Storia dell’informatica

Le radici logiche dell’informatica

Parte II: Logica in età ellenistica e medievale

Folio A2r from Euclid Elements of Geometry Printed by Erhard Ratdolt in Venice: 1482.

The Stoics

Stoicism was one of the new philosophical movements of the Hellenistic period. The name derives from the porch (*stoa poikilê*) in the Agora at Athens decorated with mural paintings, where the members of the school congregated, and their lectures were held.

We do not possess a single complete work by any of the first three heads of the Stoic school:

- **Zeno** of Citium in Cyprus (344-262 BC)
- **Cleanthes** (d. 232 BC)
- **Chrysippus** (d. ca. 206 BC).

Chrysippus was particularly prolific, composing over 165 works, but we have only fragments of his works.

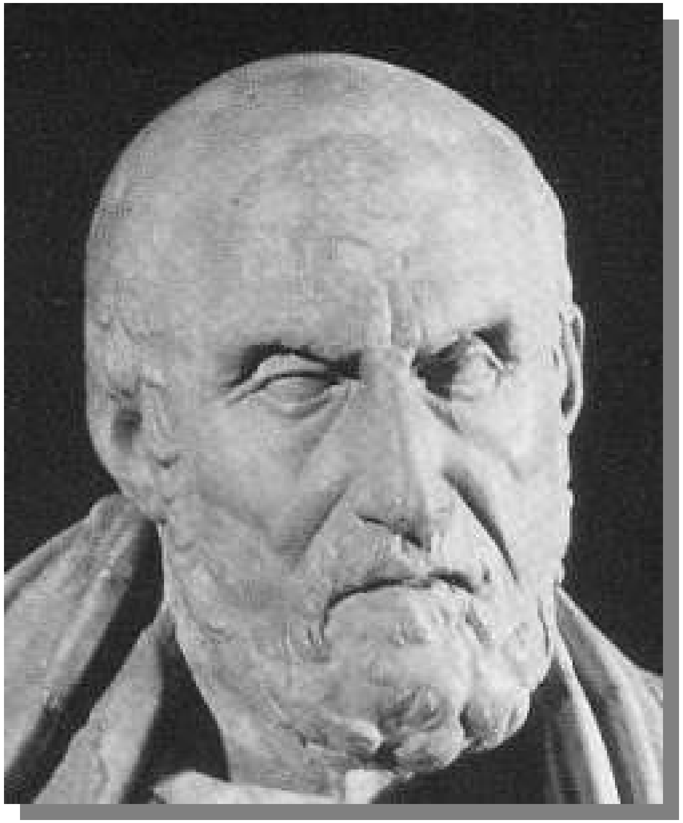
Stoic Logic

For the Stoics, the scope of what they called ‘logic’ (*logikê*, i.e. knowledge of the functions of *logos* or reason) is very wide, including not only the analysis of argument forms, but also rhetoric, grammar, the theories of concepts, propositions, perception, and thought, and what we would call epistemology and philosophy of language.

Formally, logic was standardly divided into just two parts: **rhetoric** and **dialectic** (Diog. Laert., 31A).

In general, one may say that theirs is a logic of propositions rather than a logic of terms, like the Aristotelian syllogistic

Chrysippus, the Father of Propositional Logic



Between Aristotle, writing in the fourth century BC, and Boole (1847), writing more than two millennia later, only one logician published a system of logic.

That was **Chrysippus** (c. 280-207 BC), the third head of the Stoic school. Chrysippus' system of propositional logic was dominant for 400 years, until bits of it were eventually absorbed into a confused amalgamation with Aristotle's categorical logic

Stoic Logic

Let us shortly resume what we know about Stoic logic:

1. The Stoics paid attention to the form of an expression, seeing that in some cases this determines meaning.

2. The Stoics had deep insights about what is important in logic. They discussed, for instance, the nature of implication and distinguished various ways of understanding conditional statements.

3. The Stoics had created the calculus for the purpose of reasoning and focused their attention on what conclusion follows from premises. So they used arguments. This is a prototype of today's sequent, meaning an ordered pair of sequences of formulas.

Propositions

We have Chrysippus’ own definition of a proposition (*axiôma*):

«A proposition is that which is either true or false, or a thing complete in itself which is assertible insofar as concerns itself, as Chrysippus says in his Dialectical Definitions: ‘A proposition is that which is assertible or affirmable insofar as concerns itself, for example It is day, Dion is walking.’ ... A proposition is what we assert when speaking, which is either true or false.»

(Diocles 7.65-66; cf. A.L. 2.73-74, Gellius 16.8.4 = FDS 877)

Stoic "Axiomata" *vs* Aristotle's "Protaseis"

Stoic propositions should be sharply distinguished from the so-called "propositions" (*protaseis*) of Aristotle, which are sentences in which one thing is affirmed or denied of one thing (*Prior Analytics* A1.24a16-17).

An Aristotelian *protasis* is a certain kind of simple linguistic entity.

A Stoic *axiôma* in contrast is a non-linguistic and non-existent incorporeal which need not be simple.

Much confusion has resulted in the western logical tradition from the use of the same word "proposition" for both Aristotle's *protasis* and the Stoics' *axiôma*.

Non-simple Propositions

Propositions can be combined to get non-simple propositions.

Four types of non-simple propositions are noticed in the primitives of Stoic propositional logic:

- negations (*apophatika*),
- conjunctions (*sumpeplegmena*),
- conditionals (*sunêmmena*) and
- disjunctions (*diezeugmena*).

Negations

The **negation** of a proposition is the proposition formed by prefixing a negative, *apophasis* (*ouk* or *ouchi*, English *not*) to the proposition (Diocles 7.69, A.L. 2.88-90).

This formation rule reflects a clear understanding that the scope of the negative *not* is an entire proposition and not, for example, the predicate in a simple proposition.

The Stoic practice was to write *Not it is day* for the negation of the proposition *It is day*.

Negation is classically truth-functional: the negation of a true proposition is false, and of a false proposition true (A.L. 2.103).

Conjunctions

A **conjunction** is "a proposition which is conjoined by some conjunctive connectives, for example, *Both it is day and it is light*" (Diocles 7.72).

The formation rule allows more than two conjuncts, as other examples in our sources attest (cf. e.g. Gellius 16.8.10 = FDS 967).

Diocles' example indicates that an initial conjunctive connective was required, as is necessary to avoid syntactic ambiguity when a conjunction is negated.

The conjunctive connective is classically truth-functional: a conjunction is true if all its conjuncts are true and false if a conjunct is false (A.L. 2.125, Gellius 16.8.11 = FDS 967).

Disjunctions

A **disjunction** is “that which is disjoined by the disjunctive connective *either*, for example *Either it is day or it is night*.” (Diocles 7.72).12

As with the conjunction, the initial *either* (Greek *êtoi*) prevents syntactic ambiguity, e.g. when a disjunction is negated.

The definition, and examples elsewhere, indicate that there are disjunctions with more than two disjuncts, e.g. *Either pleasure is evil or pleasure is good or both not pleasure is good and not pleasure is bad* (Gellius 16.8,12 = FDS 976).

Our sources convey a confused message about the truth conditions for a disjunction.

According to Diocles, the disjunctive connective “declares that one or the other of the <disjoined> propositions is false” (7.72). This is truth-functional exclusive disjunction.

Conditionals

A **conditional** is:

«as Chrysippus says in his Dialectical Definitions ..., that which is put together by the conditional connective if. This connective declares that the second follows from the first, for example If it is day, it is light.»

(Diocles 7.71; cf. A.L. 2.109-111, Gellius 16.8.9 = FDS 953)

Stoic Interpretations of Conditionals

Stoic logicians took the assertion of a conditional as a statement that its consequent follows from its antecedent. And they took this condition to be met if and only if the contradictory of the consequent “conflicts with” (*machetai*, literally “battles with”) its antecedent:

« A conditional is true in which the contradictory of the consequent conflicts with the antecedent, for example If it is day, it is light. This is true, for Not it is light, the contradictory of the consequent, conflicts with It is day.

A conditional is false in which the contradictory of the consequent does not conflict with the antecedent, for example If it is day, Dion is walking. For Not Dion is walking does not conflict with It is day. »

(Diocles 7.73; cf. P.H. 2.111)

Other Interpretations

The contrast is to the criteria of **Philo** and **Diodorus Cronus**, predecessors of the Stoics in the so-called "dialectical school".

A conditional is true for Philo whenever it does not have a true antecedent and a false consequent, and for Diodorus if it never has a true antecedent and a false consequent (P.H. 2.110-11).

Philo's criterion is met at any time when a conditional has either a false antecedent or a true consequent, e.g. *If it is day, Dion is walking* whenever *It is day* is false or *Dion is walking* is true.

Diodorus' criterion is met if a conditional has either an always false antecedent or an always true consequent, e.g. *If not there are partless elements of existents, Dion is walking* or *If Dion is walking, there are partless elements of existents*.

Philo's "Material" Implication is Modern Implication

Philo's interpretation corrsponds to the modern one used today in logic and Computer Science.

p	q	$p \rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

Some Odd Properties of the Material Conditional

Any conditional $p \rightarrow q$ with a false antecedent p is true, no matter what the consequent q is, and no matter whether there is any kind of link between the two.

For example, the proposition 'If Sydney is the capital of Australia then Shakespeare wrote *Hamlet*' is true, simply because of the falsehood of its antecedent (the capital is in fact Canberra).

We could replace the consequent by any other proposition, even its own negation, and the material conditional would remain true.

The “Undemonstrated”…

1. *If the first, then the second; but the first; therefore the second.*
2. *If the first, then the second; but not the second; therefore, not the first.*
3. *Not both the first and the second; but the first; therefore, not the second.*
4. *Either the first or the second [and not both]; but the first; therefore, not the second.*
5. *Either the first or the second; but not the second; therefore the first.*

… in Modern Notations

(R1)	$A \to B,\ A \vdash B$	*modus ponens*
(R2)	$A \to B,\ \neg B \vdash \neg A$	*rule of contraposition*
(R3)	$\neg(A \wedge B),\ A \vdash \neg B$	*disjunctive syllogism*
(R3′)	$\neg(A \wedge B),\ B \vdash \neg A$	
(R4)	$A \veebar B,\ A \vdash \neg B$	*rules for the (excluding) disjunction*
(R4′)	$A \veebar B,\ B \vdash \neg A$	
(R5)	$A \veebar B,\ \neg A \vdash B$	
(R5′)	$A \veebar B,\ \neg B \vdash A$	

The “Themata”

Besides the above-mentioned undemonstrated (primitive) arguments, the Stoics also used four rules called *themata*.

We know only two of these four rules:

$$\text{(MT1)} \quad \frac{X, A, B \vdash C}{X, A, \neg C \vdash \neg B} \qquad\qquad \text{(MT1}'\text{)} \quad \frac{X, A, B \vdash C}{X, B, \neg C \vdash \neg A}$$

$$\text{(MT3)} \quad \frac{X, A \vdash C \qquad Y \vdash A}{X, Y \vdash C}$$

The rule (MT3) is known as the *cut rule*.

According to some scholars a certain version of the third *thema* or maybe one of the missing *themata* was the so-called *theorema:*

$$\text{(TH)} \quad \frac{X \vdash A \qquad X, A \vdash C}{X \vdash C}$$

Euclid’s Elements and Deductive Sciences

Euclid of Alexandria (ca. 325 BC–265 BC) was a Hellenistic mathematician who lived in Alexandria, Egypt almost certainly during the reign of Ptolemy I (323 BC–283 BC).

Often considered as the "father of geometry", his most popular work is ***Elements***, which is considered to be one of the most successful textbooks in the history of mathematics.

Definitions, Axioms, and Postulates

In *Posterior Analytics*, Aristotle presents a detailed discussion of the role of *first principles* in demonstrative sciences. First principles are those concepts or assertions which remain unproved. Their truth is assumed and from them other assertions are proved.

The first principles of Aristotle may be classified into three types: *definitions*, *axioms*, and *postulates*.

A **definition** is a statement which requires only an understanding of the terms being used. It says nothing about the existence of the thing being defined; this must be proved separately. For example, defining what is meant by the term ``circle'' does not imply that such an object exists.

An **axiom** or (*common notion*) is an assertion, the truth of which is taken for granted as being blatantly obvious, and which is applicable -- by analogy, at least -- in all sciences. An example is that things equal to the same thing are equal to each other; this is the first axiom in the *Elements*.

Definitions, Axioms, and Postulates / 2

Postulates, like axioms, are assumed without proof. However, whereas modern mathematicians tend not to make any distinction between the two, the ancient Greeks did.

Aristotle gives three ways of differentiating between postulates and axioms:

1. Postulates are not self-evident, as are axioms.

2. Postulates are applicable only to the specific science being considered, whereas axioms are more general.

3. Postulates assert that something exists, whereas axioms do not.

First Principles

Euclid based his work in Book I on

- 23 **definitions** (such as point, line and surface)

- 5 **postulates**

- 5 "**common notions**" (both of which are today called axioms).

Example Definitions from Book I

Definition 1. A *point* is that which has no part.

Definition 2. A *line* is breadthless length.

Definition 3. The ends of a line are points.

[…]

Definition 10. When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right,* and the straight line standing on the other is called a *perpendicular* to that on which it stands.

Definition 11. An *obtuse angle* is an angle greater than a right angle.

Definition 12. An *acute angle* is an angle less than a right angle.

[…]

Definition 15. A *circle* is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.

Definition 16. And the point is called the *center* of the circle.

Definition 17. A *diameter* of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.

[…]

Postulates in Book I

1. A straight line segment can be drawn by joining any two points.
2. A straight line segment can be extended indefinitely in a straight line.
3. Given a straight line segment, a circle can be drawn using the segment as radius and one endpoint as center.
4. All right angles are equal to one another.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

Common Notions in Book I

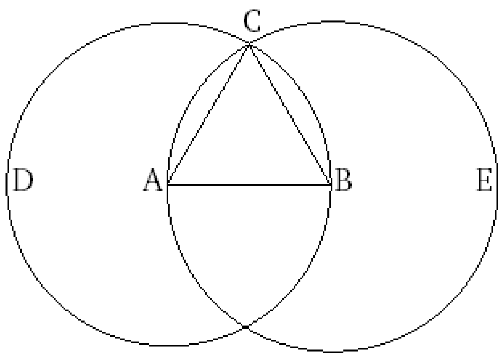
1. Things which equal the same thing are equal to one another.
2. If equals are added to equals, then the sums are equal.
3. If equals are subtracted from equals, then the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

These basic principles reflect the interest of Euclid, along with his contemporary Greek and Hellenistic mathematicians, in constructive geometry.

The first three postulates basically describe the constructions one can carry out with a compass and an unmarked straightedge.

A marked ruler is forbidden

Book I, Proposition I



To construct an equilateral triangle on a given finite straight-line.

Proof. Let AB be the given finite straight-line. So it is required to construct an equilateral triangle on the straight-line AB.

Let the circle BCD with center A and radius AB have been drawn [Post. 3], and again let the circle ACE with center B and radius BA have been drawn [Post. 3]. And let the straight-lines CA and CB have been joined from the point C, where the circles cut one another, to the points A and B (respectively) [Post. 1].

And since the point A is the center of the circle CDB, AC is equal to AB [Def. 1.15]. Again, since the point B is the center of the circle CAE, BC is equal to BA [Def. 1.15]. But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, CA is also equal to CB. Thus, the three (straight-lines) CA, AB, and BC are equal to one another.

Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB. (Which is) the very thing it was required to do.

Critiques of the Proof and the Need for Rigor

It is surprising that such a short, clear, and understandable proof can be so full of holes. These are logical gaps where statements are made with insufficient justification.

Having the first proof in the *Elements* this proposition has probably received more criticism over the centuries than any other.

Example: Why does the point *C* exist? Near the beginning of the proof, the point *C* is mentioned where the circles are supposed to intersect, but there is no justification for its existence. The only one of Euclid's postulate that says a point exists the parallel postulate, and that postulate is not relevant here. Thus, there is no assurance that the point *C* actually exists.

Critiques of the Proof and the Need for Rigor / 2

Why does *ABC* contain an equilateral triangle? Proclus relates that early on there were critiques of the proof and describes that of Zeno of Sidon, an Epicurean philosopher of the early first century B.C.E. (not to be confused with Zeno of Elea famous of the paradoxes who lived long before Euclid), and whose criticisms, Proclus says, were refuted in a book by Posidonius. The critique is sound, however, and the refutation faulty.

Zeno of Sidon criticized the proof because it was not shown that the sides do not meet before they reach the vertices. Suppose *AC* and *BC* meet at *E* before they reach *C*, that is, the straight lines *AEC* and *BEC* have a common segment *EC*. Then they would contain a triangle *ABE* which is not equilateral, but isosceles.

After Aristotle and Chrysippus…

Advances in logic were undertaken in small steps in the centuries that followed.

This work was done by, for example, the second century logician **Galen** (roughly 129-210 CE), the sixth century philosopher **Boethius** (roughly 480-525 CE) and later by medieval thinkers such as **Peter Abelard** (1079-1142) and **William of Ockham** (1288-1347), and others.

Much of their work involved producing better formalizations of the principles of Aristotle or Chrysippus, introducing improved terminology and furthering the discussion of the relationships between operators.

Abelard, for example, seems to have been the first to differentiate clearly exclusive from inclusive disjunction, and to suggest that inclusive disjunction is the more important notion for the development of a relatively simple logic of disjunctions.

Medieval Theories of the Syllogism

Historically, medieval logic is divided into the old logic (*logica vetus*), the tradition stretching from Boethius (c. 480-525) until Abelard (1079-1142), and the new logic (*logica nova*), from the late twelfth century until the Renaissance.

The division reflects the availability of ancient logical texts. Before Abelard, medieval logicians were only familiar with Aristotle's *Categories* and *On Interpretation* and Porphyry's *Isagoge* or *Introduction* to the Categories and not the *Prior Analytics* — though they did know something of his theory through secondary sources.

Once the *Prior Analytics* reappeared in the West in the middle of the twelfth century, commentaries on it began appearing in the late twelfth and early thirteenth centuries.

Progress in Modal Syllogistics

Medieval logicians could not add much to Arostotle’s theory of assertoric syllogisms, though small changes were sometimes made.

It was not until the mid-fourteenth century, when John Buridan reworked logic in general and placed the theory of the syllogism in the context of the more comprehensive logic of consequence, that people's understanding of syllogistic logic began to change.

The theory of the modal syllogism, however, was incomplete in the *Prior Analytics*, and in the hands of medieval logicians it saw a remarkable development.

The first commentators tried to save Aristotle's original theory and in the course of doing so produced some interesting solutions.

The next generation of logicians simply abandoned the idea of saving Aristotle and instead introduced new distinctions and developed a completely new theory that subsumed the logic of syllogisms.

William of Ockham (c. 1288–1348)

William of Ockham was an English Franciscan friar and scholastic philosopher, from Ockham, a small village in Surrey, near East Horsley. As a Franciscan, William was devoted to a life of extreme poverty.

In mathematical logic Ockham worked towards what would later be called De Morgan's Laws and considered ternary logic, that is a logical system with three truth values, a concept that would become important in 20th century mathematics.

The ***Summa Logicae***, or **Sum of Logic**, is a textbook on logic by William of Ockham. Using the theory of inference as its uniting theme, it described and developed syllogistic logic. It was written some time before 1327, and published in Paris in 1487.

Ockham, *Summa*, II, 32

“Copulativa è quella proposizione che si compone di più categoriche unite dalla congiunzione ‘e’ o da una qualche particella equivalente a siffatta congiunzione”.

“Alla verità della copulativa si richiede che entrambe le parti siano vere, e dunque se una delle due parti è falsa, la copulativa è falsa”.

p	q	p&q
1	1	1
0	1	0
1	0	0
0	0	0

Ockham, *Summa*, II, 33

“Disgiuntiva è quella proposizione che si compone di più categoriche mediante la congiunzione ‘o’ [vel], oppure mediante un’espressione equivalente”.

“Alla verità di una disgiuntiva si richiede che una qualche sua parte sia vera”

p	q	p o q
1	1	1
0	1	1
1	0	1
0	0	0

Ockham, *Summa*, II, 32-33

"Bisogna sapere che l'opposta secondo contraddizione della copulativa è una disgiuntiva composta dalle contraddittorie delle parti della copulativa".

"Bisogna sapere che l'opposta secondo contraddizione di una disgiuntiva è una copulativa composta dalle contraddittorie delle parti della disgiuntiva".

Leggi di de Morgan

[Augustus de Morgan: 1806-1871]

Non (α & β) equivale a (non-α o non-β)

Non (α o β) equivale a (non-α & non-β)

Walter Burleigh (1275–1345)

“Bisogna sapere che, poiché condizione necessaria e sufficiente per la verità della copulativa è che entrambe le sue parti siano vere, perciò i contraddittori delle parti della copulativa sono la causa della verità della contraddittoria della copulativa; e poiché le cause della verità della proposizione disgiuntiva si convertono con tale proposizione, affermo perciò che il contraddittorio della copulativa equivale logicamente a una disgiuntiva composta dai contraddittorii delle parti della copulativa.”

“Cioè, il contraddittorio della copulativa: “Socrate corre e Platone corre” equivale alla disgiuntiva “Socrate non corre o Platone non corre”. Per cui, in breve, il contraddittorio di una copulativa è una disgiuntiva composta dalle contraddittorie delle parti della copulativa”.

(W. Burleigh, *De puritate artis logicae*, pp. 112-13)

References

Euclid's Elements.
http://aleph0.clarku.edu/~djoyce/java/elements/toc.html