(Q) = M,$$

for all λ-terms Q such that $P_{\mathcal{T}} \sqsubseteq_\sigma Q_{\mathcal{T}}$.

Proof: Let Q be a λ-term such that $P_{\mathcal{T}} \sqsubseteq_\sigma Q_{\mathcal{T}}$. Then, by definition of $\sqsubseteq_\sigma$, there exist a λ-term $P' \in P_{\mathcal{T}}$ (i.e., $\mathcal{T} \vdash P = P'$) and a λ-term $Q' \in Q_{\mathcal{T}}$ (i.e., $\mathcal{T} \vdash Q = Q'$) such that $P' \sqsubseteq_\sigma Q'$ (i.e., $\sigma(P') \leq \sigma(Q')$).
Since the relation $\sqsubseteq_\sigma$ is compatible, the fact that $P' \sqsubseteq_\sigma Q'$ implies that

$$t(P') \sqsubseteq_\sigma t(Q'). \tag{4}$$

From (4) and from $\mathcal{T} \vdash t(P') = t(P) = M$ it follows that

$$M_{\mathcal{T}} \sqsubseteq_\sigma t(Q')_{\mathcal{T}}.$$

By hypothesis M is a generic λ-term and thus, by Lemma 3.14, $M_{\mathcal{T}}$ is maximal. So,

$$M_{\mathcal{T}} = t(Q')_{\mathcal{T}}$$

which implies

$$\mathcal{T} \vdash M = t(Q').$$

The conclusion follows because $\mathcal{T} \vdash t(Q) = t(Q')$. □

Let $\mathcal{T}$ be a generic lambda theory. To prove the main theorem we must find an orderable LAA_I $\mathbf{B}$ such that there exists an embedding $\mathbf{\Lambda}_I^{\mathcal{T}} \hookrightarrow \mathbf{B}$.

Consider the variety $\mathsf{LAA}_I^{\mathcal{T}}$ generated by the term algebra $\mathbf{\Lambda}_I^{\mathcal{T}}$ of $\mathcal{T}$. Since $\mathbf{\Lambda}_I^{\mathcal{T}}$ has no proper subalgebras and it is term-generated (i.e., every element is the value of a term without context variables), then $\mathbf{\Lambda}_I^{\mathcal{T}}$ is the free algebra in $\mathsf{LAA}_I^{\mathcal{T}}$ over the empty set of generators (see Prop. 1.8). Then, given a set X of context variables, the free extension $\mathbf{\Lambda}_I^{\mathcal{T}}[X]$ of $\mathbf{\Lambda}_I^{\mathcal{T}}$ by X in the variety $\mathsf{LAA}_I^{\mathcal{T}}$ is just the free algebra in $\mathsf{LAA}_I^{\mathcal{T}}$ over X. Consider an identity between λ-contexts (with context variables in X):

$$t(\xi_1, \dots, \xi_n) = u(\xi_1, \dots, \xi_n). \tag{5}$$

Then by well known results of Universal Algebra we have:

$$\begin{aligned} \mathsf{LAA}_I^{\mathcal{T}} \models t = u \quad &\text{iff} \quad \mathbf{\Lambda}_I^{\mathcal{T}}[X] \models t = u, \quad \text{since } \mathbf{\Lambda}_I^{\mathcal{T}}[X] \text{ is free algebra} \\ &\text{iff} \quad \mathbf{\Lambda}_I^{\mathcal{T}} \models t = u, \quad \text{since } \mathbf{\Lambda}_I^{\mathcal{T}} \text{ generates } \mathsf{LAA}_I^{\mathcal{T}} \\ &\text{iff} \quad \mathcal{T} \vdash t(M_1, \dots, M_n) = u(M_1, \dots, M_n) \quad \forall M_1, \dots, M_n \in \Lambda_I. \end{aligned}$$

Now we must find an orderable algebra $\mathbf{B}$ and define an embedding of $\mathbf{\Lambda}_I^{\mathcal{T}}$ into $\mathbf{B}$.

Let ξ_1 and ξ_2 be two context variables and let $\mathbf{\Lambda}_I^{\mathcal{T}}[\xi_1, \xi_2]$ be the free extension of $\mathbf{\Lambda}_I^{\mathcal{T}}$. The elements of $\mathbf{\Lambda}_I^{\mathcal{T}}[\xi_1, \xi_2]$ are equivalence classes of λ-contexts with at most two context variables ξ_1 and ξ_2. We denote by $\overline{\xi}_i$ the equivalence class of the context variable ξ_i.

Consider the smallest reflexive compatible relation $\lhd$ on $\mathbf{\Lambda}_I^{\mathcal{T}}[\xi_1, \xi_2]$ such that

$$\overline{\xi}_1 \lhd \overline{\xi}_2.$$

If a and b are elements of $\mathbf{\Lambda}_I^{\mathcal{T}}[\xi_1, \xi_2]$, we have $a \lhd b$ iff there exists a unary polynomial operation f such that $a = f(\xi_1)$ and $b = f(\xi_2)$, iff there exists a λ-context $t(\mu_1, \mu_2, \mu_3)$ such that $t(\xi_1, \xi_1, \xi_2) \in a$ and $t(\xi_1, \xi_2, \xi_2) \in b$.

Lemma 3.16 *Let $\mathcal{T}$ be a generic lambda theory. Then $\overline{\xi}_2$ is a maximal element with respect to $\lhd$, that is*

$$\overline{\xi}_2 \lhd a \implies a = \overline{\xi}_2.$$

Proof: Since $\overline{\xi}_2 \lhd a$ there exists a λ-context $t(\mu_1, \mu_2, \mu_3)$ such that

$$t(\xi_1, \xi_1, \xi_2) \in \overline{\xi}_2 \tag{6}$$

and

$$t(\xi_1, \xi_2, \xi_2) \in a.$$

Since $\mathbf{\Lambda}_I^{\mathcal{T}}[\xi_1, \xi_2]$ is a free algebra in the variety $\mathsf{LAA}_I^{\mathcal{T}}$, from condition (6) it follows that the identity

$$t(\xi_1, \xi_1, \xi_2) = \xi_2 \tag{7}$$

holds in the variety $\mathsf{LAA}_I^{\mathcal{T}}$.
Let $x_1 \dots x_k$ be the finite sequence of λ-variables which contains all the λ-variables occurring in $t(\mu_1, \mu_2, \mu_3)$ as constants x_i or as λ-abstractions λx_i. Let $y_1, y_2 \in I$ be two λ-variables distinct from $x_1 \dots x_k$. From Thm. 13 in [21] it follows that identity (7) holds in the variety $\mathsf{LAA}_I^{\mathcal{T}}$ generated by $\mathbf{\Lambda}_I^{\mathcal{T}}$ if and only if

$$\mathcal{T} \vdash t(y_1 x_1 \dots x_k, y_1 x_1 \dots x_k, y_2 x_1 \dots x_k) = y_2 x_1 \dots x_k.$$

By assumption $\mathcal{T}$ is a generic lambda theory and so, by Def. 3.5 there exists a λ-term P such that $\sigma(P) = \bot$, where σ is a semimorphism. Then, by (7) we have

$$\mathcal{T} \vdash t(P, P, y_2 x_1 \dots x_k) = y_2 x_1 \dots x_k.$$

Consider the λ-context $u(\mu) = t(P, \mu, y_2 x_1 \dots x_k)$. Then we have:

$$\mathcal{T} \vdash u(P) = y_2 x_1 \dots x_k,$$

where $y_2 x_1 \dots x_k$ is a generic λ-term.
On the other hand, P satisfies the condition $\sigma(P) = \bot$, so that $P_{\mathcal{T}} \sqsubseteq_\sigma (y_2 x_1 \dots x_k)_{\mathcal{T}}$. By applying Lemma 3.15 we obtain

$$\mathcal{T} \vdash u(y_2 x_1 \dots x_k) = y_2 x_1 \dots x_k,$$

that is

$$\mathcal{T} \vdash t(P, y_2 x_1 \dots x_k, y_2 x_1 \dots x_k) = y_2 x_1 \dots x_k.$$

Consider now the λ-context $w(\mu) = t(\mu, y_2 x_1 \dots x_k, y_2 x_1 \dots x_k)$ satisfying

$$\mathcal{T} \vdash w(P) = y_2 x_1 \dots x_k.$$

Again, by applying Lemma 3.15 to

$$P_{\mathcal{T}} \sqsubseteq_\sigma (y_1 x_1 \dots x_k)_{\mathcal{T}}$$

we obtain

$$\mathcal{T} \vdash w(y_1 x_1 \dots x_k) = y_2 x_1 \dots x_k,$$

that is

$$\mathcal{T} \vdash t(y_1x_1 \dots x_k, y_2x_1 \dots x_k, y_2x_1 \dots x_k) = y_2x_1 \dots x_k. \quad (8)$$

Since $\mathbf{\Lambda}_I^{\mathcal{T}}$ generates the variety $\mathsf{LAA}_I^{\mathcal{T}}$, by [21, Thm. 13] condition (8) is equivalent to

$$\mathsf{LAA}_I^{\mathcal{T}} \models t(\xi_1, \xi_2, \xi_2) = \xi_2,$$

that is equivalent to

$$t(\xi_1, \xi_2, \xi_2) \in \overline{\xi}_2$$

in the free algebra $\mathbf{\Lambda}_I^{\mathcal{T}}[\xi_1, \xi_2]$. At the beginning of this proof we have assumed that $t(\xi_1, \xi_2, \xi_2) \in a$. This implies the conclusion $a = \overline{\xi}_2$, that is a and $\overline{\xi}_2$ are the same $\mathcal{T}$-block. □

We recall what we have explained at the beginning of this section: the smallest preorder $\preceq$ on $\mathbf{\Lambda}_I^{\mathcal{T}}[\xi_1, \xi_2]$ such that $\overline{\xi}_1 \preceq \overline{\xi}_2$, is the transitive closure of $\lhd$. The following result holds.

Lemma 3.17 *Let $\mathcal{T}$ be a generic lambda theory. Then $\overline{\xi}_2$ is a maximal element with respect to $\preceq$, that is, if there exists a $\mathcal{T}$-block a such that $\overline{\xi}_2 \preceq a$ then $\overline{\xi}_2 = a$.*

Proof: Since $\preceq$ is the transitive closure of $\lhd$, if there exists a $\mathcal{T}$-block a such that $\overline{\xi}_2 \preceq a$ then there exist elements $b_1, \dots, b_n$ $(n \geq 0)$ such that

$$\overline{\xi}_2 \lhd b_1 \lhd b_2 \lhd \dots \lhd b_n \lhd a.$$

From Lemma 3.16 it follows that

$$\overline{\xi}_2 = b_1 = b_2 = \dots = b_n = a.$$

□

The Proof of main theorem (Thm. 3.12) is now immediate. By ([25, Lemma 3.7]) $\mathbf{\Lambda}_I^{\mathcal{T}}$ is absolutely unorderable if, and only if, $\overline{\xi}_2 \preceq \overline{\xi}_1$ in the free extension $\mathbf{\Lambda}_I^{\mathcal{T}}[\xi_1, \xi_2]$ of $\mathbf{\Lambda}_I^{\mathcal{T}}$. From Lemma 3.17 we know that $\overline{\xi}_2$ is a maximal element (w.r.t. $\preceq$) and so $\mathbf{\Lambda}_I^{\mathcal{T}}$ is absolutely unorderable if, and only if $\overline{\xi}_2 = \overline{\xi}_1$. However, $\mathbf{\Lambda}_I^{\mathcal{T}}[\xi_1, \xi_2]$ is the free algebra in the variety $\mathsf{LAA}_I^{\mathcal{T}}$ and the equality $\overline{\xi}_2 = \overline{\xi}_1$ does not hold in a free algebra. Therefore there exists an orderable LAA_I $\mathbf{B}$ such that $\mathbf{\Lambda}_I^{\mathcal{T}}$ can be embedded into $\mathbf{B}$, that is $\mathbf{\Lambda}_I^{\mathcal{T}}$ is not absolutely unorderable.

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