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Graph lambda theories†

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A longstanding open problem in lambda calculus is whether there exist continuous models of the untyped lambda calculus whose theory is exactly the $\lambda\beta$ or the least sensible λ-theory $\mathcal{H}$ (generated by equating all the unsolvable terms). A related question is whether, given a class of lambda models, there are a minimal λ-theory and a minimal sensible λ-theory represented by it. In this paper, we give a positive answer to this question for the class of graph models à la Plotkin-Scott-Engeler. In particular, we build two graph models whose theories are respectively the set of equations satisfied in any graph model and in any sensible graph model. We conjecture that the least sensible graph theory, where "graph theory" means "λ-theory of a graph model", is equal to $\mathcal{H}$, while in one of the main results of the paper we show the non-existence of a graph model whose equational theory is exactly the $\lambda\beta$ theory.
Another related question is whether, given a class of lambda models, there is a maximal sensible λ-theory represented by it. In the main result of the paper we characterize the greatest sensible graph theory as the λ-theory $\mathcal{B}$ generated by equating λ-terms with the same Böhm tree. This result is a consequence of the main technical theorem of the paper: all the equations between solvable λ-terms, which have different Böhm trees, fail in every sensible graph model. A further result of the paper is the existence of a continuum of different sensible graph theories strictly included in $\mathcal{B}$.

Keywords. *Lambda calculus, lambda theories, graph models, minimum graph theory, maximum graph theory, $\lambda\beta$-theory.*

Contents

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1. Introduction

The untyped lambda calculus was introduced around 1930 (Church 1933; Church 1941) as part of an investigation in the formal foundations of mathematics and logic. Although lambda calculus is a very basic language, it is sufficient to express all computable functions. The process of application and evaluation reflects the computational behavior of many modern functional programming languages, which explains the interest in the lambda calculus among computer scientists.

Lambda theories are equational extensions of the untyped lambda calculus closed under derivation. They arise by syntactical or semantic considerations. Indeed, a λ-theory may correspond to a possible operational (observational) semantics of the lambda calculus, as well as it may be induced by a model of lambda calculus through the kernel congruence relation of the interpretation function. Although researchers have mainly focused their interest on a limited number of them, the class of λ-theories constitutes a very rich and complex structure, see e.g. (Barendregt 1984; Berline 2000). Syntactical techniques are usually difficult to use in the study of λ-theories. Therefore, semantic methods have been extensively investigated.

Topology is at the center of the known approaches to giving models of the untyped lambda calculus. The first model, found by Scott in 1969 in the category of complete lattices and Scott continuous functions, was successfully used to show that all unsolvable λ-terms can be consistently equated. After Scott, a large number of mathematical models for lambda calculus, arising from syntax-free constructions, have been introduced in various categories of domains and were classified into semantics according to the nature of their representable functions, see e.g. (Abramsky 1991; Barendregt 1984; Berline 2000; Plotkin 1993). Scott's continuous semantics

(Scott 1972) is given in the category whose objects are complete partial orders and morphisms are Scott continuous functions. The stable semantics (Berry 1978) and the strongly stable semantics (Bucciarelli and Ehrhard 1991) are a strengthening of the continuous semantics, introduced to capture the notion of "sequential" Scott continuous function. All these semantics are structurally and equationally rich in the sense that it is possible to build up $2^{\aleph_0}$ models in each of them inducing pairwise distinct λ-theories (Kerth 1998; Kerth 2001). Nevertheless, the above denotational semantics are *equationally incomplete*: they do not match all possible operational semantics of lambda calculus. The problem of the equational incompleteness was positively solved by Honsell-Ronchi della Rocca for the continuous semantics (Honsell and Ronchi della Rocca 1992), and by Bastonero-Gouy for the stable semantics (Gouy 1995; Bastonero and Gouy 1999). Salibra has recently shown in a uniform way that all semantics, which involve monotonicity with respect to some partial order and have a bottom element, are equationally incoimplete (Salibra 2001; Salibra 2003). From this it follows the incompleteness of the strongly stable semantics, which had been conjectured by Bastonero-Gouy and by Berline (Bastonero and Gouy 1999; Berline 2000).

If a semantics is incomplete, then there exists a λ-theory T that is not induced by any model in the semantics. In such a case we say that the semantics *omits* the λ-theory T. More generally, a semantics *omits* (*forces*, respectively) an equation if the equation fails (holds) in every model of the semantics. The set of equations forced by a semantics $\mathcal{C}$ constitutes a λ-theory. It is the minimal (with respect to the inclusion order) λ-theory of $\mathcal{C}$ if it is induced by a model of $\mathcal{C}$.

The following natural questions arise (Berline 2000): given a class $\mathcal{C}$ of models of lambda calculus,

1 Is there a minimal λ-theory represented by $\mathcal{C}$?
2 Is there a minimal sensible (i.e., equating all unsolvable λ-terms) λ-theory represented by $\mathcal{C}$?

In (Di Gianantonio *et al.* 1995) it was shown that the above question (1) admits a positive answer for Scott's continuous semantics, at least if we restrict to extensional models. Another result, in the same spirit, is the construction of a model whose theory is $\lambda\beta\eta$, *a fortiori* minimal, in the ω_1-semantics. However, the proofs of (Di Gianantonio *et al.* 1995) use logical relations, and since logical relations do not allow to distinguish terms with the same applicative behavior, the proofs do not carry out to non-extensional models.

In this paper we show that both question (1) and question (2) admit a positive answer for the *graph semantics*, that is, the semantics of lambda calculus given in terms of *graph models*. These models, isolated in the seventies by Scott and Engeler (Barendregt 1984) within the continuous semantics, have been proved useful for giving proofs of consistency of extensions of lambda calculus and for studying operational features of lambda calculus. For example, the simplest graph model, namely Engeler's model, has been used in (Berline 2000) to give concise proofs of the head-normalization theorem and of the left-normalization theorem of lambda calculus, while a semantical proof of the "easiness" of $(\lambda x.xx)(\lambda x.xx)$ was obtained in (Baeten and Boerboom 1979). It is well known that the graph semantics is incomplete, since it trivially omits the axiom of extensionality. The main technical device used in the proof of the existence of the least (sensible) graph theory is the notion of *weak product* of graph models. Roughly speaking, the weak product of a family of graph models is a new graph model which is the "canonical completion" of the disjoint union of the models in the family. We show that the theory of a weak product is always semisensible (i.e., it does not equate solvable and unsolvable terms) and it is included in the intersection of the theories of its factors (the inclusion is in general strict). The *least graph theory* (where "graph theory" means "λ-theory of a graph model") is the theory of the weak product of the family $(D_e : e \in I)$, where I is the set of equations between λ-terms which fail to hold in some graph model, and D_e is a fixed graph model not satisfying the equation e.

Two further questions naturally arise: what equations between λ-terms belong to the minimal graph theory? And to the minimal sensible one? The answer to the second difficult question is still unknown; we conjecture that the λ-theory $\mathcal{H}$, generated by equating all unsolvable λ-terms, is the least sensible graph theory. The first question is related to a longstanding open problem in lambda calculus, asking whether there exists a non-syntactic model whose equational theory is equal to the least λ-theory $\lambda\beta$. In this paper we show that this model cannot be found within graph semantics (this result negatively answers Question 1 in (Berline 2000, Section 6.2) for the class of graph models). From this result it follows that the minimal graph theory is not equal to $\lambda\beta$, so that graph semantics forces equations between non-β-equivalent λ-terms. In this paper we provide an example of an equation of this kind.

The set of all sensible λ-theories constitutes a bounded lattice. The least sensible λ-theory is the λ-theory $\mathcal{H}$ (generated by equating all the unsolvable terms), while the greatest sensible λ-theory is the λ-theory $\mathcal{H}^*$ (generated by equating terms with the same Böhm tree up to possibly infinite η-equivalence). In (Kerth 1998) it was shown that there exists a continuum of different sensible graph theories. Then it make sense to ask whether there exists a maximal λ-theory represented by graph semantics. In one of the main results of the paper we show that the λ-theory $\mathcal{B}$ (generated by equating λ-terms with the same Böhm tree) is the greatest sensible graph theory. This result is a consequence of the main technical theorem of the paper: the graph semantics omits all equations $M = N$ between λ-terms satisfying the following conditions:

$$\mathcal{H}^* \vdash M = N \text{ and } \mathcal{B} \nvdash M = N. \tag{1}$$

In other words, the graph semantics omits all equations $M = N$ between λ-terms which do not have the same Böhm tree, but have the same Böhm tree up to (possibly infinite) η-equivalence.

The following are other consequences of the main result of the paper.

(i) There exists a continuum of different sensible graph theories strictly included in $\mathcal{B}$ (this result positively answers Question 2 in (Berline 2000, Section 6.3));

(ii) For every closed term P, the λ-theory generated by $\Omega = P$, where Ω is the paradigmatic unsolvable term $(\lambda x.xx)(\lambda x.xx)$, contains no equation satisfying condition (1).

The paper is organized as follows. In Section 2 we review the basic definitions of lambda calculus and graph models. In particular, we recall the formal definition of the canonical completion of a partial model. The notion of a weak product of graph models is introduced and studied in Section 3. The proof of the existence of a minimal (sensible) graph theory is presented in Section 4, while in Section 5 it is shown that the least graph theory is not equal to $\lambda\beta$. Sections 6-8 are devoted to the characterization of the maximal sensible graph theory. Conclusions and future work are presented in Section 9.

2. Preliminaries

To keep this article self-contained, we summarize some definitions and results concerning lambda calculus and graph models. With regard to the lambda calculus we follow the notation and terminology of (Barendregt 1984).

2.1. *Lambda calculus*

The set Λ of λ-terms over an infinite set of variables is constructed as usual: every variable is a λ-term; if M and N are λ-terms, then so are (MN) and $\lambda x.M$ for each variable x. Λ^o denotes the set of closed λ-terms.

The symbol $\equiv$ denotes syntactic equality. The following are some well-known λ-terms:

$$\Omega \equiv (\lambda x.xx)(\lambda x.xx); \quad \Omega_3 \equiv (\lambda x.xxx)(\lambda x.xxx);$$

$$\mathbf{i} \equiv \lambda x.x; \quad \mathbf{k} \equiv \lambda xy.x; \quad \mathbf{1} \equiv \lambda xy.xy.$$

A *compatible λ-relation* T is any set of equations between λ-terms that is closed under the following two rules:

(i) If $M = N \in T$ and $P = Q \in T$, then $MP = NQ \in T$;
(ii) If $M = N \in T$ then $\lambda x.M = \lambda x.N \in T$ for every variable x.

We will write either $T \vdash M = N$ or $M =_T N$ for $M = N \in T$.

A *λ-theory* T is any compatible λ-relation which is an equivalence relation and includes (α)- and (β)-conversion. The set of all λ-theories is naturally equipped with a lattice structure, with meet defined as set theoretical intersection. The join of two λ-theories T and S is the least equivalence relation including $T \cup S$. $\lambda\beta$ denotes the minimal λ-theory, while $\lambda\beta\eta$ denotes the minimal extensional λ-theory (axiomatized by $\mathbf{i} = \mathbf{1}$).

Solvable λ-terms can be characterized as follows: a λ-term M is solvable if, and only if, it has a *head normal form*, that is, $M =_{\lambda\beta} \lambda x_1 \dots x_n.yM_1 \dots M_k$ for some $n, k \geq 0$ and λ-terms $M_1, \dots, M_k$. $M \in \Lambda$ is *unsolvable* if it is not solvable.

The λ-theory $\mathcal{H}$, generated by equating all unsolvable λ-terms, is consistent by (Barendregt 1984, Thm. 16.1.3) and admits a unique maximal consistent extension $\mathcal{H}^*$ (Barendregt 1984, Thm. 16.2.6). A λ-theory T is called *sensible* if it is consistent and $\mathcal{H} \subseteq T$. The set of all sensible λ-theories is naturally equipped with a structure of bounded lattice. $\mathcal{H}$ is the least sensible λ-theory, while $\mathcal{H}^*$ is the greatest one. $\mathcal{H}^*$ is an extensional λ-theory. A λ-theory is *semisensible* if no solvable term is equivalent to an unsolvable term. It is easy to prove that sensible theories are semisensible. It is also possible to characterize semisensible λ-theories as follows: a λ-theory T is semisensible if, and only if, $T \subseteq \mathcal{H}^*$ (Barendregt 1984, Section 16.2).

2.2. *Böhm trees*

A λ-term M is called *a projection term* if $M \equiv \lambda x_1 \dots x_n.y$ ($n \geq 0$). A *Böhm-like tree* is a finite branching labelled tree, whose inner nodes are labelled by projection terms and leaves either by projection terms or by $\perp$. Equality of Böhm-like trees is defined up to α-equivalence of corresponding labels.

The Böhm tree $BT(M)$ of a λ-term M is a finite or infinite Böhm-like tree. If M is unsolvable, then $BT(M) = \bot$, that is, $BT(M)$ is a tree with a unique node labelled by $\bot$. If M is solvable and $\lambda x_1 \dots x_n.yM_1 \dots M_k$ is the principal head normal form of M (Barendregt 1984, Def. 8.3.20) then we have

$$BT(M) = \lambda x_1 \dots x_n.y$$

$$BT(M_1) \dots\dots\dots BT(M_k)$$

The λ-theory $\mathcal{B}$, generated by equating λ-terms with the same Böhm tree, is sensible and non-extensional. $\mathcal{B}$ is distinct from $\mathcal{H}$ and $\mathcal{H}^*$, so that $\mathcal{H} \subset \mathcal{B} \subset \mathcal{H}^*$. Notice that (Barendregt 1984, Exercise 16.5.5) gives a non-extensional λ-theory T satisfying $\mathcal{B} \subset T \subset \mathcal{H}^*$ (see the remark after Thm. 46 in this paper).

In the remaining part of this section we characterize the λ-theory $\mathcal{H}^*$ in terms of Böhm trees.

For all λ-terms M and N, we write $M \leq_\eta N$ if $BT(N)$ is a (possibly infinite) η-*expansion* of $BT(M)$ (Barendregt 1984, Def. 10.2.10). For example, let $J \equiv \Theta(\lambda jxy.x(jy))$, where Θ is the Turing's fixpoint combinator. Then, $x \leq_\eta Jx$ (Barendregt 1984, Example 10.2.9), since

$$Jx =_{\lambda\beta} \lambda z_0.x(Jz_0) =_{\lambda\beta} \lambda z_0.x(\lambda z_1.z_0(Jz_1))$$

$$=_{\lambda\beta} \lambda z_0.x(\lambda z_1.z_0(\lambda z_2.z_1(Jz_2))) =_{\lambda\beta} \dots$$

The following is the Böhm tree of Jx:

$$BT(Jx) = \lambda z_0.x$$

$$\lambda z_1.z_0$$

$$\lambda z_2.z_1$$

$$\dots\dots$$

We write $N \approx_\eta M$ if there exists a Böhm-like tree A such that $BT(M) \leq_\eta A$ and $BT(N) \leq_\eta A$. We refer the reader to (Barendregt 1984, Def. 10.2.25) and to the proof of the point (i $\Rightarrow$ ii) in (Barendregt 1984, Thm. 10.2.31). It is well known that

$$M =_{\mathcal{H}^*} N \;\Leftrightarrow\; M \approx_\eta N \quad \text{(Barendregt 1984, Thm. 19.2.9).}$$

2.3. *Graph models*

The class of graph models belongs to Scott's continuous semantics. Historically, the first graph model was Scott's P_ω, which is also known in the literature as "the graph model". "Graph" referred to the fact that the continuous functions were encoded in the model via (a sufficient fragment of) their graph.

As a matter of notation, for every set D, D^* is the set of all finite subsets of D, while $\mathcal{P}(D)$ is the powerset of D. If C is a complete partial ordering (cpo, for short), then $[C \to C]$ denotes the cpo of all Scott continuous functions from C into C.

Definition 1. A *graph model* D is a pair $(|D|, c_D)$, where $|D|$ is an infinite set, called the *web* of D, and $c_D : |D|^* \times |D| \to |D|$ is an injective total function.

We use the same notation D for the graph model and its web. Thus, for example, $\alpha \in D$ means $\alpha \in |D|$. In what follows, $a, b, a_1, \ldots$ denote finite subsets of $|D|$.

As a matter of notation, we write $a \to_D \alpha$, or also simply $a \to \alpha$, for $c_D(a, \alpha)$. When parenthesis are omitted, then association to the right is assumed. For example, $a \to b \to \alpha$ stands for $c_D(a, c_D(b, \alpha))$. If $\overline{a} = a_1 \ldots a_n$ is a sequence of finite subsets of D, then we write $\overline{a} \to \alpha$ (or $\overline{a}_n \to \alpha$ when we put in evidence the length of the sequence) for $a_1 \to a_2 \to \cdots \to a_n \to \alpha$.

The function c_D is useful to encode a fragment of the graph of a Scott continuous function $f : \mathcal{P}(D) \to \mathcal{P}(D)$ as a subset $G(f)$ of D:

$$G(f) = \{a \to_D \alpha \;:\; \alpha \in f(a) \text{ and } a \in D^*\}. \tag{2}$$

Any graph model D is used to define a model of lambda calculus through the reflexive cpo $(\mathcal{P}(D), \subseteq)$ determined by two Scott continuous mappings $G : [\mathcal{P}(D) \to \mathcal{P}(D)] \to \mathcal{P}(D)$ and $F : \mathcal{P}(D) \to [\mathcal{P}(D) \to \mathcal{P}(D)]$. The function G is defined in (2), while F is defined as follows:

$$F(X)(Y) = \{\alpha \in D : (\exists a \subseteq Y)\; a \to_D \alpha \in X\}.$$

We sometimes write XY for $F(X)(Y)$ and $XY_1 \ldots Y_n$ for $(\ldots((XY_1)Y_2)\ldots)Y_n)$. For more details we refer the reader to (Berline 2000) and (Barendregt 1984, Chapter 5).

Let Env_D be the set of D-environments ρ mapping the variables of lambda calculus into $\mathcal{P}(D)$. If $Y \subseteq D$, then the environment $\rho[x := Y]$ is defined by: $\rho[x := Y](x) = Y$; $\rho[x := Y](z) = \rho(z)$ for $z \neq x$. The interpretation M^D of a λ-term M in an environment ρ is defined as follows.

- $x^D_\rho = \rho(x)$
- $(MN)^D_\rho = \{\alpha \in D : (\exists a \subseteq N^D_\rho)\; a \to \alpha \in M^D_\rho\}$
- $(\lambda x.M)^D_\rho = \{a \to \alpha : \alpha \in M^D_{\rho[x:=a]}\}$

If $\overline{x} \equiv x_1 \ldots x_n$ is a sequence of variables and $\overline{a} = a_1 \ldots a_n$ is a sequence of finite subsets of D, then we have

$$(\lambda\overline{x}.M)^D_\rho = \{\overline{a} \to \alpha : \alpha \in M^D_{\rho[x_1:=a_1]\ldots[x_n:=a_n]}\}.$$

We turn now to the interpretation of Ω in graph models. The following remark gives a necessary condition and a sufficient condition for α to be in the interpretation of Ω in a graph model.

Lemma 2. (Baeten and Boerboom 1979) Let D be a graph model and $\alpha \in D$. Then we have:
(i) If $\alpha \in \Omega^D$, then there exists a such that $a \to \alpha \in a$.
(ii) If there exists $\beta \in D$ such that $\{\beta\} \to \alpha = \beta$, then $\alpha \in \Omega^D$.

Given a graph model D, we have that $M^D = N^D$ if, and only if, $M^D_\rho = N^D_\rho$ for all environments ρ. The λ-theory $Th(D)$ induced by D is defined as

$$Th(D) = \{M = N : M^D = N^D\}.$$

A λ-theory induced by a graph model will be called a *graph theory*. The graph model D is called *sensible* if $Th(D)$ is a sensible λ-theory. In (Kerth 1998) it was shown that there exists a continuum of different (sensible) graph theories. It is well known that the graph theory $Th(D)$ is never extensional because $(\lambda x.x)^D \neq (\lambda xy.xy)^D$.

In (Di Gianantonio and Honsell 1993) it was shown that graph models are related to filter models (Coppo and Dezani 1980; Barendregt *et al.* 1984), since the class of graph theories is included within the class of λ-theories induced by non-extensional filter models. This inclusion is strict (Alessi *et al.* 2001), namely there exists an equation between λ-terms, which is omitted in graph semantics, whilst it is satisfied in some non-extensional filter model.

A graph theory T will be called

1 *the minimal graph theory* if $T \subseteq Th(D)$ for all graph models D;
2 *the minimal sensible graph theory* if T is sensible and $T \subseteq Th(D)$ for all sensible graph models D;
3 *the maximal sensible graph theory* if T is sensible and $Th(D) \subseteq T$ for all sensible graph models D.

Definition 3. A class $\mathcal{C}$ of graph models *omits* a λ-theory T if $T \neq Th(D)$ for all graph models $D \in \mathcal{C}$. More generally, $\mathcal{C}$ *omits* (*forces*, respectively) an equation if it fails (holds) in all models of $\mathcal{C}$.

If a class $\mathcal{C}$ of graph models omits an equation $M = N$, then it omits all λ-theories including $M = N$. It is easy to verify that the set of equations 'forced' by $\mathcal{C}$ constitutes a λ-theory. It is the minimal λ-theory of $\mathcal{C}$ if it is induced by a model of $\mathcal{C}$.

The completion method for building graph models from "partial pairs" was initiated in (Longo 1983) and developed on a wide scale in (Kerth 1998; Kerth 2001). This method is useful to build models satisfying prescribed constraints, such as domain equations and inequations, and it is particularly convenient for dealing with the equational theories of graph models.

Definition 4. A *partial pair* A is given by a set $|A|$ and by a partial, injective function $c_A : |A|^* \times |A| \to |A|$.

Notice that the underlying set of a partial pair may be finite. As for graph models, we use the same notation A for the partial pair and its underlying set.

A partial pair is a graph model if and only if c_A is total. We always suppose that no element of A is a pair. This is not restrictive because partial pairs can be considered up to isomorphism.

Lambda terms can be interpreted by induction in a partial pair A in the obvious way. For example, we have that $(MN)^A_\rho = \{ \alpha \in A : (\exists a \subseteq N^A_\rho)\ [(a,\alpha) \in dom(c_A) \wedge c_A(a,\alpha) \in M^A_\rho]\}$ and $(\lambda x.M)^A_\rho = \{ c_A(a,\gamma) \in A : (a,\gamma) \in dom(c_A) \wedge \gamma \in M^A_{\rho[x:=a]} \}$.

Definition 5. Let A be a partial pair. The *canonical completion*[†] of A is the graph model E defined as follows:

— $E = \bigcup_{n\in\omega} E_n$, where $E_0 = A$, $E_{n+1} = E_n \cup ((E^*_n \times E_n) - dom(c_A))$.
— Given $a \in E^*$, $\alpha \in E$,

$$c_E(a,\alpha) = \begin{cases} c_A(a,\alpha) & \text{if } c_A(a,\alpha) \text{ is defined,} \\ (a,\alpha) & \text{otherwise.} \end{cases}$$

[†] Canonical completions are termed *free completions* in (Berline 2006), and *Engeler completions* in (Bucciarelli and Salibra 2003; Bucciarelli and Salibra 2004).

It is easy to check that the canonical completion of a given partial pair A is actually a graph model. The canonical completion of a total pair A is equal to A.

A notion of *rank* can be naturally defined on the canonical completion E of a partial pair A. The elements of A are the elements of rank 0, while an element $\alpha \in E - A$ has rank n if $\alpha \in E_n$ and $\alpha \notin E_{n-1}$.

Classic graph models, such as Scott's P_ω (Barendregt 1984), Park's $\mathcal{P}$ (Berline 2000) and Engeler's $\mathcal{E}_B$ (Berline 2000), where B is an arbitrary non-empty set, can be viewed as the canonical completions of suitable partial pairs. In fact, P_ω, $\mathcal{P}$ and $\mathcal{E}_B$ are respectively isomorphic to the canonical completions of $A = (\{0\}, c_A)$ (with $c_A(\emptyset, 0) = 0$), $D = (\{p\}, c_D)$ (with $c_D(\{p\}, p) = p$) and $E = (B, c_E)$ (with c_E the empty function).

Let $\overline{x} = x_1 \dots x_n$ be a sequence of variables and ρ be a D-environment such that $\rho(x_i)$ is a finite set. As a matter of notation, we write $\rho(\overline{x}) \to \alpha$ for $\rho(x_1) \to \rho(x_2) \to \cdots \to \rho(x_n) \to \alpha$.

3. Weak product

In this section we introduce the notion of *weak product* of graph models, which is the main technical device used in the proof of the existence of the least (sensible) graph theory. The idea of a weak product is the following: given two graph models D_1 and D_2, construct the partial pair whose web is the disjoint union of the webs of D_1 and D_2, and whose coding function is the disjoint union of their coding functions. The canonical completion of this partial pair is the weak product of D_1 and D_2.

As a matter of notations, given two sets A_1 and A_2, we write $A_1 \uplus A_2$ their disjoint union, $in_i : A_i \to A_1 \uplus A_2$ the canonical injections and $pr_i : 2^{A_1 \uplus A_2} \to 2^{A_i}$ the canonical projections.

Definition 6. Let D_1 and D_2 be graph models. We define the partial pair $D_1 \uplus D_2$ by

$$|D_1 \uplus D_2| = |D_1| \uplus |D_2|$$

$$c_{D_1 \uplus D_2}(b, \beta) = \begin{cases} in_i(c_{D_i}(a, \alpha)) & \text{if } b = \{in_i(\alpha') \mid \alpha' \in a\}, \beta = in_i(\alpha), \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Definition 7. Let D_1 and D_2 be graph models. The graph model $D_1 \diamond D_2$, called the *weak product* of D_1 and D_2, is the canonical completion of the partial pair $D_1 \uplus D_2$ defined above.

These definitions extend to countable products by considering countable disjoint unions of webs. Countable weak products are denoted by $\diamond_{i \in \omega} D_i$.

For the sake of visibility of statements and proofs, we will suppose that, when forming weak products, the factors' webs are disjoint, and that the canonical injections are replaced by set inclusions. So, for instance, if M is a λ-term and D_i is a factor of a weak product E, it makes sense to write $M^{D_i} \subseteq M^E$.

The rest of this section is devoted to the proof of the main properties of this construction:

(i) The theory of a weak product is included in the intersection of the theories of its factors (Section 3.1).

(ii) The theory of a weak product is semisensible (Section 3.2).

(iii) The inclusion in (i) is strict in general (Section 3.3).

3.1. *The theory of a weak product and of its factors*

In this section we show that the theory $Th(E)$ of a weak product E is included in the theory $Th(D_i)$ of each of its factor D_i. The idea is to prove that, for all closed λ-terms M

$$M^{D_i} = M^E \cap D_i. \tag{3}$$

This takes a structural induction on M, and hence the analysis of open terms too. Roughly, we are going to show that equation (3) holds for open terms as well, provided that the environments satisfy a suitable closure property introduced below.

In the rest of this section D_i is a factor of a (finite or countable) weak product E.

Definition 8. We call i-*flattening* the function $f_i : E \to E$ defined by induction on the rank of $\alpha \in E$ as follows:
if $rank(\alpha) = 0$ then $f_i(\alpha) = \alpha$,
if $rank(\alpha) > 0$ and $\alpha = (a, \beta)$ then :

$$f_i(\alpha) = \begin{cases} c_{D_i}(f_i(a) \cap D_i, f_i(\beta)) & \text{if } f_i(\beta) \in D_i, \\ \alpha & \text{otherwise,} \end{cases}$$

where $f_i(a) = \{f_i(\gamma) : \gamma \in a\}$.

The following easy facts will be useful:

Fact 9.

(a) For all $\alpha \in E$, $f_i(\alpha) \neq \alpha$ if, and only if, $f_i(\alpha) \in D_i$ and $\alpha \notin D_i$.
(b) If $a \cup \{\beta\} \subseteq E$ and $f_i(\beta) \in D_i$, then $f_i(c_E(a, \beta)) \in D_i$.

We notice that Fact 9(b) holds, a fortiori, if $\beta \in D_i$.

Definition 10. For $a \subseteq E$ let $\hat{a} = a \cup f_i(a)$; we say that a is i-*closed* if $\hat{a} = a$.

In other words, a is i-closed if $f_i(a) \subseteq a$.

Lemma 11. For all $a \subseteq E$, $\hat{a} \cap D_i = f_i(a) \cap D_i$.

Proof. By definition, $\hat{a} = a \cup f_i(a)$, hence

$$\hat{a} \cap D_i = (a \cap D_i) \cup (f_i(a) \cap D_i).$$

Since f_i restricted to D_i is the identity function, we have $a \cap D_i \subseteq f_i(a) \cap D_i$, and we are done. □

Definition 12. Let $\rho : Var \to \mathcal{P}(E)$ be an E-environment. We define the i-*restriction* ρ_i of ρ by $\rho_i(x) = \rho(x) \cap D_i$, while we say that ρ is i-*closed* if for every variable x, $\rho(x)$ is i-closed.

The following is the key technical lemma of the section.

Lemma 13. Let M be a λ-term and ρ be an i-closed E-environment; then

(a) M^E_ρ is i-closed.
(b) $M^E_\rho \cap D_i \subseteq M^E_{\rho_i}$.

Proof. We prove (a) and (b) simultaneously by induction on the structure of M. If $M \equiv x$, both statements are trivially true.

(a) Let $M \equiv \lambda x.N$. Given $\alpha \in M^E_\rho$, we have to show that $f_i(\alpha) \in M^E_\rho$. First we remark that, if $rank(\alpha) = 0$ or if $\alpha = (a, \beta)$ and $f_i(\beta) \notin D_i$, then by Fact 9(a) $f_i(\alpha) = \alpha$ and we are done. Then, let $\alpha = (a, \beta)$ and $f_i(\beta) \in D_i$; we have:

	$\alpha \in M^E_\rho$	
$\Rightarrow$	$\beta \in N^E_{\rho[x:=a]}$	by definition of $(\_)^E$
$\Rightarrow$	$\beta \in N^E_{\rho[x:=\hat{a}]}$	by monotonicity of $(\_)^E$ w.r.t. environments
$\Rightarrow$	$f_i(\beta) \in N^E_{\rho[x:=\hat{a}]}$	by ind. hyp. (a), remark that $\rho[x := \hat{a}]$ is i-closed
$\Rightarrow$	$f_i(\beta) \in N^E_{(\rho[x:=\hat{a}])_i}$	by ind. hyp. (b), since $f_i(\beta) \in D_i$
$\Rightarrow$	$f_i(\beta) \in N^E_{\rho_i[x:=f_i(a)\cap D_i]}$	by Lemma 11
$\Rightarrow$	$c_E(f_i(a) \cap D_i, f_i(\beta)) \in M^E_{\rho_i}$	by definition of M^E
$\Rightarrow$	$c_{D_i}(f_i(a) \cap D_i, f_i(\beta)) \in M^E_{\rho_i}$	by definition of c_E
$\Rightarrow$	$f_i(\alpha) \in M^E_{\rho_i}$	by definition of f_i
$\Rightarrow$	$f_i(\alpha) \in M^E_\rho$	by monotonicity of $(\_)^E$

(b) Let $M \equiv \lambda x.N$. Then we have:

	$\alpha \in M^E_\rho \cap D_i$	
$\Rightarrow$	$(\exists a \cup \{\beta\} \subseteq D_i)\ \alpha = c_{D_i}(a, \beta)$ and $\beta \in N^E_{\rho[x:=a]}$	by def. of M^E and since $\alpha \in D_i$
$\Rightarrow$	$\beta \in N^E_{(\rho[x:=a])_i}$	by ind. hyp. (b), remark that $\hat{a} = a$
$\Rightarrow$	$\beta \in N^E_{\rho_i[x:=a]}$	since $a \subseteq D_i$
$\Rightarrow$	$\alpha \in M^E_{\rho_i}$	by definition of $(\_)^E$

(a) Let $M \equiv PQ$ and $\beta \in (PQ)^E_\rho$. If $f_i(\beta) = \beta$ we are done; otherwise by Fact 9(a) we have that $f_i(\beta) \in D_i$ and $\beta \notin D_i$. Moreover, by definition of interpretation in a graph model $\exists a \subseteq E$ such that $c_E(a, \beta) \in P^E_\rho$ and $a \subseteq Q^E_\rho$. Applying Fact 9(b) and the induction hypothesis we get

$$f_i(c_E(a, \beta)) = c_{D_i}(f_i(a) \cap D_i, f_i(\beta)) = c_E(f_i(a) \cap D_i, f_i(\beta)) \in P^E_\rho.$$

Applying the induction hypothesis to Q we get $f_i(a) \subseteq Q^E_\rho$. Hence $f_i(\beta) \in M^E_\rho$.

(b) Let $M \equiv PQ$. If $\beta \in (PQ)^E_\rho \cap D_i$, then $\exists a \subseteq E$ such that $c_E(a, \beta) \in P^E_\rho$ and $a \subseteq Q^E_\rho$. Since ρ is i-closed, then by (a) (induction hypothesis) we get $f_i(c_E(a, \beta)) \in P^E_\rho$ and $f_i(a) \subseteq Q^E_\rho$. Since $\beta \in D_i$, then by Fact 9(b) we get $f_i(c_E(a, \beta)) = c_{D_i}(f_i(a) \cap D_i, \beta) \in P^E_\rho$. Now, by (b) (induction hypothesis) we obtain $c_{D_i}(f_i(a) \cap D_i, \beta) \in P^E_{\rho_i}$ and $f_i(a) \cap D_i \subseteq Q^E_{\rho_i}$, and we conclude $\beta \in (PQ)^E_{\rho_i}$. □

Lemma 14. Let M be a λ-term and $\rho : Var \to \mathcal{P}(D_i)$ be a D_i-environment; then we have $M^E_\rho \cap D_i = M^{D_i}_\rho$.

Proof. We prove by induction on the structure of M that $M^E_\rho \cap D_i \subseteq M^{D_i}_\rho$. The converse is ensured by $M^{D_i}_\rho \subseteq M^E_\rho$ and $M^{D_i}_\rho \subseteq D_i$, both trivially true.

If $M \equiv x$, the statement trivially holds.

Let $M \equiv \lambda x N$.

$$
\begin{array}{lll}
& \alpha \in M^E_\rho \cap D_i & \\
\Leftrightarrow & (\exists a \cup \{\beta\} \subseteq D_i)\ \alpha = c_{D_i}(a,\beta) \text{ and } \beta \in N^E_{\rho[x:=a]} & \text{by def. of } M^E \text{ and since } \alpha \in D_i \\
\Leftrightarrow & (\exists a \cup \{\beta\} \subseteq D_i)\ \alpha = c_{D_i}(a,\beta) \text{ and } \beta \in N^{D_i}_{\rho[x:=a])} & \text{by induction hypothesis} \\
\Leftrightarrow & \alpha \in M^{D_i}_\rho & \text{by definition of } M^{D_i}
\end{array}
$$

Let $M \equiv PQ$. If $\alpha \in (PQ)^E_\rho \cap D_i$, then $\exists a \subseteq E$ such that $c_E(a,\alpha) \in P^E_\rho$ and $a \subseteq Q^E_\rho$. Since ρ is i-closed and $\alpha \in D_i$, we can use Fact 9(b) and Lemma 13(a) to obtain

$$f_i(c_E(a,\alpha)) = c_{D_i}(f_i(a) \cap D_i, \alpha) \in P^E_\rho.$$

Hence we can use the induction hypothesis to get $c_{D_i}(f_i(a) \cap D_i, \alpha) \in P^{D_i}_\rho$. Moreover, $f_i(a) \cap D_i \subseteq Q^{D_i}_\rho$ by using again Lemma 13(a) and the induction hypothesis on Q. Hence $\alpha \in (PQ)^{D_i}_\rho$. The opposite direction is trivial. □

Theorem 15. $Th(E) \subseteq Th(D_i)$.

Proof. Let $M^E = N^E$. By the previous proposition we have

$$M^{D_i} = M^E \cap D_i = N^E \cap D_i = N^{D_i}.$$

□

The existence of the least (resp. the least sensible) graph theory will be a consequence of Thm. 15 (see Section 4).

The following easy properties of weak products will be used in Section 4.2:

Proposition 16. Let $E = \Diamond_{i \in I} D_i$. For all $\alpha \in E$ there exists a unique $j \in I$ such that $f_j(\alpha) \in D_j$.

Proof. By induction on the rank of α. □

Proposition 17. Let $E = \Diamond_{i \in I} D_i$ and M be a closed λ-term. For all $\alpha \in M^E$ there exists a unique $j \in I$ such that $f_j(\alpha) \in M^{D_j}$.

Proof. By Prop. 16 we know that there is a unique j such that $f_j(\alpha) \in D_j$, while by Lemma 13(a) we have that $f_j(\alpha) \in M^E$. The conclusion follows from Lemma 14. □

3.2. *The theory of a weak product is semisensible*

In this section we show that *stratified* graph models have semisensible theories. A graph model is stratified if it is the completion of a proper partial pair, i.e. one whose coding function is not total. Since weak products are particular stratified graph models, then the theory of a weak product is also semisensible.

Semisensibility of the theory of a stratified graph model is proved by case analysis, on the order of unsolvable terms (see Def. 24 for the definition of order of an unsolvable). The fact that unsolvables of order 0 cannot be equated to a solvable in a stratified graph model is shown in Lemma 25 by using the approximation theorem below.

Concerning unsolvable of finite order, we introduce the notion of *height* of an element of the model, and then rely on the previous case (Lemma 27).

For the unsolvable of infinite order, we rely on a general property of graph models, their non-extensionality, to show that such terms cannot be equated to solvables in any graph model (Lemma 29).

3.2.1. *An Approximation Theorem.* Approximation theorems are an important tool in the analysis of the λ-theories induced by models of lambda calculus. In this section we provide an approximation theorem for the class of stratified graph models: we show that the interpretation of a λ-term in a stratified graph model is the union of the interpretations of its direct approximants. This approximation theorem will be applied in Section 3.2.2 to show that the interpretation of an unsolvable of order 0 in a stratified graph model is a set of elements of rank 0. We do not claim any particular originality for the approximation theorem we prove in this section, since it is a very similar to that in (Honsell and Ronchi della Rocca 1992) and it is a particular case of that in (Bastonero and Gouy 1999). However, for the sake of completeness, we provide a proof.

Let D be a stratified graph model, which is the completion of the partial pair A. Recall that $D_0 = A$ and $D_{n+1} = D_n \cup (D_n^* \times D_n) - \mathrm{dom}\, c_A$. For every $X \subseteq D$, we denote by $X_{\underline{n}} = X \cap D_n$.

The underlined natural numbers $\underline{n}$ are called *labels*. Lambda terms with occurrences of labels are called *labelled-terms*. For example, $(\lambda x.x_{\underline{n}})_{\underline{m}} y$ and $(y_{\underline{n}})_{\underline{m}}$ are labelled-terms. Note that the set of ordinary λ-terms is a proper subset of the set of labelled terms (those without any label). If N is a labelled term, we denote by $|N|$ the λ-term obtained by erasing all labels of N. For example, we have that $|(\lambda x.x_{\underline{n}})_{\underline{m}} y| = (\lambda x.x)y$.

Labelled terms are interpreted in D: the interpretation function of labelled terms is the unique extension of the interpretation function of λ-terms such that, for every labelled term M and label $\underline{n}$, $(M_{\underline{n}})^D_\rho = (M^D_\rho)_{\underline{n}}$.

As a matter of notation, we write $M =_{D,\rho} N$ for $M^D_\rho = N^D_\rho$ and $M \subseteq_{D,\rho} N$ for $M^D_\rho \subseteq N^D_\rho$.

An easy fact that we will use later is that, for all labelled terms M, N and environments ρ, if N is obtained by erasing some of the labels of M, then , $M \subseteq_{D,\rho} N$. In particular, for every labelled term M and environment ρ, $M \subseteq_{D,\rho} |M|$.

Definition 18. The *weak direct approximant* (w.a.) of a λ-term is defined by induction as follows:

— $x^w = x$;
— $(\lambda x.M)^w = \lambda x.M^w$;
— $(MN)^w = M^w N^w$ if MN is not a redex;
— $((\lambda x.M)N)^w = (\lambda x.M^w)_{\underline{0}} N^w$.

The weak direct approximant M^w of a λ-term M is a labelled term such that $|M^w| = M$. Moreover, it is easy to show that $M^w \subseteq_{D,\rho} M$ for every λ-term M and environment ρ, so that we have

$$\bigcup \{(N^w)^D_\rho : M =_{\lambda\beta} N\} \subseteq M^D_\rho.$$

The remaining part of this section is devoted to prove that the inclusion above is actually an equality.

Theorem 19. *(The Approximation Theorem)* Let D be a stratified graph model. For every λ-term

M and environment ρ, we have

$$M^D_\rho = \bigcup\{(N^w)^D_\rho : M =_{\lambda\beta} N\}.$$

Proof. The proof is divided into claims.

We say that a labelled-term N is *completely labelled* if every subterm of N has at least a label. For example, $((\lambda x.x_{\underline{n}})_{\underline{0}} y_{\underline{m}})_{\underline{0}}$ and $((\lambda x.x_{\underline{n}})_{\underline{0}}(y_{\underline{m}})_{\underline{0}})_{\underline{0}}$ are two completely labelled versions of the λ-term $(\lambda x.x)y$.

Claim 20. For every λ-term M and for every environment ρ we have:

$$M^D_\rho = \bigcup\{N^D_\rho : N \text{ is a completely labelled term}, |N| = M\}.$$

It is sufficient to show by induction on M that, if $\alpha \in (M)^D_\rho \cap D_n$, then there is a completely labelled term N such that $|N| = M$ and $\alpha \in (N)^D_\rho$.

Claim 21. The rewriting system generated by the rules

$$(\lambda x.P)_{\underline{n+1}} Q \to_{lab} P_{\underline{n}}[x := Q_{\underline{n}}]; \quad (P_{\underline{n}})_{\underline{m}} \to_{lab} P_{\underline{min(n,m)}}$$

is Church-Rosser and strongly normalizing.

The proof is in (Barendregt 1984, Section 14.1); remark that:

- if M, N are labelled terms and $M \to^*_{lab} N$, then $|M| \to^*_\beta |N|$.
- every $\to_{lab}$ reduct of a completely labelled term is completely labelled.
- the usual substitution lemma holds for labelled terms: for all labelled terms P and Q and environments ρ, $(P[x := Q])^D_\rho = P^D_{\rho[x:=Q^D_\rho]}$.

The next claim shows that the interpretation of a labelled term does not decrease along $\to_{lab}$ reduction paths.

Claim 22. For all labelled λ-terms P and Q and environments ρ,

$$(\lambda x.P)_{\underline{n+1}} Q \subseteq_{D,\rho} P_{\underline{n}}[x := Q_{\underline{n}}]$$

Let $\alpha \in ((\lambda x.P)_{\underline{n+1}} Q)^D_\rho$. Then there exist $b \cup \{\alpha\} \subseteq D$ such that $b \to \alpha \in ((\lambda x.P)_{\underline{n+1}})^D_\rho$ and $b \subseteq Q^D_\rho$. Hence $b \cup \{\alpha\} \subseteq |D|_n$, and $\alpha \in P^D_{\rho[x:=b]}$. By these two last relations and by $b \subseteq (Q_{\underline{n}})^D_\rho$ we obtain that $\alpha \in (P_{\underline{n}})^D_{\rho[x:=(Q_{\underline{n}})^D_\rho]}$. By the substitution lemma we conclude that $\alpha \in P_{\underline{n}}[x := Q_{\underline{n}}]^D_\rho$.

Finally, we have the approximation theorem.

Claim 23. For all λ-terms M and environments ρ,

$$M^D_\rho = \bigcup\{(N^w)^D_\rho : M =_{\lambda\beta} N\}.$$

Let N be a completely labelled term such that $|N| = M$. By Claim 22 we get $N \subseteq_{D,\rho} N_1$, where N_1 is the normal form of N w.r.t. the rewriting rules $\to_{lab}$. Since N_1 has no redexes w.r.t. $\to_{lab}$, and it is completely labelled as remarked above, then every redex of the λ-term $|N_1|$ should occur in N_1 as $(\lambda x.P)_{\underline{0}} Q$. Let N_2 be the λ-term obtained from N_1 by erasing all labels $n > 0$; we have $N_1 \subseteq_{D,\rho} N_2$. Finally, we get a new term N_3 by erasing from N_2 all occurrences of the label $\underline{0}$ which are not in the position $(\lambda x.P)_{\underline{0}} Q$. Note that N_3 is the direct approximant of $|N_3|$.

In conclusion, we have

$$N \subseteq_{D,\rho} N_1 \subseteq_{D,\rho} N_2 \subseteq_{D,\rho} N_3; \quad N_3 = |N_3|^w. \tag{4}$$

Moreover, as remarked above, we also have

$$M \to_\beta^* |N_3|. \tag{5}$$

In conclusion,

M	$=$	$\bigcup\{N_\rho^D : N$ completely labelled, $\lvert N\rvert = M\}$	by Claim 20
	$\subseteq_{D,\rho}$	$\bigcup\{Q^w : M =_\beta Q\}$	by (4) and (5)
	$\subseteq_{D,\rho}$	M	as remarked after Definition 18.

This concludes the proof of the approximation theorem. □

3.2.2. *The theory of a stratified graph model.* We apply the approximation theorem to show that stratified graph models have semisensible theories. Let us recall the definition of *order* of an unsolvable λ-term:

Definition 24. An unsolvable λ-term U has

1 order 0 if it is not β-equivalent to an abstraction term;
2 order n if $U =_{\lambda\beta} \lambda x_1 \dots x_n.T$ and T has order 0;
3 order ω if it has no finite order.

For example, Ω and Ω_3 are unsolvable of order 0, $\lambda x.\Omega$ has order 1, while $Y\mathbf{k}$ has order ω, where Y is any fixpoint combinator.

Lemma 25. Let D be a stratified model, and U be an unsolvable of order 0. Then, for every environment ρ, we have:

$$U_\rho^D \subseteq D_0.$$

Proof. If $N =_{\lambda\beta} U$ then N is also an unsolvable of order 0. Hence, $N \equiv (\lambda x.P)Q_1 \dots Q_m$, so that $N^w \equiv (\lambda x.P^w)_{\underline{0}} Q_1^w \dots Q_m^w$. The conclusion follows from the approximation theorem because $((\lambda x.P^w)_{\underline{0}} Q_1^w \dots Q_m^w)_\rho^D \subseteq D_0$. □

An easy corollary of this lemma is that, in stratified graph models, unsolvables of order 0 cannot be equated to solvables, since the interpretation of any solvable contains elements of arbitrary rank (see Lemma 28).

In order to deal with unsolvable of arbitrary order, we introduce the notion of *height* in a stratified model.

Definition 26. Let D be a stratified model and $\alpha \in D$. Then we define by induction over the rank the notion of *height* $h(\alpha)$ of α:

— If $rank(\alpha) = 0$, then $h(\alpha) = 0$;
— If $rank(\alpha) > 0$ and $\alpha = (b, \beta)$, then $h(\alpha) = 1 + h(\beta)$.

Notice that, whenever $\alpha = a_1 \to \dots \to a_n \to \beta$ with $rank(\beta) = 0$, then $h(\alpha) \leq n$.

Lemma 27. Let D be a stratified model and U be an unsolvable of order n. Then, for every environment ρ, we have:

$$\alpha \in U_\rho^D \ \Rightarrow\ h(\alpha) \leq n.$$

Proof. By hypothesis $U =_{\lambda\beta} \lambda x_1 \dots x_n.T$ with T of order 0. If $\alpha \in U_\rho^D$ then $\alpha = a_1 \to \dots \to a_n \to \beta$ and $\beta \in T_\sigma^D$, where $\sigma(x_i) = a_i$ and $\sigma(y) = \rho(y)$ for all $y \neq x_i$. By Lemma 25 we have that $rank(\beta) = 0$. Then the conclusion follows by the remark after Def. 26. □

We have shown that, for any unsolvable U of finite order, the interpretation of U in a stratified graph model contains only elements whose height is not bigger than the order of U.

The next lemma shows that the interpretation of any solvable terms contains elements of arbitrary height:

Lemma 28. Let D be a stratified model and $S \in \Lambda^o$ be a solvable λ-term. Then, for every natural number k, there is $\alpha \in D$ such that $\alpha \in S^D$ and $h(\alpha) \geq k$.

Proof. Let $S =_{\lambda\beta} \lambda x_1 \dots x_n.x_j P_1 \dots P_m$. It is easy to show that $\alpha = \emptyset^{j-1} \to (\emptyset^m \to \beta) \to \emptyset^{n-j} \to \beta \in S^D$ for all $\beta \in D$. If we choose $h(\beta) = k$, then $h(\alpha) \geq h(\beta) = k$ and we get the conclusion. □

So far, we have seen that in a stratified model the interpretation of an unsolvable term of finite order is different from the interpretation of any solvable term.

We show now that unsolvable terms of infinite order cannot be consistently equated to solvable terms in graph models.

Lemma 29. Let D be a graph model, $U \in \Lambda^o$ be an unsolvable λ-term of infinite order and $S \in \Lambda^o$ be a solvable λ-term. Then

$$U^D \neq S^D.$$

Proof. Assume, by the way of contradiction, that $U^D = S^D$. Since S is solvable, there exist λ-terms $M_1, \dots, M_k$ such that $SM_1 \dots M_k =_{\lambda\beta} x$, for an arbitrary variable x. Then we have, for any environment ρ,

$$x =_{D,\rho} UM_1 \dots M_k.$$

Since U is unsolvable of infinite order, then $UM_1 \dots M_k$ is also an unsolvable of infinite order. This implies that $UM_1 \dots M_k =_{\lambda\beta} \lambda y.T$ for suitable y and T. However, the equation $x = \lambda y.T$ does not hold in any graph model: consider an environment ρ such that $\rho(x) = \{a \to \alpha\}$ for given finite a and $\alpha \in D$. Then $(\lambda y.T)_\rho^D = \{a \to \alpha\}$. This is not possible because, for all finite $b \subseteq D$, we have that $(b \cup a) \to \alpha \in (\lambda y.T)_\rho^D$. Contradiction. □

Summing up, we have proved the following result:

Theorem 30. The theory of any stratified graph model is semisensible.

Corollary 31. The theory of any weak product is semisensible.

3.3. *Self weak product*

Thm. 15 states that the theory of a weak product is included in the intersection of those of its factors. In this section we show that this inclusion is strict in general. Moreover, in Thm. 33 below we show that self weak products do not preserve in general equations between unsolvable terms. Then it is not in general true that $Th(D \diamond D) = Th(D)$, whenever $Th(D)$ is semisensible.

Proposition 32. Let D be a graph model satisfying the equation $\Omega = \mathbf{i}$. The model $D \diamond D$, that we call *self weak product* of D, does not satisfy $\Omega = \mathbf{i}$.

Proof. By Cor. 31 the theory of $D \diamond D$ is semisensible. □

Theorem 33. There exists a graph model D satisfying the following two conditions:

(i) $D \models \Omega = \lambda x.\Omega$
(ii) $D \diamond D \not\models \Omega = \lambda x.\Omega$.

Proof. The proof is divided into claims. For the sake of clarity, we denote by D_1 the first copy of D in $D \diamond D$ and by D_2 the second copy. Moreover, we assume that these (isomorphic) copies are disjoint.

Recall that every weak product is a stratified graph model.

Claim 34. Let D be a graph model and let $E \equiv D \diamond D$ be the self weak product of D. Then we have:

$$\Omega^D = \emptyset \iff E \models \Omega = \lambda x.\Omega.$$

($\Leftarrow$) Assume $\Omega^D \neq \emptyset$. By Lemma 14 we have that $\Omega^{D_1} = \Omega^E \cap D_1$. Thus the hypothesis implies $\Omega^E \neq \emptyset$. Let $\beta \in \Omega^E$ be an arbitrary element and let $b \subseteq E$ be a finite set containing elements of rank 1. Then $b \to \beta \in (\lambda x.\Omega)^E = \{a \to \alpha : \alpha \in \Omega^E\}$. In conclusion, by Lemma 25 we have that $\Omega^E \subseteq E_0$ is a set of elements of rank 0, while $(\lambda x.\Omega)^E$ contains elements of rank greater than 0. We get the conclusion $E \not\models \Omega = \lambda x.\Omega$.

($\Rightarrow$) The conclusion follows from the following relations: $\Omega^E \subseteq E_0$ (see Lemma 25); $E_0 = D_1 \cup D_2$; $\emptyset = \Omega^{D_i} = \Omega^E \cap D_i$ $(i = 1, 2)$ (see Lemma 14).

This concludes the proof of Claim 34.

Claim 35. There exists a graph model D satisfying the following two conditions:

1 $D \models \Omega = \lambda x.\Omega$;

2 $\Omega^D \neq \emptyset$.

We construct a graph model by using the technique of forcing introduced in (Baeten and Boerboom 1979). In the following proof we follow (Berline and Salibra 2006).

Let D be any infinite countable set. We are going to define by "forcing" the injective total function $c_D : D^* \times D \to D$

We fix an enumeration of D, and an enumeration of $D^* \times D$. Let p be the first element in the enumeration of D.

We are going to build an infinite sequence of elements $\alpha_n \in D \cup \{v\}$ $(n \geq 0)$, where v is some new element, and an infinite sequence of partial pairs A_n $(n \geq 1)$ such that $|A_n|$ is a finite set and $c_{A_n} \subseteq c_{A_{n+1}}$ (i.e., the graph of c_{A_n} is contained within the graph of $c_{A_{n+1}}$). D becomes a graph model by defining $c_D =_{def} \cup_{n\in\omega} c_{A_n}$.

We start from $|A_1| = \{p\}$, $c_{A_1}(\{p\}, p) = p$ and $\alpha_0 = p$ (note that the canonical completion of the partial pair A_1 is Park's model (see Section 2.3)). It is not difficult to verify that $\Omega^{A_1} = \{p\} = (\lambda x.\Omega)^{A_1}$ (recall that the interpretation of a λ-term in a partial pair is defined in Section 2.3).

Assume that the partial pair A_n and $\alpha_0, \ldots, \alpha_{n-1}$ have been built.

Let α_n be the first element of $(\lambda x.\Omega)^{A_n} - \{\alpha_0, \ldots, \alpha_{n-1}\}$ if this set is non-empty, and v otherwise.

Let (b_n, δ_n) be the first element in $D^* \times D \ - \ dom(c_{A_n})$ and γ_n be the first element in $D \ - \ (range(c_{A_n}) \cup b_n)$.

Case 1. $\alpha_n = v$. Then $|A_{n+1}| = |A_n| \cup b_n \cup \{\delta_n, \gamma_n\}$ and $c_{A_{n+1}}$ is a proper extension of c_{A_n} defined as follows in the new pair (b_n, δ_n):

$$c_{A_{n+1}}(b_n, \delta_n) = \gamma_n$$

Case 2. $\alpha_n \in D$. Then $|A_{n+1}| = |A_n| \cup b_n \cup \{\delta_n, \gamma_n, \beta_n, \alpha_n\}$ and $c_{A_{n+1}}$ is a proper extension of c_{A_n} defined as follows in the new pairs (b_n, δ_n) and $(\{\beta_n\}, \alpha_n)$:

$$c_{A_{n+1}}(b_n, \delta_n) = \gamma_n; \quad c_{A_{n+1}}(\{\beta_n\}, \alpha_n) = \beta_n,$$

where β_n is the first element of D such that :

$$\begin{array}{rcl}(\{\beta_n\}, \alpha_n) & \in & D^* \times D \ - \ (dom(c_{A_n}) \cup \{(b_n, \delta_n)\}) \text{ and} \\ \beta_n & \in & D \ - \ (range(c_{A_n}) \cup \{\gamma_n\}).\end{array}$$

It is clear that c_{A_n} is a strictly increasing sequence of well-defined partial injective maps and that $c_D = \cup c_{A_n}$ is total.

There remains to see that the graph model D satisfies the equation $\Omega = \lambda x.\Omega = B$, where $B =_{def} \{ \alpha_n : n \in \omega \} \cap D$.

$B \subseteq (\lambda x.\Omega)^D$ follows from $\alpha_0 = p \in (\lambda x.\Omega)^{A_1}$, from the definition of α_n ($n > 0$) and from the fact that $(\lambda x.\Omega)^{A_n} \subseteq (\lambda x.\Omega)^D$.

$(\lambda x.\Omega)^D \subseteq B$: suppose $\gamma \in (\lambda x.\Omega)^D$; then $\gamma \in (\lambda x.\Omega)^{A_m}$ for some m (and for all the larger ones). If $\gamma \notin B$ then, for all $n \geq m$, $\alpha_n \neq v$ (i.e., $\alpha_n \in D$) is smaller than γ in the enumeration of D, contradicting the fact that there is only a finite number of such elements.

$B \subseteq \Omega^D$: $\alpha_n \in \Omega^D$ follows immediately from $\alpha_0 = p \in \Omega^{A_1} \subseteq \Omega^D$, from the fact that $c_D(\{\beta_n\}, \alpha_n) = \beta$ and from Lemma 2.

$\Omega^D \subseteq B$: if $\varepsilon \in \Omega^D$ then there is an $a \in D^*$ such that $c_D(a, \varepsilon) \in a$ (by Lemma 2). Since $c_D = \cup c_{A_n}$, then either $\varepsilon = \gamma_n$ or $\varepsilon = \alpha_n$ for some n. Because of the choices of the γ_n, the first possibility is not possible.

This concludes the proof of Claim 35.

The conclusion of the theorem is now a simple corollary of Claim 34 and Claim 35. □

Corollary 36. There exist graph models D satisfying the following condition:

$$Th(D \diamond D) \neq Th(D) \cap T, \quad \text{for every sensible } \lambda\text{-theory } T.$$

4. Weak product and graph theories

In this section we show the existence of a minimal graph theory and of a minimal sensible graph theory. The main technical device is that of weak product studied in the above section.

4.1. *The minimal graph theory*

Let I be the set of equations between λ-terms which fail to hold in some graph model. For every equation $e \in I$, we consider a fixed graph model D_e, where the equation e fails to hold.

Then, we consider the weak product $E = \Diamond_{e \in I} D_e$.

By Thm. 15, $Th(E) \subseteq Th(D_e)$, for all $e \in I$. In particular, $e \notin Th(E)$, for all $e \in I$; hence:

Theorem 37. The theory of the graph model E is the minimal graph theory.

4.2. *The minimal sensible graph theory*

We proceed as before: let I_s be the set of equations which fail to hold in some sensible graph model. For every $e \in I_s$, let D_e be a sensible graph model where the equation e fails to hold.

Then, we consider the weak product $E_s = \Diamond_{e \in I} D_e$.

By Thm. 15 the theory $Th(E_s)$ is contained within any sensible graph theory. If $Th(E_s)$ is sensible, then we are done.

In the remaining part of this section we show that $Th(E_s)$ is actually sensible.

The proof of the following lemma can be found in (Kerth 1995, Example 5.3.7).

Lemma 38. (Kerth 1995) Let D be a graph model. If $\alpha \in (\Omega_3)^D$, then there exists a natural number $k \geq 1$ such that

$$\alpha = b_1 \to \cdots \to b_k \to \alpha$$

for suitable finite subsets b_i contained in the interpretation of $\lambda x.xxx$.

Lemma 39. If all closed unsolvable λ-terms have the same interpretation in a graph model, then it must be the empty set.

Proof. Let D be a graph model and let X be a nonempty subset of D, that is the common interpretation of all closed unsolvables. Since Ω and $\lambda x.\Omega$ are both unsolvables, then we have that

$$X = (\lambda x.\Omega)^D = \{a \to \alpha : \alpha \in \Omega^D\} = \{a \to \alpha : \alpha \in X\}. \tag{6}$$

It follows that $a \to \alpha \in X$ for all finite subsets a of D and all $\alpha \in X$. Let γ be an element of X. Then $a \to \gamma \in (\Omega_3)^D$ by (6), since Ω_3 is unsolvable and $(\Omega_3)^D = X$. From Lemma 38 it follows that

$$a \to \gamma = b_1 \to \cdots \to b_k \to a \to \gamma,$$

where $b_1, \ldots, b_k$ are finite subsets contained in the interpretation of $\lambda x.xxx$. It follows that $b_1 = a$. By the arbitrariness of a we can conclude that $(\lambda x.xxx)^D = D$. This is not possible, because, for example, $\emptyset \to \beta \notin (\lambda x.xxx)^D$. □

Theorem 40. The theory of E_s is the minimal sensible graph theory.

Proof. By construction, $Th(E_s)$ is contained within any sensible graph theory. In order to prove that $Th(E_s)$ is sensible, let us suppose that a closed unsolvable term U has a non-empty interpretation in E_s, i.e., there exists $\alpha \in U^{E_s}$. By Prop. 16 there exists a unique $e \in I_s$ such that $f_e(\alpha) \in D_e$. By Lemma 13(a) we have that $f_e(\alpha) \in U^{E_s}$, and finally, by Lemma 13(b), that $f_e(\alpha) \in U^{D_e}$. Since D_e is sensible, this is impossible by Lemma 39. Hence $U^{E_s} = \emptyset$ for any closed unsolvable U (and actually for any unsolvable in any environment). □

5. The minimal graph theory is not $\lambda\beta$

A longstanding open problem is whether there exists a non-syntactic model of lambda calculus whose equational theory is equal to the least λ-theory $\lambda\beta$. In Thm. 42 below we show that this model cannot be found within graph semantics. This result negatively answers Question 1 in (Berline 2000, Section 6.2) for the class of graph models.

We start with a lemma.

Lemma 41. All graph models satisfy the inequality $\Omega_3 \leq \lambda y.\Omega_3 y$.

Proof. Let D be an arbitrary graph model and $\alpha \in (\Omega_3)^D$. From Lemma 38 it follows that there exists a natural number $k \geq 1$ such that $\alpha = b_1 \to b_2 \to \cdots \to b_k \to \alpha$ for suitable finite subsets b_i contained in the interpretation of $\lambda x.xxx$. We have that $\alpha = b_1 \to b_2 \to \cdots \to b_k \to \alpha \in (\lambda y.\Omega_3 y)^D$ iff there exists a finite set d such that $d \to b_2 \to \cdots \to b_k \to \alpha \in (\Omega_3)^D$ and $d \subseteq b_1$. This last relation is true by defining $d \equiv b_1$, so that $\alpha \in (\lambda y.\Omega_3 y)^D$. In conclusion, we get $(\Omega_3)^D \subseteq (\lambda y.\Omega_3 y)^D$. □

We would like to point out here that Berline et al. have recently shown in (Berline *et al.* 2007) that the order theory of a graph model is never recursively enumerable.

Theorem 42. There exists no graph model whose equational theory is $\lambda\beta$.

Proof. Assume that there exists a graph model D whose equational theory is $\lambda\beta$. By Cor. 2.4 in (Selinger 2003) the denotations of two non-$\lambda\beta$-equivalent closed λ-terms must be incomparable in every model of lambda calculus whose equational theory is $\lambda\beta$. Then, for all closed λ-terms M and N such that $M \neq_{\lambda\beta} N$, we have that neither $M^D \subseteq N^D$ nor $N^D \subseteq M^D$. We get a contradiction because of Lemma 41. □

In Thm. 37 we have shown that there exists a minimal graph theory. By Thm. 42 we have that $\lambda\beta$ is strictly included within the minimal graph theory. Thus, there exist equations between non-$\lambda\beta$-equivalent terms satisfied by all graph models. In Thm. 44, whose proof is based on technical results that can be found in (Selinger 2003), we characterize an equation of this kind.

Let f be any λ-term satisfying, via a fixpoint combinator, the recursive equation $fxy =_{\lambda\beta} fx(fx(fxy))$ for variables x, y (in other words, any three applications of fx are equivalent to a single application) and let $A \equiv \lambda xyzwv.fx(fy(fz(fwv)))$. The λ-terms f and A were defined in (Selinger 2003). In (Selinger 2003, Prop. 2.1) it was shown that

$$Axxxy =_{\lambda\beta} Axyyy \tag{7}$$

and

$$Axxxy \neq_{\lambda\beta} Axxyy. \tag{8}$$

This inequality has a ingenious proof based on the notion of a finite lambda reduction model.

For the sake of completeness, we recall (Selinger 2003, Lemma 2.2) that will be used in the proof of Thm. 44.

Lemma 43. (Selinger 2003) Let $P_1, \dots, P_n$ be λ-terms that are distinct in $\lambda\beta$, and let x be a variable not free in $P_1, \dots, P_n$. Then, for all terms M, N for which x is not free in M and N, and for all variables $y_1, \dots, y_n$, we have:

$$M(xP_1)(xP_2)\dots(xP_n) =_{\lambda\beta} N(xP_1)(xP_2)\dots(xP_n) \Rightarrow My_1y_2\dots y_n =_{\lambda\beta} Ny_1y_2\dots y_n.$$

As a matter of notation, let $t \equiv \Omega_3$ and $u \equiv \lambda y.\Omega_3 y$ in the following theorem.

Theorem 44. Let T be the minimal graph theory (whose existence has been shown in Thm. 37). Then we have, for a variable x,

$$A(xt)(xt)(xt)(xu) =_T A(xt)(xt)(xu)(xu), \tag{9}$$

while

$$A(xt)(xt)(xt)(xu) \neq_{\lambda\beta} A(xt)(xt)(xu)(xu). \tag{10}$$

Proof. By compatibility, by $t \leq u$ (see Lemma 41) and by (7) we obtain that the following relations hold in every graph model:

$$A(xt)(xt)(xt)(xu) \leq A(xt)(xt)(xu)(xu) \leq A(xt)(xu)(xu)(xu) =_{\lambda\beta} A(xt)(xt)(xt)(xu).$$

It easily follows (9). It remains to show the inequality (10). Assume, by the way of contradiction, the opposite: $A(xt)(xt)(xt)(xu) =_{\lambda\beta} A(xt)(xt)(xu)(xu)$. We can apply the hypotheses of Lemma 43 to $M \equiv \lambda xy.Axxxy$, $N \equiv \lambda xy.Axxyy$, $P_1 \equiv t$ and $P_2 \equiv u$. Then we get the conclusion of Lemma 43: $My_1y_2 =_{\lambda\beta} Ny_1y_2$, that implies $Ay_1y_1y_1y_2 =_{\lambda\beta} Ay_1y_1y_2y_2$. This contradicts (8). □

6. Omitting equations and theories

In this section we prove the main results of the paper:

- The λ-theory $\mathcal{B}$ of Böhm trees is the greatest sensible graph theory.
- Graph semantics omits all equations $M = N$ between solvable λ-terms which have distinct Böhm trees.

The following two theorems are the main results of the paper. The proof of Thm. 45 is postponed to Section 8.

Theorem 45. The graph semantics omits all equations $M = N$ satisfying the following conditions:

$$M =_{\mathcal{H}^*} N \text{ and } M \neq_{\mathcal{B}} N. \tag{11}$$

The above result shows the intrinsic non-extensionality of graph models. In particular, any equation $M = N$ between λ-terms which are $\lambda\beta\eta$-equivalent, but they do not have the same Böhm tree, fails in every graph model. This is not anymore true if M and N are $\lambda\beta\eta$-equivalent and the η-redexes in M and N are pushed to infinite. For example, let M be any λ-term such that $Mz =_{\lambda\beta} \lambda x.x(Mz)$. Then the Böhm tree of $M\mathbf{1}$ does not contain any η-redex, because $\mathbf{1}$ is pushed to infinite when we generate its Böhm tree:

$$M\mathbf{1} =_{\lambda\beta} \lambda x.x(M\mathbf{1}) =_{\lambda\beta} \lambda x.x(\lambda x.x(M\mathbf{1})) =_{\lambda\beta} \ldots \lambda x.x(\lambda x.x(\lambda x.x(\lambda x.x(\ldots))))$$

This is the reason why the equation $M\mathbf{i} = M\mathbf{1}$ is true, for example, in Scott's graph model P_ω (Barendregt 1984, Section 19.1).

Theorem 46. pThe λ-theory $\mathcal{B}$ is the unique maximal sensible graph theory.

Proof. $\mathcal{B}$ is the equational theory of Scott's graph model P_ω (Barendregt 1984, Section 19.1) and of Engeler's graph model $\mathcal{E}_A$ (Berline 2000). Let T be a sensible graph theory and suppose $M =_T N$. We have that $M =_{\mathcal{H}^*} N$, because $\mathcal{H}^*$ is the unique maximal sensible λ-theory. Since graph semantics does not omit the equation $M = N$, then from $M =_{\mathcal{H}^*} N$ and from Thm. 45 it follows that $M =_{\mathcal{B}} N$, so that $T \subseteq \mathcal{B}$. □

It is well known that every graph theory is non-extensional. We remark that Thm. 46 is not trivial, because there exist non-extensional sensible λ-theories that strictly include $\mathcal{B}$ (Barendregt 1984, Exercize 16.5.5).

(Berline 2000) asked whether there is a non-syntactic sensible model of lambda calculus whose theory is strictly included in $\mathcal{B}$. The answer is positive as shown in the following corollary.

Theorem 47. There exists a continuum of different sensible graph theories strictly included in $\mathcal{B}$.

Proof. Based on a syntactic difficult result (conjectured in (Kerth 1998) and proved in (David 2001)), In (Kerth 1998) it was shown that there exists a continuum of sensible graph theories. Then the conclusion follows from Thm. 46. □

It is well known that the λ-term Ω is easy, that is, it can be consistently equated to every other closed λ-term M. We denote by $(\Omega = M)^+$ the λ-theory generated by the equation $\Omega = M$.

Theorem 48. Let M be an arbitrary closed λ-term. Then we have:

$$P =_{\mathcal{H}^*} Q,\ P \neq_{\mathcal{B}} Q\ \Rightarrow\ (\Omega = M)^+ \not\vdash P = Q.$$

In other words, $(\Omega = M)^+ \cap \mathcal{H}^* \subseteq \mathcal{B}$.

Proof. By (Baeten and Boerboom 1979) the λ-theory $(\Omega = M)^+$ is contained within a graph theory. Then the conclusion follows from Thm. 45. □

7. Incompatible sets

In the proof of the main theorem we are going to use families of points of a graph model, which are not only pairwise distinct, but also "incompatible" in the sense expressed by Def. 49 below. Then, in Lemma 53 we show that such families actually exist in all graph models.

Let D be a graph model and $\alpha, \beta \in D$ and $m \geq 1$. We say that

— β *m-constructs* α if there exists a sequence $\overline{a} = a_1 \ldots a_m$ of finite subsets of D such that $\alpha = a_1 \to \cdots \to a_m \to \beta$.

— β *constructs* α if β m-constructs α for some m.

Let $P \equiv yP_1 \ldots P_m$ be a λ-term, and ρ be a D-environment. Given $\beta \in D$, we would like to check whether $\beta \in (yP_1 \ldots P_m)^D_\rho = \rho(y)(P_1)^D_\rho \ldots (P_m)^D_\rho$. In general, we have to consider all elements $\alpha \in \rho(y)$ such that β m-constructs α. The goal of the definition of a incompatible set is to reduce these elements to just one, say $a_1 \to \ldots a_m \to \beta$. In this case, to check that $\beta \notin (yP_1 \ldots P_m)^D_\rho$, it would be sufficient to find an i such that $a_i \not\subseteq (P_i)^D_\rho$.

Definition 49. An infinite subset X of D is called *incompatible* if, for all $\alpha, \beta \in X$, we have that β does not construct α (notice that α may be equal to β).

Before proving that incompatible sets indeed exist, we explain how they will be used in the following.

Lemma 50. Let X be an incompatible set. For every $\alpha \in D$, there exists at most one element β of X such that β constructs α.

We denote by $q_X : D \to X$ the partial function such that $q_X(\alpha)$ is the unique element of X which constructs α when this element exists.

Definition 51. Let X be an incompatible set. A subset Y of D is called X-*good* if the function $q_X : D \to X$ restricted to Y is total and injective.

Lemma 52. Let X be an incompatible set, $P \equiv yP_1 \ldots P_m$ be a λ-term and ρ be a D-environment. If $\rho(y)$ is X-good, then the following two conditions are equivalent for all $\alpha \in \rho(y)$:

1 $q_X(\alpha) \in \rho(y)(P_1)^D_\rho \ldots (P_m)^D_\rho$;
2 $q_X(\alpha) \in \{\alpha\}(P_1)^D_\rho \ldots (P_m)^D_\rho$.

Proof. $q_X(\alpha) \in \rho(y)(P_1)^D_\rho \ldots (P_m)^D_\rho$ if, and only if, there exists $\gamma \in \rho(y)$ such that $\gamma \equiv a_1 \to \cdots \to a_m \to q_X(\alpha) \in \rho(y)$ and $a_i \subseteq (P_i)^D_\rho$. Since $\rho(y)$ is X-good, by Lemma 50 $\gamma = \alpha$ is the unique possibility. □

As a consequence of the above lemma, to prove that $q_X(\alpha) \notin (yP_1 \ldots P_m)^D_\rho$, it is sufficient to find an i such that $a_i \not\subseteq (P_i)^D_\rho$. This is the reason why we have introduced the incompatible sets.

Lemma 53. There exists an incompatible set.

Proof. Let D be a graph model. Given $\alpha \in D$, we define the *degree* of α as the least natural number $k > 0$ such that there exist finite subsets $b_1, \ldots, b_k$ of D satisfying $\alpha = b_1 \to \cdots \to b_k \to \alpha$. If such a natural number does not exist, we say that the degree of α is infinite.

We now show that there exists an element of D whose degree is infinite.

By the way of contradiction, assume that every element of D has a finite degree. Let α be an arbitrary element of D. Then there exists $b_1, \ldots, b_k$ ($k \geq 1$) such that $\alpha = b_1 \to \cdots \to b_k \to \alpha$. Define $\beta_i \equiv b_i \to \cdots \to b_k \to \alpha$ ($i = 1, \ldots, k$). Then $\alpha = b_1 \to \cdots \to b_{i-1} \to \beta_i$. In particular, $\alpha = b_1 \to \beta_2$. Since there exist infinite elements of D^* distinct from b_1, we consider the family of elements $c \to \beta_2$ with $c \neq b_1$. Let c such that $c \to \beta_2 \neq \beta_1, \ldots, \beta_k$. Then it is not possible that $c \to \beta_2$ has a finite degree.

We now conclude the proof. Let $\alpha \in D$ whose degree is infinite. Given a family $\{a_n\}_{n \in \omega}$

of pairwise distinct, finite subsets of D, define $\beta_n = a_n \to \alpha$ ($n \geq 0$). We prove that the set $\{\beta_n : n \geq 0\}$ is an incompatible set. By the way of contradiction, assume that $\beta_i = b_1 \to \cdots \to b_t \to \beta_j$ for some i and j, i.e.,

$$a_i \to \alpha = b_1 \to \cdots \to b_t \to a_j \to \alpha.$$

It follows that $\alpha = b_2 \to \cdots \to b_t \to a_j \to \alpha$. We get a contradiction because the degree of α is infinite. $\square$

7.1. *Incompatible sets at work*

In the following X is a fixed incompatible set.

As a matter of notation, we define $\overline{\emptyset}_n \to \alpha$ as follows:

$$\overline{\emptyset}_0 \to \alpha \equiv \alpha; \quad \overline{\emptyset}_{n+1} \to \alpha \equiv \emptyset \to (\overline{\emptyset}_n \to \alpha).$$

Recall that, if ρ is an environment and $\overline{x} \equiv x_1 \dots x_n$ is a sequence of variables, then we write $\rho(\overline{x}) \to \alpha$ for $\rho(x_1) \to \cdots \to \rho(x_n) \to \alpha$. Notice that it is implicitly assumed that $\rho(x_i)$ is a finite set for all i.

In the following lemma we show how to separate in a graph model two solvable λ-terms.

Lemma 54. Let D be a graph model, X be an incompatible set, and:

- $S \equiv \lambda\overline{x}.z\overline{S}$, where $\overline{S} \equiv S_1 \dots S_t$;
- $T \equiv \lambda\overline{x}\lambda\overline{y}.z\overline{T}\,\overline{Q}$, where $\overline{y} = y_1 \dots y_u$ ($u \geq 0$), $\overline{T} \equiv T_1 \dots T_t$ and $\overline{Q} \equiv Q_1 \dots Q_r$ ($r \geq 1$);
- $\delta \equiv \overline{\emptyset}_{t+r} \to \beta$ and $\alpha \equiv \rho(\overline{x}) \to \rho(\overline{y}) \to \beta$ with $\beta \in X$;
- ρ and σ be environments such that $\rho(z), \sigma(z)$ are X-good (z is the head variable of S), $\delta \in \rho(z) \cap \sigma(z)$, and $\rho(y_i) \neq \emptyset$ for some $1 \leq i \leq u$.

Then we have $\alpha \in T^D_\sigma$ and $\alpha \notin S^D_\sigma$.

Proof. We define $\tau \equiv \sigma[\overline{x} := \rho(\overline{x})][\overline{y} := \rho(\overline{y})]$. Notice that $\tau(z) = \rho(z)$ if $z \in \overline{xy}$; otherwise $\tau(z) = \sigma(z)$. Then we have:

$$\begin{aligned} \alpha \in T^D_\sigma \quad & \text{iff} \quad \beta \in z^D_\tau\, \overline{T}^D_\tau\, \overline{Q}^D_\tau \\ & \text{iff} \quad \beta \in \tau(z)\, \overline{T}^D_\tau\, \overline{Q}^D_\tau \\ & \text{iff} \quad \beta \in \{\delta\}\overline{T}^D_\tau \overline{Q}^D_\tau, \\ & \qquad\quad \text{by } \delta \in \rho(z) \cap \sigma(z) \text{ and by Lemma 52} \\ & \text{iff} \quad \beta \in \{\delta\}\overline{\emptyset}_{t+r}, \\ & \qquad\quad \text{by def. } \delta \end{aligned}$$

Since this last condition is true, then $\alpha \in T^D_\sigma$.

Let $\tau' \equiv \sigma[\overline{x} := \rho(\overline{x})]$.

$$
\begin{aligned}
\alpha \in S_{\sigma}^{D} \quad & \text{iff} \quad \rho(\overline{y}) \to \beta \in z_{\tau'}^{D} \overline{S}_{\tau'}^{D} \\
& \text{iff} \quad \rho(\overline{y}) \to \beta \in \{\delta\} \overline{S}_{\tau'}^{D}, \\
& \qquad \text{by } \delta \in \rho(z) \cap \sigma(z) \text{ and by Lemma 52} \\
& \text{iff} \quad \rho(\overline{y}) \to \beta \in \{\delta\} \overline{\emptyset}_t, \\
& \qquad \text{by def. } \delta \text{ and } \overline{S} = S_1 \dots S_t \\
& \text{iff} \quad \rho(\overline{y}) \to \beta = \overline{\emptyset}_r \to \beta, \\
& \qquad \text{by def. } \delta
\end{aligned}
$$

Since $\beta \in X$ (and thus β does not construct β), then this last condition holds if, and only if, $u = r$, where $\overline{y} = y_1 \dots y_u$, and $\rho(y_j) = \emptyset$ for all $1 \le j \le r$. By hypothesis $(\exists i)\ \rho(y_i) \neq \emptyset$. Then $\alpha \notin S_{\sigma}^{D}$. □

Lemma 55. Let D be a graph model, X be an incompatible set, and:

— $S \equiv \lambda \overline{x}.z\overline{S}$, where $\overline{S} \equiv S_1 \dots S_t$;
— $\delta = \overline{\emptyset}_{i-1} \to \{\alpha'\} \to \overline{\emptyset}_{t-i} \to \beta$ for some $i \le t$ and $\beta \in X$;
— ρ and σ be environments such that $\rho(z)$ and $\sigma(z)$ are X-good (z is the head variable of S) and $\delta \in \rho(z) \cap \sigma(z)$;
— $\alpha = \rho(\overline{x}) \to \beta$.

Then we have:

$$\alpha \in S_{\sigma}^{D} \ \Leftrightarrow\ \alpha' \in (S_i)_{\sigma[\overline{x}:=\rho(\overline{x})]}^{D}.$$

Proof. Let $\tau \equiv \sigma[\overline{x} := \rho(\overline{x})]$. Notice that $\tau(z) = \rho(z)$ if $z \in \overline{x}$; otherwise $\tau(z) = \sigma(z)$.

$$
\begin{aligned}
\alpha \in S_{\sigma}^{D} \quad & \text{iff} \quad \beta \in z_{\tau}^{D} \overline{S}_{\tau}^{D}, \\
& \qquad \text{by def. } \alpha \\
& \text{iff} \quad \beta \in \{\delta\} \overline{S}_{\tau}^{D}, \\
& \qquad \text{by } \delta \in \sigma(z) \cap \rho(z) \text{ and by Lemma 52} \\
& \text{iff} \quad \beta \in \{\delta\} \overline{\emptyset}_{i-1} (S_i)_{\tau}^{D} \overline{\emptyset}_{t-i}, \\
& \qquad \text{by def. } \delta \\
& \text{iff} \quad \alpha' \in (S_i)_{\tau}^{D}, \\
& \qquad \text{by def. } \delta.
\end{aligned}
$$

The conclusion of the lemma is now immediate. □

8. The proof of the main theorem

In this section we provide the proof of Thm. 45.

We recall that a node of a tree is a sequence of natural numbers and that the level of a node is the length of the sequence. The empty sequence will be denoted by ε.

Let M, N be closed λ-terms such that $M =_{\mathcal{H}^*} N$ and $M \neq_{\mathcal{B}} N$. This last condition expresses the fact that the Böhm tree $BT(M)$ of M is different from the Böhm tree $BT(N)$ of N. The first condition $M =_{\mathcal{H}^*} N$ implies that, if a given node has the same label in $BT(M)$ and $BT(N)$, then this node has the same number of sons in $BT(M)$ and $BT(N)$.

Let us introduce now some notations and definitions needed in the proof.

Let $u = r_1 \dots r_k$ be a node at least level, where the labels of $BT(M)$ and $BT(N)$ are different. The sequence ε, r_1, r_1r_2, $r_1r_2r_3, \dots, r_1 \dots r_k$ is the sequence of nodes that are in the path from the root ε to u. These nodes will be denoted by $u_0, u_1, u_2, \dots, u_k$. Then, for example, $u_0 = \varepsilon$, $u_2 = r_1r_2$ and $u_k = u$. From the hypothesis of minimality of u it follows that

(i) The label of the node u_j ($0 \le j < k$) in the Böhm tree of M is equal to the corresponding one in the Böhm tree of N;

(ii) The labels of the node u in $BT(M)$ and $BT(N)$ are different.

We define two sequences M_{u_j} and N_{u_j} ($0 \le j \le k$) of λ-terms whose Böhm trees $BT(M_{u_j})$ and $BT(N_{u_j})$ are the subtrees of $BT(M)$ and $BT(N)$ at root u_j, respectively. Without loss of generality, we may assume that M and N are in head normal form (hnf, for short). Then we define:

$$M_{u_0} \equiv M; \quad N_{u_0} \equiv N.$$

If $k = 0$ we have finished. Otherwise, assume by induction hypothesis that we have already defined two λ-terms M_{u_j} and N_{u_j} ($j < k$) in hnf and that the Böhm trees of M_{u_j} and N_{u_j} are respectively the subtrees of $BT(M)$ and $BT(N)$ at root u_j. Assume that M_{u_j} and N_{u_j} have respectively the following form:

$$M_{u_j} \equiv \lambda x_1^j \dots x_{n_j}^j . z_j M_1^j \dots M_{s_j}^j; \tag{12}$$

$$N_{u_j} \equiv \lambda x_1^j \dots x_{n_j}^j . z_j N_1^j \dots N_{s_j}^j .$$

To abbreviate the notation we will write M_{u_j} and N_{u_j} as follows:

$$M_{u_j} \equiv \lambda \overline{x}_{n_j}^j . z_j M_1^j \dots M_{s_j}^j; \quad N_{u_j} \equiv \lambda \overline{x}_{n_j}^j . z_j N_1^j \dots N_{s_j}^j .$$

Then the node u_j in the Böhm trees of M and N has s_j sons. Since $u_{j+1} = u_j r_{j+1}$ is a son of u_j in the Böhm trees of M and N, then we have $r_{j+1} \le s_j$. Then we define $M_{u_{j+1}}$ and $N_{u_{j+1}}$ as the principal head normal forms (principal hnfs, for short) of $M_{r_{j+1}}^j$ and $N_{r_{j+1}}^j$ respectively (see (Barendregt 1984, Def. 8.3.20) for the definition of principal hnf).

The Böhm trees of $M_{u_{j+1}}$ and $N_{u_{j+1}}$ are respectively the subtrees of $BT(M)$ and $BT(N)$ at root u_{j+1}. The principal hnfs of M_{u_k} and N_{u_k} (recall that $u_k = u$ is the node where the Böhm trees are different) cannot have the same number of abstractions. Without loss of generality, we assume that

$$M_{u_k} \equiv \lambda \overline{x}_{n_k}^k . z_k M_1^k \dots M_{s_k}^k; \tag{13}$$

$$N_{u_k} \equiv \lambda \overline{x}_{n_k}^k \lambda y_1 \dots y_r . z_k N_1^k \dots N_{s_k}^k Q_1 \dots Q_r \ (r \ge 1), \tag{14}$$

where $y_i \le_\eta Q_i$ ($1 \le i \le r$), i.e., Q_i is a possibly infinite η-expansion of the fresh variable y_i.

Let D be an arbitrary graph model. First we will show that the terms N_{u_k} and M_{u_k} have different interpretations in D, that is, there exist an element $\alpha_k \in D$ and a D-environment σ_k such that

$$\alpha_k \in (N_{u_k})_{\sigma_k}^D \text{ and } \alpha_k \notin (M_{u_k})_{\sigma_k}^D . \tag{15}$$

Second we will show that this difference at level k can be propagated upward, that is, there exists an element $\alpha_0 \in D$ and a D-environment σ_0 such that

$$\alpha_k \in (N_{u_k})^D_{\sigma_k} \text{ iff } \alpha_0 \in (N_{u_0})^D_{\sigma_0}$$

and

$$\alpha_k \in (M_{u_k})^D_{\sigma_k} \text{ iff } \alpha_0 \in (M_{u_0})^D_{\sigma_0}.$$

From $\alpha_k \in (N_{u_k})^D_{\sigma_k}$ and from $\alpha_k \notin (M_{u_k})^D_{\sigma_k}$ it follows the conclusion: $\alpha_0 \in (N_{u_0})^D_{\sigma_0}$ (i.e., $\alpha_0 \in N^D_{\sigma_0}$) and $\alpha_0 \notin (M_{u_0})^D_{\sigma_0}$ (i.e., $\alpha_0 \notin M^D_{\sigma_0}$).

8.1. *The definition of the environments*

Let X be an incompatible set, let $\beta_0, \ldots, \beta_k, \beta_{k+1}$ be distinct elements of X and $\Delta = \{\delta_0, \delta_1, \ldots, \delta_k\}$ be an X-good set. Let

— $z_0 \ldots z_k$ be the sequence of the head variables of the λ-terms $M_{u_0}, \ldots, M_{u_k}$ (notice that not all these head variables will be necessarily distinct);
— y_r be the last variable occurring in the external abstractions of N_{u_k} (notice that $y_r \neq z_0, \ldots, z_k$);
— $\overline{x}^j_{n_j}$ be the sequence of variables occurring in the external abstractions of M_{u_j}.

We now define a sequence of environments ρ_j $(0 \leq j \leq k)$ and σ_i $(0 \leq i \leq k+1)$.

$$\rho_j(x) = \begin{cases} \{\delta_i : j \leq i \leq k \text{ and } z_i = x\} & \text{if } x \neq y_r \\ \{\delta_k\} & \text{if } x = y_r \end{cases}$$

$$\sigma_0 \equiv \rho_0; \quad \sigma_{i+1} = \sigma_i[\overline{x}^i_{n_i} := \rho_i(\overline{x}^i_{n_i})].$$

As a matter of notation, if τ and ρ are environments, we write $\tau \leq \rho$ for $\tau(x) \subseteq \rho(x)$ for all variables x.

Lemma 56.

(a) $i \leq j \;\Rightarrow\; \rho_i \geq \rho_j$.
(b) $\rho_j \leq \sigma_{j+1} \leq \rho_0$ for every $0 \leq j \leq k$.
(c) $\delta_j \in \sigma_{j+1}(z_j)$ for all $0 \leq j \leq k$.

Proof. (b) By definition we have $\sigma_1 = \rho_0$. Assume by induction hypothesis that $\rho_{j-1} \leq \sigma_j$. Then by (a) we have $\rho_j \leq \sigma_j$. The conclusion is an immediate consequence of the definition of σ_{j+1}. □

We now define by induction the structure of the elements δ_j (belonging to the X-good set Δ) and of the elements α_j $(0 \leq j \leq k)$.

(i) $\delta_k \equiv \overline{\emptyset}_{s_k+r} \to \beta_k$;
(ii) $\alpha_k \equiv \rho_k(\overline{x}^k_{n_k}) \to \rho_k(\overline{y}_r) \to \beta_k$.

Assume we have defined δ_{j+1} and α_{j+1} $(j < k)$. We define δ_j and α_j as follows.

(i) $\delta_j \equiv \overline{\emptyset}_{r_j-1} \to \{\alpha_{j+1}\} \to \overline{\emptyset}_{s_j-r_j} \to \beta_j$;
(ii) $\alpha_j \equiv \rho_j(\overline{x}^j_{n_j}) \to \beta_j$.

8.2. *The conclusion of the proof*

In the following lemma we show that N_{u_k} and M_{u_k} have different interpretations.

Lemma 57. We have $\alpha_k \in (N_{u_k})^D_{\sigma_k}$ and $\alpha_k \notin (M_{u_k})^D_{\sigma_k}$.

Proof. We apply Lemma 54 by putting $S \equiv M_{u_k}, T \equiv N_{u_k}, \delta \equiv \delta_k, t \equiv s_k, \beta \equiv \beta_k, \rho \equiv \rho_k$, $\sigma \equiv \sigma_k$, $u \equiv r$ and $i = r$. □

The different interpretation of N_{u_k} and M_{u_k} can be propagated upward as shown in the following lemma.

Lemma 58. For every $k > j \geq 0$ we have

$$\alpha_j \in (N_{u_j})^D_{\sigma_j} \;\Leftrightarrow\; \alpha_{j+1} \in (N_{u_{j+1}})^D_{\sigma_{j+1}}$$

and

$$\alpha_j \in (M_{u_j})^D_{\sigma_j} \;\Leftrightarrow\; \alpha_{j+1} \in (M_{u_{j+1}})^D_{\sigma_{j+1}}.$$

Proof. We apply Lemma 55 to N_{u_j}, by putting $S \equiv N_{u_j}, \overline{x} \equiv \overline{x}^i_{n_j}, \delta \equiv \delta_j, i \equiv r_j, \alpha' \equiv \alpha_{j+1}$, $\alpha \equiv \alpha_j$, $\rho \equiv \rho_j$ and $\sigma \equiv \sigma_j$, and by observing that $\sigma_{j+1} \equiv \sigma_j[\overline{x}^j_{n_j} := \rho_j(\overline{x}^i_{n_j})]$. We similarly apply Lemma 55 to M_{u_j}. □

Lemma 59. We have $\alpha_0 \in N^D_{\sigma_0}$, while $\alpha_0 \notin M^D_{\sigma_0}$.

Proof. Recall that $N \equiv N_{u_0}$ and $M \equiv M_{u_0}$. By applying Lemma 58 it is easy to show that $\alpha_0 \in N^D_{\sigma_0} \;\Leftrightarrow\; \alpha_k \in (N_{u_k})^D_{\sigma_k}$, and $\alpha_0 \in M^D_{\sigma_0} \;\Leftrightarrow\; \alpha_k \in (M_{u_k})^D_{\sigma_k}$. Then the conclusion is immediate, because by Lemma 57 we have that $\alpha_k \in (N_{u_k})^D_{\sigma_k}$ and $\alpha_k \notin (M_{u_k})^D_{\sigma_k}$. □

9. Conclusion and future work

In this paper, we have collected in an organized manner several already published results and some new material: the existence of the minimum (resp. minimun sensible) graph-theory appeared originally in (Bucciarelli and Salibra 2003) (resp. (Bucciarelli and Salibra 2004)). The new presentation of Section 3 stresses the relevance and generality of the *weak product* construction, underlying these results, and adds some new results (for instance, the fact that the theory of weak products is semisensible and it is in general strictly finer than the intersection of the factors' theories, obtained via the notion of *self weak product*).

Section 6 covers the main result of (Bucciarelli and Salibra 2004), namely the fact that the maximal sensible graph theory is $\mathcal{B}$.

The content of Section 5, a negative answer to the question of whether $\lambda\beta$ is the minimal graph theory, also appeared in (Bucciarelli and Salibra 2004). Actually, this negative result opens the way to the investigation of the minimal graph theory.

Section 3.2 and Section 3.3 present new results. First, the fact that stratified graph models, which are those obtained by canonical completion of partial pairs, i.e., virtually all known graph models, apart from those constructed by *forcing* (Baeten and Boerboom 1979; Berline and Salibra 2006), have semisensible theories. Then we show that the theory of a weak product is in general strictly finer than the intersection of the factors' theories. Finally, we provide equations between unsolvable terms which are not preserved in weak products.

Several questions about graph λ-theories remain open. Among them, we wish to address that concerning the minimal sensible graph theory: Is it $\mathcal{H}$ (the minimum sensible theory) or is it bigger? For the time being, we are able to separate in a graph model some typical example of $\mathcal{B}$-equivalent, $\mathcal{H}$-distinct λ-terms, like Yx and Θx.

The notion of *effective* graph model is a natural one: it is enough to ask that the coding function be total recursive w.r.t. given enumerations of the model's web, finite sets and pairs of natural numbers. Then one recasts classical recursion theory results in the framework of graph models, and this seems particularly compelling since those are models of the λ-calculus. A paper on this subject is (Berline *et al.* 2007).

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