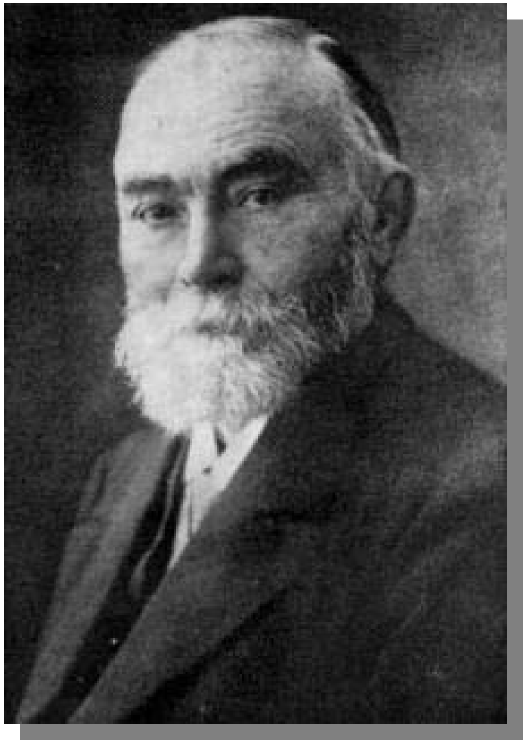
Storia dell’informatica

Le radici logiche dell’informatica.

Parte IV: Da Frege a Gödel

Rettili, di M. C. Escher (litografia, 1943)

Gottlob Frege (1848-1925)



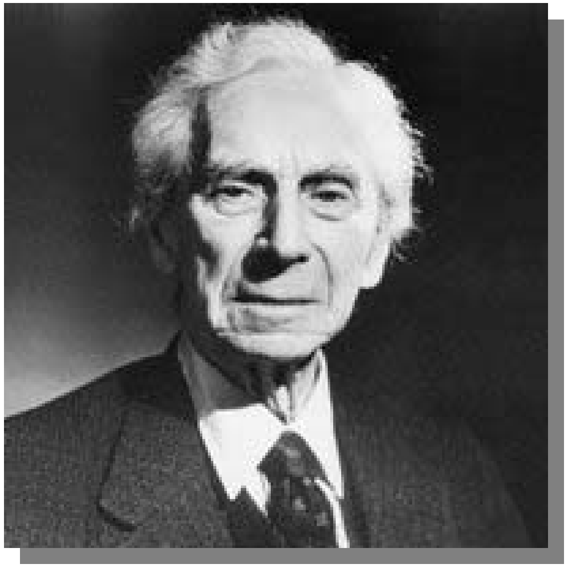
Gottlob Frege (1848-1925) was a German logician, mathematician and philosopher who played a crucial role in the emergence of modern logic and analytic philosophy.

Frege's logical works were revolutionary, and are often taken to represent the fundamental break between contemporary approaches and the older, Aristotelian tradition.

He invented modern quantificational logic, and created the first fully axiomatic system for logic.

His theory of meaning, especially his distinction between the sense and reference of linguistic expressions, was groundbreaking in semantics and the philosophy of language. He had a profound and direct influence on such thinkers as Russell, Carnap and Wittgenstein.

1902: Russell Writes to Frege



In June 1902 a letter arrived in Jena, a medieval town later to be part of Communist East Germany, addressed to the 53-year-old Gottlob Frege from the young British philosopher **Bertrand Russell**.

Although Frege believed that he had made important and fundamental discoveries, his work had been almost totally ignored. It must then have been with some pleasure that he read,

“I find myself in agreement with you in all essentials […] I find in your work discussions, distinctions, and definitions that one seeks in vain in the work of other logicians. […] There is just one point where I have encountered a diffculty.”

Frege soon realized that this one “diffculty” seemed to lead to the collapse of his life’s work. It cannot have helped too much that Russell went on to write,

“The exact treatment of logic in fundamental questions has remained very much behind; in your works I find the best I know of our time, and therefore I have permitted myself to express my deep respect to you.”

«A scientist can hardly meet with anything more undesirable than to have the foundations give way just as the work is finished. I was put in this position by a letter from Mr. Bertrand Russell when the work was nearly through the press.»

(Gottlob Frege,

Quoted in *Scientific American (May 1984) 77.)*

Russell’s Paradox

Russell's paradox is the most famous of the logical or set-theoretical paradoxes.

Some sets, such as the set of all teacups, are not members of themselves. Other sets, such as the set of all non-teacups, are members of themselves.

Call the set of all sets that are not members of themselves *R*.

If *R* is a member of itself, then by definition it must not be a member of itself.

Similarly, if *R* is not a member of itself, then by definition it must be a member of itself.

Discovered in 1901, the paradox has prompted much work in logic, set theory and the philosophy and foundations of mathematics.

Significance of the Paradox

The significance of Russell's paradox can be seen once it is realized that, using classical logic, all sentences follow from a contradiction.

For example, assuming both *P* and ~*P,* any arbitrary proposition, *Q*, can be proved as follows: from *P* we obtain *P* v *Q* by the rule of Addition; then from *P* v *Q* and ~*P* we obtain *Q* by the rule of Disjunctive Syllogism.

Because of this, and because set theory underlies all branches of mathematics, many people began to worry that, if set theory was inconsistent, no mathematical proof could be trusted completely.

Russell's own response to the paradox was his aptly named *theory of types.* Recognizing that self-reference lies at the heart of the paradox, Russell's basic idea is that we can avoid commitment to *R* (the set of all sets that are not members of themselves) by arranging all sentences (or, equivalently, all propositional functions) into a hierarchy.

Frege, the Man / 1

«As I think about acts of integrity and grace, I realise that there is nothing in my knowledge to compare with Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work had been ignored to the benefit of men infintely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. **It was almost superhuman and a telling indication of that of which men are capable if their dedication is to creative work and knowledge instead of cruder efforts to dominate and be known.**»

Bertrand Russell

Frege, the Man / 2

«There is some irony for me in the fact that the man about whose philosophical views I have devoted, over the years, a great deal of time to thinking, was, at least at the end of his life, a virulent racist, specifically an anti-semite. . . .

[His] diary shows Frege to have been a man of extreme right-wing opinions, bitterly opposed to the parliamentary system, democrats, liberals, Catholics, the French and, above all, Jews, who he thought ought to be deprived of political rights and,preferably, expelled from Germany.

I was deeply shocked, because I had revered Frege as an absolutely rational man . . .»

Michael Dummet

The *Begriffsschrift*

Begriffsschrift is the title of a short book of Frege, published in 1879, and is also the name of the formal system set out in that book.

Begriffsschrift is usually translated as *concept writing* or *concept notation*; the full title of the book identifies it as "a formula language, modelled on that of arithmetic, of pure thought."

"If the task of philosophy is to break the domination of words over the human mind [...], then my concept notation, being developed for these purposes, can be a useful instrument for philosophers [...] I believe the cause of logic has been advanced already by the invention of this concept notation." (Preface to the *Begriffsschrift*)

The *Begriffsschrift* was arguably the most important publication in logic since Aristotle founded the subject. Frege's motivation for developing his formal approach to logic resembled Leibniz's motivation for his calculus ratiocinator.

Frege’s Logic of Quantification

Frege's functional analysis of predication coupled with his understanding of generality freed him from the limitations of the ‘subject-predicate’ analysis of ordinary language sentences that formed the basis of Aristotelian logic and it made it possible for him to develop a more general treatment of inferences involving ‘every’ and ‘some’.

In traditional Aristotelian logic, the subject of a sentence and the direct object of a verb are not on a logical par.

The rules governing the inferences between statements with different but related subject terms are different from the rules governing the inferences between statements with different but related verb complements.

For example, in Aristotelian logic, the rule which permits the valid inference from ‘John loves Mary’ to ‘Something loves Mary’ is different from the rule which permits the valid inference from ‘John loves Mary’ to ‘John loves something’.

In Frege's logic a single rule governs both the inference from ‘John loves Mary’ to ‘Something loves Mary’ and the inference from ‘John loves Mary’ to ‘John loves something’.

That's because the subject John and the direct object Mary are both considered on a logical par, as arguments of the function *loves*.

In effect, Frege saw no logical difference between the subject ‘John’ and the direct object ‘Mary’.

What is logically important is that ‘loves’ denotes a function of 2 arguments. Thus, Frege analyzed the above inferences in the following general way:

John loves Mary. Therefore, some *x* is such that *x* loves Mary.

John loves Mary. Therefore, some *x* is such that John loves *x*.

Frege saw the formulae of mathematics as the paradigm of clear, unambiguous writing. In order to make his logical language suitable for purposes other than arithmetic, he expanded the notion of function to allow arguments and values other than numbers.

He defined a *concept* (*Begriff*) as a function that has a truth-value, either of the abstract objects the True or the False, as its value for any object as argument.

The concept *being human* is understood as a function that has the True as value for any argument that is human, and the False as value for anything else.

The values of such concepts could then be used as arguments to other functions.

Frege would analyze the sentence

"all horses are mammals"

using the logical relationship *if* . . . *then* . . . :

"*if x* is a horse, *then x* is a mammal"

Likewise, he would analyze the sentence

"some horses are pure-bred"

using the logical relationship . . . *and* . . . :

"*x* is a horse *and x* is pure-bred"

[....]

$(\forall x)$(*if x* is a horse, *then x* is a mammal)

$(\exists x)$(*x* is a horse *and x* is pure-bred)

[...]

$(\forall x)(\text{ horse}(x) \supset \text{mammal}(x))$

$(\exists x)(\text{ horse}(x) \wedge \text{pure-bred}(x))$

“Tutti gli uomini sono mortali” si può riscrivere:

$$\forall x (U(x) \rightarrow M(x))$$

che si può leggere:

“per tutte le x, se x è un uomo, allora x è mortale”

“Tutti i ragazzi amano qualche fanciulla” si può riscrivere:

$$\forall x \exists y (R(x) \& F(y) \rightarrow A(x,y))$$

che si può leggere:

"per tutti gli x, esiste un y tali che,
se x è un ragazzo e y una ragazza, allora x ama y"

Alla fine del primo capitolo della *Ideografia* Frege presenta la tavola delle opposizioni aristotelica con la sua scrittura:

(A)	**Tutti gli F sono G**	$\forall x (F(x) \rightarrow G(x))$
(E)	**Nessun F è G**	$\forall x (F(x) \rightarrow \neg G(x))$
(I)	**Qualche F è G**	$\exists x (F(x) \& G(x))$
(O)	**Qualche F non è G**	$\exists x (F(x) \& \neg G(x))$

Il limite di Boole, rispetto al progetto di Leibniz, è che l'algebra della logica ci fornisce solo un calcolo; la risposta di Frege è di accoppiare il calcolo a una lingua universale, secondo uno schema che potremmo inquadrare, come si fa solitamente in molti manuali di logica, nel modo seguente:

SISTEMA FORMALE

LINGUAGGIO	CALCOLO (Apparato Deduttivo)
Vocabolario	Assiomi
Regole di Buona Formazione	Regole di Trasformazione
Formule Ben Formate	Teoremi

The *Grundgesetze 1*

In 1893 **Die Grundgesetze der Arithmetik**, Volume1 (The Basic Laws of Arithmetic) appeared in which Frege set up a formal logical system with more rules of inference than that of his Begriffsschrift.

Now Frege axiomatized arithmetic with an intuitive collection of axioms, and proofs of number theory results which had only sketched earlier he now gave formally.

The main thrust of this volume was to develop the rules of number theory and in the later volumes Frege intended to extend the work to the real numbers.

His bitter disappointment at the lack of reaction to his earlier work shows explicitly in the Preface to Volume 1 where he complains about other authors being unfamiliar with his ideas.

He must have hoped that this first volume of what he viewed would be his greatest achievement would be well received, but except for one review by Peano, it was ignored by his contemporaries.

The *Grundgesetze 2*

While Volume 2 of The Basic Laws of Arithmetic was at the printers Frege received a letter (on 16 June 1902) from Bertrand Russell. Russell pointed out, with great modesty, that the Russell paradox gave a contradiction in Frege's system of axioms.

After many letters between the two, Frege modified one of his axioms and explains in an appendix to the book that this was done to restore the consistency of the system.

However with this modified axiom, many of the theorems of Volume 1 do not go through and Frege must have known this.

He probably never realised that even with the modified axiom the system is inconsistent since this was only shown by Leshniewski after Frege's death.

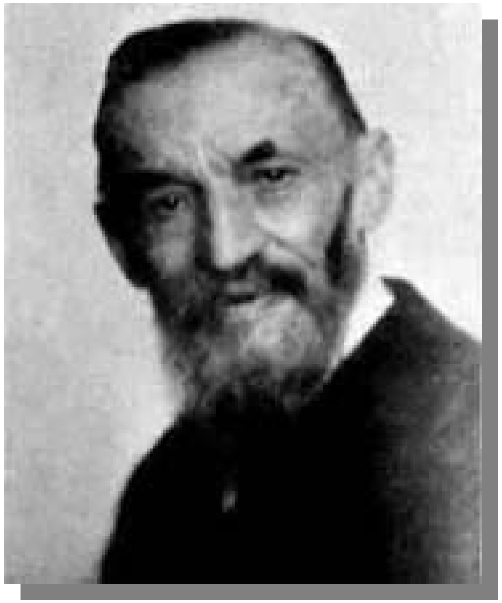
The Peano Axioms

Giuseppe Peano (1858-1932) mostrò nel 1889 che l’intera teoria dei numeri naturali può essere dedotta da tre idee primitive e da cinque proposizioni fondamentali in aggiunta a quelle della logica pura.

Le tre idee primitive della aritmetica di Peano sono:

0, numero, successore

Le cinque proposizioni primitive sono, invece:



1. 0 è un numero
2. Il successore di ogni numero è un numero
3. Due numeri distinti non possono avere lo stesso successore
4. 0 non è il successore di alcun numero.
5. Se una proprietà vale per 0, ed anche per il successore di ogni numero che abbia quella proprietà, allora vale per tutti i numeri (**principio di induzione**)

Russell and Whitehead's *Principia Mathematica*

Bertrand Russell met Peano at the 1900 International Congress of Mathematicians in Paris, and was captivated by Peano's work on foundations.

And, starting in 1900, he was studying the *Grundgesetze I* of Frege. This led to his discovery of the famous contradiction in Frege's system in June, 1901, while writing his *Principles of Mathematics (1903).* Nonetheless, Russell and Whitehead, who started their joint work on foundations in 1900, would carry out the program of Frege to a significant extent, namely the seamless development of mathematics from a few clearly stated axioms and rules of inference in pure logic.

However they opted for the more modern notation of Peano instead of Frege's Begriffsschrift.

Their work, *Principia Mathematica,* filled three volumes, almost 2,000 pages, and appeared in the years 1910-1913. Their approach was essentially that of Frege, to define mathematical entities, like numbers, in pure logic and then derive their fundamental properties. Indeed their definition of natural numbers was essentially that of Frege, but unlike him, they opted to avoid the philosophical aspects and justifications.

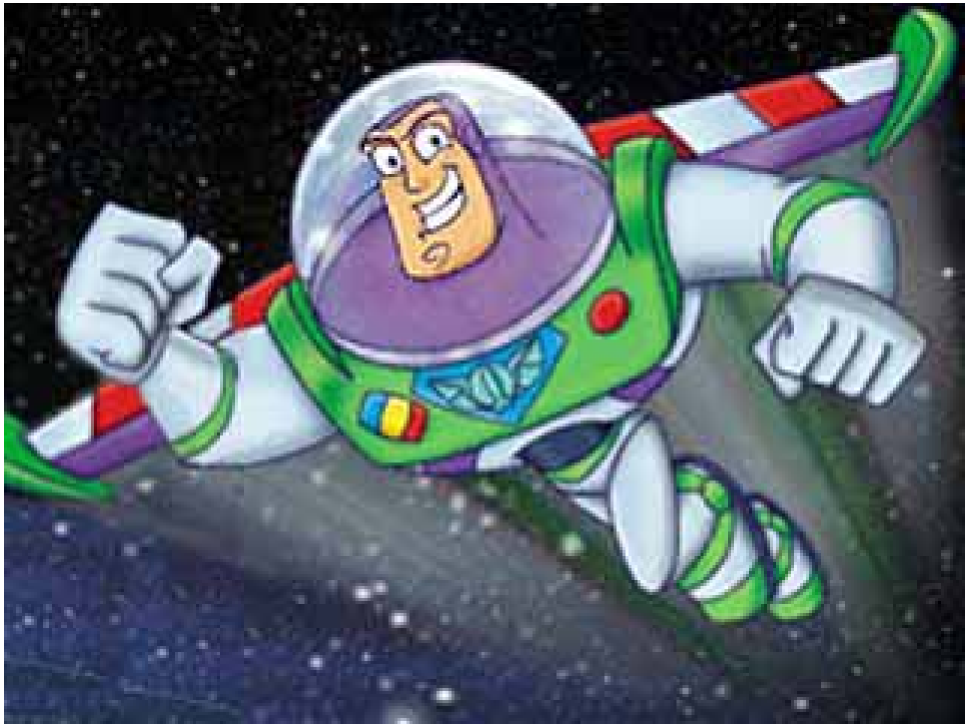
In the preface they say

- *We have avoided both controversy and general philosophy, and made our statements dogmatic in form .*
- *The general method which guides our handling of logical symbols is due to Peano. His great merit consists not so much in his definite logical discoveries nor in the details of his notations (excellent as both are), as in the fact that he first showed how symbolic logic was to be freed from its undue obsession with the forms of ordinary algebra, and thereby made it a suitable instrument for research .*
- *In all questions of logical analysis, our chief debt is to Frege.*

The main innovation of *Principia Mathematica* was to introduce a stratification of Frege's formulas into types, and to use this to restrict which of Frege's formulas would be permitted in their logic.

Having salvaged Frege's logic, they proceeded to develop some of the elementary theorems of mathematics, covering far more ground than Frege.

“To Infinity, and Beyond!”



Georg Cantor (1845-1918)

Georg Ferdinand Ludwig Philipp Cantor was a German mathematician who is best known as the creator of set theory.

Cantor established the importance of one-to-one correspondence between sets, defined infinite and well-ordered sets, and proved that the real numbers are "more numerous" than the natural numbers. In fact, Cantor's theorem implies the existence of an "infinity of infinities." He defined the cardinal and ordinal numbers, and their arithmetic.

Cantor's work encountered resistance from contemporaries such as Leopold Kronecker and Henri Poincaré, Nowadays, the majority of mathematicians accept Cantor's work on transfinite sets and arithmetic, recognizing it as a major paradigm shift. David Hilbert once said: **“No one shall expel us from the Paradise that Cantor has created”.**

The Paradoxes of the Infinite

«For any number there exists a corresponding even number which is its double. Hence the number of all numbers is not greater than the number of even numbers, that is, ***the whole is not greater than the part****.»*

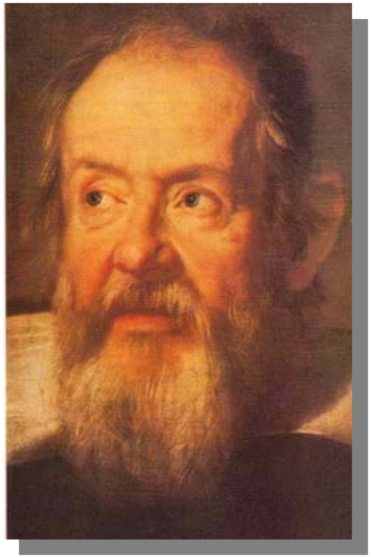
G. W. Leibniz

Cantor reasoned much as Leibniz had and faced the same dilemma: *either* it makes no sense to speak of the *number* of elements in an infinite set *or* some infinite sets will have the same number of elements as one of its subsets.

However, while Leibniz had chosen one horn of this dilemma, Cantor chose the other. He went on to develop a theory of *number* that would apply to infinite sets, and just accepted the consequence that an infinite set could have the same number of elements as one of its parts.

Galileo’s Paradox

Galileo considera i numeri naturali 0, 1, 2, 3 ... ed osserva che l'insieme (infinito) dei loro quadrati 0, 1, 4, 9, ... è certamente più piccolo e, pur tuttavia, contiene tanti elementi quanti erano i numeri di partenza, perché ad ogni numero corrisponde in modo biunivoco il suo quadrato. Galileo conclude:



*«io non veggo che ad altra decisione si possa venire che a dire infiniti essere tutti i numeri, infiniti i quadrati […]
né la moltitudine de' quadrati essere minore di quella di tutti numeri, né questa essere maggiore di quella, ed, in ultima conclusione, gli attributi di eguale, maggiore e minore non aver luogo negl'infiniti ma solo nelle quantità terminate […] queste son di quelle difficoltà che derivano dal discorrer che noi facciamo col nostro intelletto finito intorno all'infinito, dandogli quegli attributi che noi diamo alle cose finite e terminate; il che penso che sia inconveniente.»*

Discorsi e dimostrazioni matematiche intorno a due nuove scienze (1638)

Hilbert’s Grand Hotel

Suppose you are an innkeeper at a hotel with an infinite number of rooms. The hotel is full, and then a new guest arrives. It's possible to fit the extra guest in by asking the guest who was in room 1 to move to room 2, the guest in room 2 to move to room 3, and so on, leaving room 1 vacant.

We can explicity write a segment of this mapping:

$$1 \leftrightarrow 2,\ \ 2 \leftrightarrow 3,\ 3 \leftrightarrow 4,\ \ldots,\ n \leftrightarrow n+1,\ \ldots$$

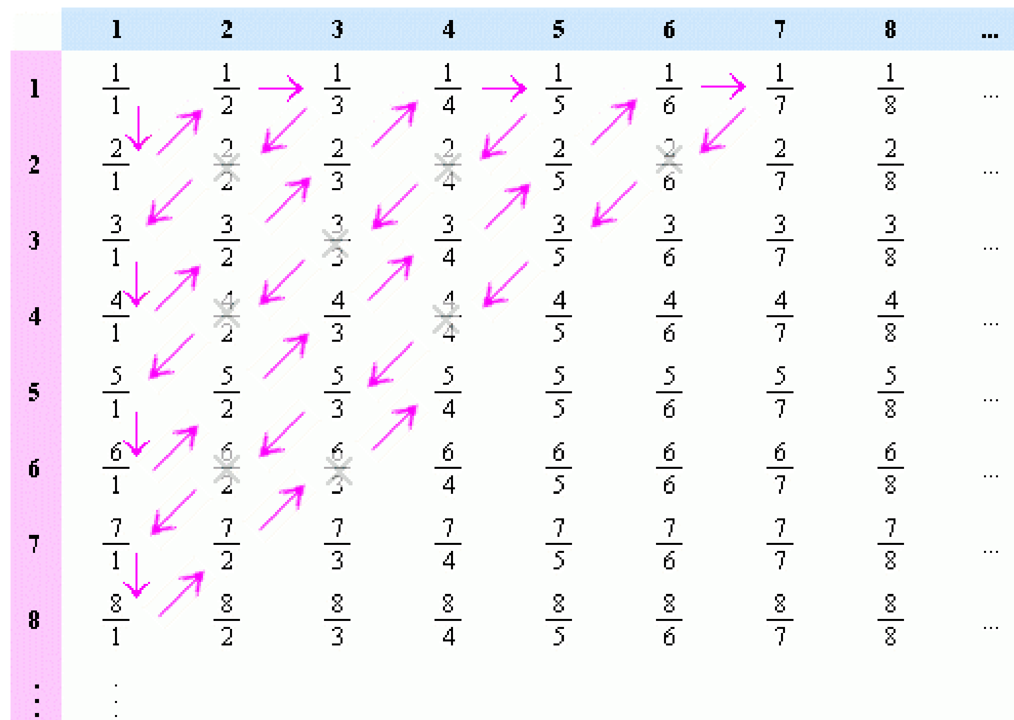
In this way we can see that the set {1,2,3,...} has the same cardinality as the set {2,3,4,...} since a bijection between the first and the second has been shown.

This motivates the definition of an infinite set being any set which has a proper subset of the same cardinality; in this case {2,3,4,...} is a proper subset of {1,2,3,...}.

There are as Many Integers as Rational Numbers!

Two sets A and B are called equivalent (A~B) if there is a bijection *f*: A −> B.

Cantor proved that the set of rational number is equivalent to (i.e. contains the same number of elements as) the set of integers.



And What About the Reals?

In a letter dated 1873 Cantor asked the German mathematician Richard Dedekind the following question:

«Take the collection of all positive whole numbers n and denote it by (n); then think of the collection of all real numbers x and denote it by (x); the question is simply whether (n) and (x) may be corresponded so that each individual of one collection corresponds to one and only one of the other?...As much as I am inclined to the opinion that (n) and (x) permit no such unique correspondence, I cannot find the reason.»

What Cantor is suggesting is that even though both the set of natural numbers and the set of real numbers are infinite, they do not have "the same number of elements."

It was Cantor himself that proved the impossibility of this correspondence and hence proved that **the set of real numbers is bigger than the set of natural numbers.**

Cantor's Diagonal Argument / 1

Cantor's diagonal argument is a proof to demonstrate that the real numbers are not countably infinite.

Cantor's original proof shows that the interval [0,1] is not countably infinite.

The proof by contradiction proceeds as follows:

Assume (for the sake of argument) that the interval [0,1] is countably infinite. There must then exist a sequence **M** in the form $(r_1, r_2, r_3, \ldots)$ that enumerates all numbers in this interval.

We may represent each of these numbers as an infinite decimal expansion.

We arrange the numbers in a list. Assume, for example, that the decimal expansions of the beginning of the sequence, **M**, are as follows:

$r_1 = 0\,.\,5\;1\;0\;5\;1\;1\;0\;\ldots$
$r_2 = 0\,.\,4\;1\;3\;2\;0\;4\;3\;\ldots$
$r_3 = 0\,.\,8\;2\;4\;5\;0\;2\;6\;\ldots$
$r_4 = 0\,.\,2\;3\;3\;0\;1\;2\;6\;\ldots$
$r_5 = 0\,.\,4\;1\;0\;7\;2\;4\;6\;\ldots$
$r_6 = 0\,.\,9\;9\;3\;7\;8\;3\;8\;\ldots$
$r_7 = 0\,.\,0\;1\;0\;5\;1\;3\;5\;\ldots$
...

We shall now construct a real number x in [0,1] by considering the k^{th} digit after the decimal point of the decimal expansion of r_k.

Cantor's Diagonal Argument / 2

The digits we will consider are underlined and in bold face, illustrating why this is called the **diagonal proof**.

$r_1 = 0\ .\ \underline{\mathbf{5}}\ 1\ 0\ 5\ 1\ 1\ 0\ \ldots$

$r_2 = 0\ .\ 4\ \underline{\mathbf{1}}\ 3\ 2\ 0\ 4\ 3\ \ldots$

$r_3 = 0\ .\ 8\ 2\ \underline{\mathbf{4}}\ 5\ 0\ 2\ 6\ \ldots$

$r_4 = 0\ .\ 2\ 3\ 3\ \underline{\mathbf{0}}\ 1\ 2\ 6\ \ldots$

$r_5 = 0\ .\ 4\ 1\ 0\ 7\ \underline{\mathbf{2}}\ 4\ 6\ \ldots$

$r_6 = 0\ .\ 9\ 9\ 3\ 7\ 8\ \underline{\mathbf{3}}\ 8\ \ldots$

$r_7 = 0\ .\ 0\ 1\ 0\ 5\ 1\ 3\ \underline{\mathbf{5}}\ \ldots$

...

From these digits we define the digits of x as follows:.

- if the k^{th} digit of r_k is 5 then the k^{th} digit of x is 4

- if the k^{th} digit of r_k is not 5 then the k^{th} digit of x is 5

For the above sequence, for example, we obtain the following decimal expansion:

$$x = 0\ .\ 4\ 5\ 5\ 5\ 5\ 5\ 4\ \ldots$$

Hence we must have $r_n = x$ for some n, since we have assumed that $(r_1, r_2, r_3, \ldots)$ enumerates all real numbers in [0, 1].

However, because of the way we have constructed it, x differs in the n^{th} decimal place from r_n, so x is not in the sequence $(r_1, r_2, r_3, \ldots)$.

This sequence is therefore not an enumeration of the set of all reals in the interval [0,1]. **A contradiction**.

It is a direct corollary of this result that the set **R** of all real numbers is uncountable.

If **R** were countable, we could enumerate all of the real numbers in a sequence, and then get a sequence enumerating [0,1] by removing all of the real numbers outside this interval. But we have just shown that this latter list cannot exist.

Alternatively, we could show that [0,1] and **R** are the same size by constructing a bijection between them. This is slightly awkward to do, though possible, for the closed interval [0,1]; for the open interval (0,1) we might use $f : (0,1) \to \mathbb{R}$ defined by

$$f(x) = \tan\left(\pi\left(x - \frac{1}{2}\right)\right).$$

The Continuum Hypothesis

The continuum hypothesis states the following:

There is no set whose size is strictly between that of the integers and that of the real numbers.

Or mathematically speaking, denoting the cardinality for the integers by $\aleph_0$ ("aleph-null"), it can be shown that the cardinality of the real numbers is $2^{\aleph_0}$. Hence, the continuum hypothesis says:

$$\neg\exists \mathbb{A} : \aleph_0 < |\mathbb{A}| < 2^{\aleph_0}.$$

This is equivalent to:

$$|\mathbb{R}| = \aleph_1$$

David Hilbert (1862-1943)

David Hilbert was one of the most influential mathematicians of the 19th and early 20th centuries.

He established his reputation by inventing or developing a broad range of ideas, such as invariant theory, the axiomization of geometry, and the notion of Hilbert space, one of the foundations of functional analysis. Hilbert and his students supplied significant portions of the mathematic infrastructure required for quantum mechanics and general relativity.

He is one of the founders of proof theory, mathematical logic, and the distinction between mathematics and metamathematics, and warmly defended Cantor's set theory and transfinite numbers.

Is 1900 it presented a set of problems that set the course for much of the mathematical research of the 20th century.

The Young Hilbert

Hilbert's first work on invariant functions led him to the demonstration in 1888 of his famous *finiteness theorem*.

Twenty years earlier, **Paul Gordan** had demonstrated the theorem of the finiteness of generators for binary forms using a complex computational approach. The attempts to generalize his method to functions with more than two variables failed because of the enormous difficulty of the calculations involved.

Hilbert realized that it was necessary to take a completely different path. As a result, he demonstrated *Hilbert's basis theorem*: showing the existence of a finite set of generators, for the invariants of quantics in any number of variables, but in an abstract form. That is, while demonstrating the existence of such a set, it was not algorithmic but an existence theorem.

Theology or Mathematics?

Hilbert sent his results to the *Mathematische Annalen*.

Gordan, the house expert on the theory of invariants for the *Mathematische Annalen*, was not able to appreciate the revolutionary nature of Hilbert's theorem and rejected the article, criticizing the exposition because it was insufficently comprehensive.

His comment was:

This is Theology, not Mathematics!

... I am not prepared to alter or delete anything, and regarding this paper, I say with all modesty, that this is my last word so long as no definite and irrefutable objection against my reasoning is raised.

D. Hilbert

Later, after the usefulness of Hilbert's method was universally recognized, Gordan himself would say:

I must admit that even theology has its merits.

Paris, 1900

The mathematicians present at an international conference in Paris in August 1900 inevitably wondered what the new century would bring to their subject.

It was on a sultry day that the 38-year-old David Hilbert, whose stunning accomplishments had taken him to the top of his profession, was delivering an invited address in which he presented, as a challenge to the mathematicians of the twentieth century, 23 problems that seemed utterly inaccessible by the methods available at the time.

Among which:

1. The continuum hyopthesis
2. The consistency of arithmetic
3. …

The Compatibility of the Arithmetical Axioms

«When we are engaged in investigating the foundations of a science, we must set up a system of axioms which contains an exact and complete description of the relations subsisting between the elementary ideas of that science. […]

Upon closer consideration the question arises: **Whether, in any way, certain statements of single axioms depend upon one another, and whether the axioms may not therefore contain certain parts in common, which must be isolated if one wishes to arrive at a system of axioms that shall be altogether independent of one another**.

But above all I wish to designate the following as the most important among the numerous questions which can be asked with regard to the axioms: **To prove that they are not contradictory, that is, that a definite number of logical steps based upon them can never lead to contradictory results**.»

Hilbert’s Program

In the early 1920s, Hilbert put forward a new proposal for the foundation of classical mathematics which has come to be known as Hilbert's Program.

Hilbert’s program calls for a formalization of all of mathematics in axiomatic form, together with a proof that this axiomatization of mathematics is consistent.

Work on the program progressed significantly in the 1920s with contributions from logicians such as Paul Bernays, Wilhelm Ackermann, John von Neumann, and Jacques Herbrand. It was also a great influence on Kurt Gödel, whose work on the incompleteness theorems were motivated by Hilbert's Program.

Gödel's work is generally taken to show that Hilbert's Program cannot be carried out.

The Dispute with the Intuitionists

«Mathematics is the only science where one never knows what one is talking about nor whether what is said is true.»

(B. Russell)

«It is diffcult to see that the word *if* acquires when written ⊃, a virtue it did not possess when written "if".»

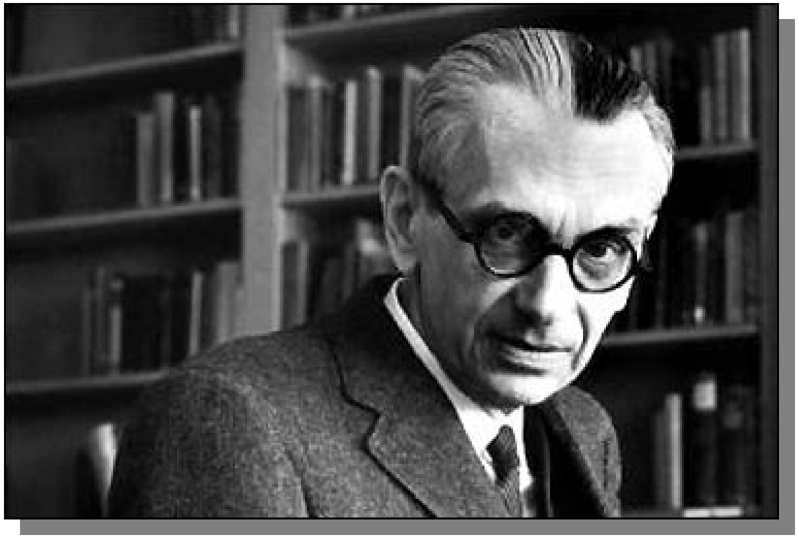
«Thus it will be readily understood that in order to demonstrate a theorem, it is not necessary or even useful to know what it means. […] we might imagine a machine where we should put in axioms at one end and take out theorems at the other, like that legendary machine in Chicago where pigs go in alive and come out transformed into hams and sausages. It is no more necessary for the mathematician than it is for these machines to know what he is doing. »

*(*H. Poincaré*, Science and Method. 1908)*

Kurt Gödel (1906-1978)

Kurt Gödel was a logician, mathematician, and philosopher of mathematics.

One of the most significant logicians of all time, Gödel's work has had immense impact upon scientific and philosophical thinking in the 20th century, a time when many, such as Bertrand Russell, A. N. Whitehead, and David Hilbert, were attempting to use logic and set theory to understand the foundations of mathematics.



Gödel is best known for his incompleteness theorems, published in 1931 when he was 25 years of age, and only one year after finishing his doctorate at the University of Vienna.

He also showed that the continuum hypothesis cannot be disproved from the accepted axioms of set theory, if those axioms are consistent. He made important contributions to proof theory by clarifying the connections between classical logic, intuitionistic logic, and modal logic.

“On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems” (1931)

«The development of mathematics towards greater exactness has, as is well-known, lead to formalization of large areas of it such that you can carry out proofs by following a few mechanical rules. The most comprehensive current formal systems are the system of Principia Mathematica (PM) on the one hand, the Zermelo-Fraenkelian axiom-system of set theory on the other hand. ***These two systems are so far developed that you can formalize in them all proof methods that are currently in use in mathematics, i.e. you can reduce these proof methods to a few axioms and deduction rules. Therefore, the conclusion seems plausible that these deduction rules are sufficient to decide all mathematical questions expressible in those systems.*** ***We will show that this is not true***, but that there are even relatively easy problem in the theory of ordinary whole numbers that can not be decided from the axioms. This is not due to the nature of these systems, but it is true for a very wide class of formal systems, which in particular includes all those that you get by adding a finite number of axioms to the above mentioned systems, provided the additional axioms don't make false theorems provable.»

The Liar Paradox

The **liar paradox** encompasses paradoxical statements such as:

“I am lying now”

“This statement is false”

Analyzing the statement "I am lying now", if what the speaker says is true, then the statement "I am lying now" is false, that means when the statement was said, the speaker was actually lying.

But then, on the contrary, if it is true that the speaker is lying, then the statement "I am lying now" is false in that the statement turns out to be true.

To avoid having a sentence directly refer to its own truth value, one can also construct the paradox as follows:

“The following sentence is true”
“The preceding sentence is false”

Eubulides and Epimenides

The oldest version of the liar paradox is attributed to the Greek philosopher **Eubulides of Miletus** who lived in the fourth century B.C. Eubulides reportedly said:

A man says that he is lying. Is what he says true or false?

Epimenides paradox is often considered an equivalent or interchangeable term for “liar paradox”.

Epimenides was a sixth century BC philosopher-poet. Himself a Cretan, he reportedly wrote:

The Cretans are always liars

The proof of Gödel's incompleteness theorem uses self-referential statements similar to the statements at work in the Liar paradox.

In the context of a sufficiently strong axiomatic system *A* of arithmetic:

This statement is not provable in A*.* (1)

Suppose (1) is provable, then what it says of itself, that it is not provable, is not true. But this conclusion is contrary to our supposition, so our supposition that (1) is provable must be false.

Suppose the contrary that (1) is not provable, then what it says of itself is true, although we cannot prove it. Therefore, there is no proof that (1) is provable, and there is also no proof that its negation is provable.

Whence, *A* is incomplete because it cannot prove all truths, namely, (1) and its negation. Statements like (1) are called **undecidable**.

Gödel Numbering / Step 1

Logical symbols	Numbers 1:12
¬	1 ("not")
∀	2 ("for all")
⊃	3 ("if,then")
∨	4 ("or")
∧	5 ("and")
(	6
)	7
S	8 ("is the successor of")
0	9
=	10
.	11
+	12
Propositional symbols	**Numbers greater than 10 but divisible by 3**
P	12
Q	15
R	18
S	21
Individual variables	**numbers greater than 10 with remainder 1 when divided by 3**
v	13
x	16
y	19
Predicate symbols	**numbers greater than 10 with remainder 2 when divided by 3**
E	14
F	17
G	20

Gödel Numbering / Step 2

Arithmetical statements are assigned unique Gödel numbers referenced to the series of prime numbers. This is based on a simple code which essentially reads:

$$\textbf{prime\_1}^{\textbf{character\_1}} * \textbf{prime\_2}^{\textbf{character\_2}} * \textbf{prime\_3}^{\textbf{character\_3}} \ldots\text{etc.}$$

For example the statement

$\forall$ **x, P(x)** becomes $\mathbf{2^{2} * 3^{16} * 5^{12} * 7^{6} * 11^{16} * 13^{7}}$,

Because {2, 3, 5, 7, 11, ...} is the prime series, and 2, 16, 12, 6, 16, 7 are the relevant character codes.

It is important to note, by the Fundamental theorem of arithmetic, this astronomical number can be decomposed into unique prime factors; thus it is possible to convert the Gödel number back to its sequence of characters.

Gödel Numbering / Step 3

Finally, sequences of arithmetical statements are assigned further Gödel numbers, such that the sequence

Statement 1 (GN1)
Statement 2 (GN2)
Statement 3 (GN3)

(where GN denotes a Gödel number) gets the Gödel number

$$2^{GN1} * 3^{GN2} * 5^{GN3}$$

which we will call GN4.

The proof of Gödel's incompleteness theorem depends on the demonstration that, in formal arithmetic, some sets of statements logically entail or *prove* other statements.

For example it might be shown that GN1, GN2, and GN3 together, i.e. GN4, *prove* GN5.

Because this is a demonstrable relationship between numbers it is entitled to its own symbol, for example **R**.

Then we could write **R(v,x)** to express "**x** *proves* **v**".

In the case where **x** and **v** are Gödel numbers GN4 and GN5 we would say **R(GN5,GN4)**.

An Informal Account of Gödel's Proof

The core of Gödel's argument is that we can write the statement

$$\forall x, \neg R(v,x)$$

which means

***no proposition of type v can be proved*.**

The Gödel number for this statement would be

$$2^{2} * 3^{16} * 5^{1} * 7^{18} * 11^{6} * 13^{13} * 17^{16} * 19^{7}$$

but we will call it GN6.

Now if we consider the statement

$$\forall\, x, \neg R(GN6,x)$$

we will realise that it says

no proposition that says
"no proposition of type v can be proved" can be proved.

This collapses into the statement:

this proposition cannot be proved.

If the statement can actually be proved, then its formal system is inconsistent because it proves something which states that it itself cannot be proven (a contradiction).

If the statement cannot be proved within its formal system, then what the statement asserts is actually true, so the statement is consistent, but since the formal system contains a statement which is semantically true but which cannot be proven (syntactically), then the system is incomplete.

Therefore, assuming that the formal system is consistent, it has to be incomplete.

J. R. Lucas:
Minds, Machines and Gödel (1961)

«Gödel's theorem seems to me to prove that Mechanism is false, that is, that minds cannot be explained as machines.

[…]

Gödel's theorem states that in any consistent system which is strong enough to produce simple arithmetic there are formulae which cannot be proved-in-the-system, but which we can see to be true. Essentially, we consider the formula which says, in effect, "This formula is unprovable-in-the-system".

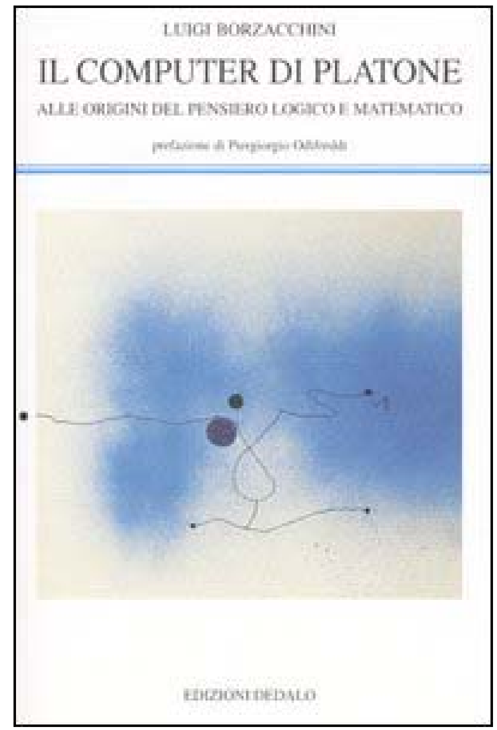
[…]

Gödel's theorem must apply to cybernetical machines, because it is of the essence of being a machine, that it should be a concrete instantiation of a formal system. It follows that given any machine which is consistent and capable of doing simple arithmetic, there is a formula which it is incapable of producing as being true---i.e., the formula is unprovable-in-the-system-but which we can see to be true.

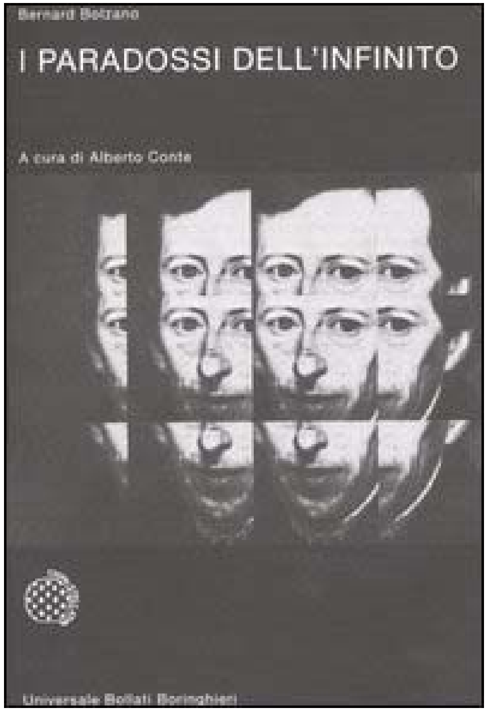
It follows that no machine can be a complete or adequate model of the mind, that minds are essentially different from machines.»

Letture (testi divulgativi)

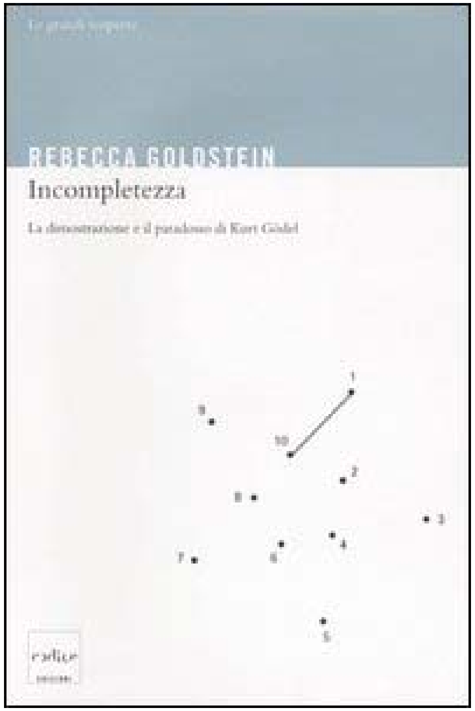
Approfondimenti



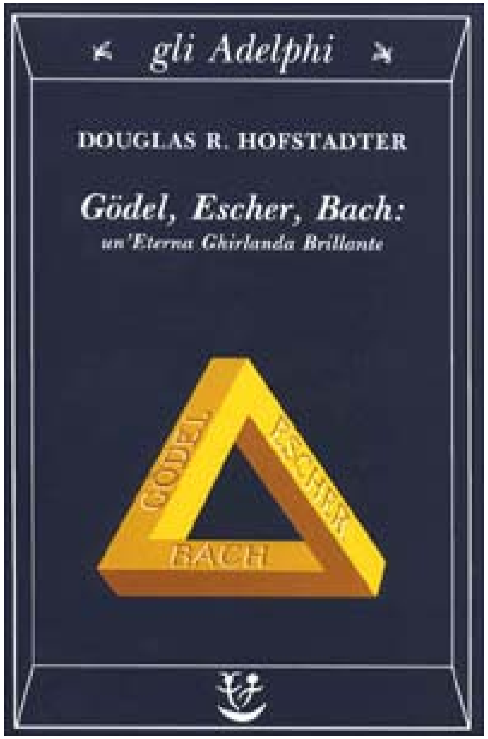
Sull'infinito

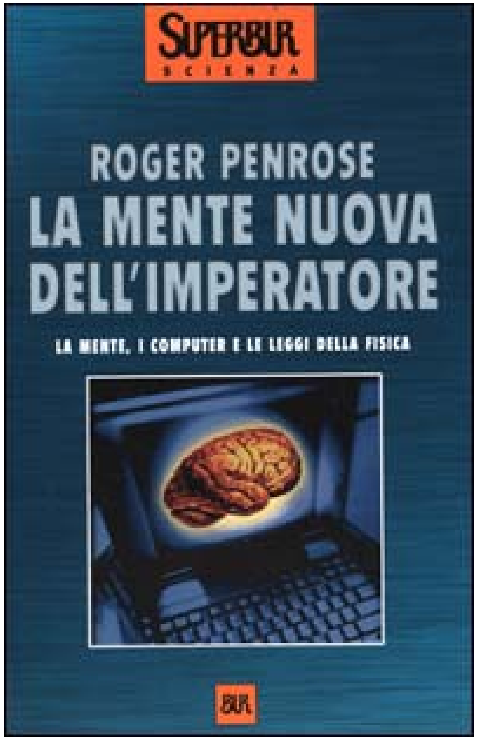


Sul Teorema di Gödel



Teorema di Gödel e Intelligenza Artificiale





Logicismo vs Intuizionismo