The Maximum Clique Problem (MCP)

You are given:

- An undirected graph $G = (V,E)$, where
  - $V = \{1,\ldots,n\}$
  - $E \subseteq V \times V$

and are asked to

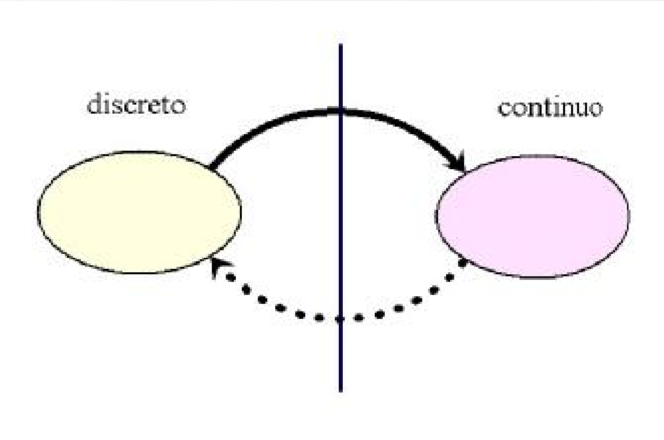
- Find the largest complete subgraph (clique) of G

The problem is known to be NP-hard, and so is problem of determining just the size of the maximum clique. Pardalos and Xue (1994) provide a review of the MCP with 260 references.

The Maximum Clique Problem (MCP)

Affrontando il problema MCP in termini di rete neurale:

- Trasformare MCP da problema discreto a problema continuo



Nell'esempio del TSP con il modello di Hopfield, non è detto che ci sia il percorso inverso (potremmo ottenere ad esempio una matrice che non ha significato); in questo nuovo problema MCP, la bidirezionalità è d'obbligo.

Some Notation

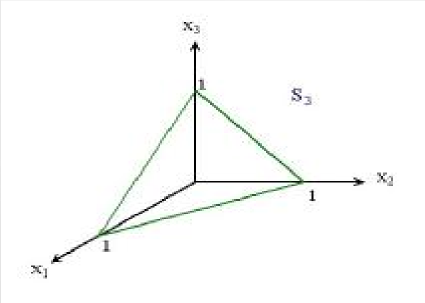
Given an arbitrary graph $G = (V,E)$ with n nodes:

- If $C \subseteq V$, x^c will denote its characteristic vector which is defined as

$$x_i^c = \begin{cases} 1/|C|, & if \;\; i \in C \\ 0, & otherwise \end{cases}$$

- S_n is the standard simplex in R^n :

$$S_n = \left\{ x \in R^n : \sum_{i=1}^{n} x_i = 1 \;\; and \;\; x_i \geq 0, \forall i \right\}$$



- $A=(a_{ij})$ is the adjacency matrix of G:

$$a_{ij} = \begin{cases} 1, & if \;\; v_i \sim v_j \\ 0, & otherwise \end{cases}$$

Infeasible Maxima in Motzkin-Straus

Si consideri la funzione continua:

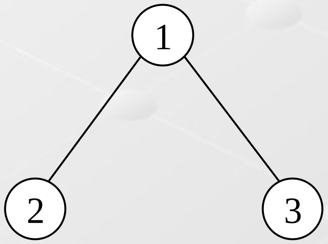
$$f(x) = x'Ax = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i\, x_j$$

dove x' è il vettore trasposto e *A* è la matrice di adiacenza.

Lagrangiano del grafo:

$$f(\bar{x}) = \sum_{i,j \in E} x_i\, x_j$$

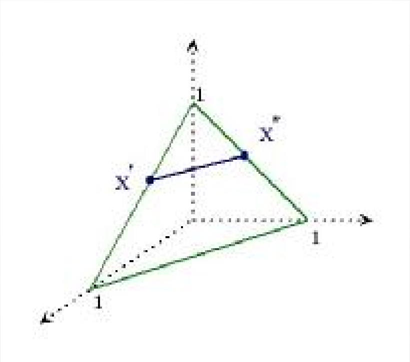
esempio:



$$f(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3$$

Continuous Formulation of MAX-CLIQUE

Il ponte che crea Motzkin-Straus è unidirezionale; solo se il vettore restituito è nella forma di vettore caratteristico allora c'è bidirezionalità.
Nell'esempio visto ci sono due massimi globali :



$$x' = \left(\frac{1}{2}, \frac{1}{2}, 0\right)^T \qquad x'' = \left(\frac{1}{2}, 0, \frac{1}{2}\right)^T$$

Si dimostra che sono massimi globali anche tutti i punti del segmento x' - x''
ovvero tutti i punti $\left(\frac{1}{2}, \frac{a}{2}, \frac{1-a}{2}\right)^T \quad \forall a \in [0,1]$; non essendo vettori caratteristici (soluzioni spurie) non è possibile estrarre la clique massima. La soluzione consiste nel sommare $\frac{1}{2}$ alla diagonale principale di *A*

$$A' = A + \frac{1}{2} I \quad \Longrightarrow \quad f(x) = x^T A' x \quad \Longrightarrow \quad f(x) = x^T \left(A + \frac{1}{2} I\right) x$$

Infeasible Maxima in Motzkin-Straus

Teorema

Dato $C \subseteq V$ e x^c vettore caratteristico allora:

- C è una clique massima di $G \Longleftrightarrow x^c$ è un massimo globale di f in S_n

- C è una clique massimale di G $\Longleftrightarrow x^c$ è un massimo locale di f in S_n

- tutti i massimi locali sono stretti e sono vettori caratteristici

Evolutionary Games

Developed in evolutionary game theory to model the evolution of behavior in animal conflicts.

Assumptions

- A large population of individuals belonging to the same species which compete for a particular limited resource
- This kind of conflict is modeled as a game, the players being pairs of randomly selected population members
- Players do not behave "rationally" but act according to a pre-programmed behavioral pattern, or *pure strategy*
- Reproduction is assumed to be asexual
- Utility is measured in terms of Darwinian fitness, or reproductive success

Notations

- $J = \{1, \ldots, n\}$ is the set of pure strategies
- $x_i(t)$ is the proportion of population members playing strategy i at time t
- The state of population at a given instant is the vector $x = (x_1, \ldots, x_n)'$
- Given a population state x, the *support* of x, denoted $s(x)$, is defined as the set of positive components of x, i.e.,

$$s(x) = \{ i \in J \ : \ x_i > 0 \}$$

Payoffs

Let $A=\left(a_{ij}\right)$ be the $n\times n$ *payoff* (or fitness) matrix.

a_{ij} represents the payoff of an individual playing strategy i against an opponent playing strategy j $\left(i, j\in J\right)$.

If the population is in state x, the expected payoff earnt by an i – strategist is:

$$p_i\left(x\right)=\sum_{j=1}^{n}a_{ij}\,x_j=\left(Ax\right)_i$$

while the mean payoff over the entire population is:

$$p\left(x\right)=\sum_{i=1}^{n}x_i\,p_i\left(x\right)=x'Ax$$

Replicator Equations

Developed in evolutionary game theory to model the evolution of behavior in animal conflicts (Hofbauer & Sigmund, 1998; Weibull, 1995).

Let $W = \left(w_{ij} \right)$ be a non-negative real-valued $n \times n$ matrix, and let

$$p_i(t) = \sum_{j=1}^{n} w_{ij}\, x_j(t)$$

Continuous-time version:

$$\frac{d}{dt} x_i(t) = x_i(t)\left(p_i(t) - \sum_{j=1}^{n} x_j(t) p_j(t) \right)$$

Discrete-time version:

$$x_i(t+1) = \frac{x_i(t) p_i(t)}{\sum_{j=1}^{n} x_j(t) p_j(t)}$$

Replicator Equations & Fundamental Theorem of Selection

S_n is invariant under both dynamics, and they have the same stationary points.

Theorem: *If* $W = W'$*, then the function*

$$F(x) = x'Wx$$

is strictly increasing along any non-constant trajectory of both continuous-time and discrete-time replicator dynamics

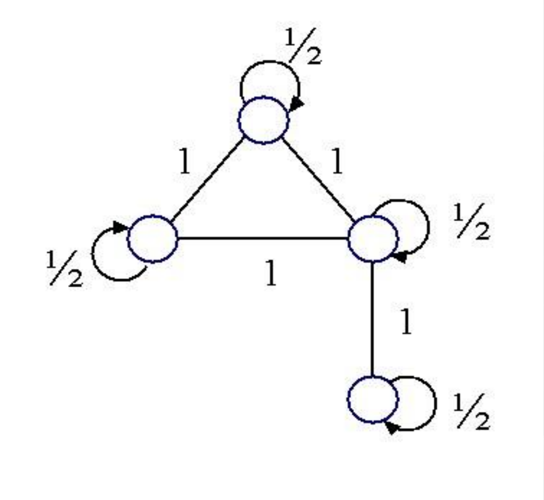
Mapping MCP's onto Relaxation Nets

To (approximately) solve a MCP by relaxation, simply construct a net having n units, and a $\{0,1\}$-weight matrix given by

$$W = A + \frac{1}{2} I_n$$

where A is the adjacency matrix of *G.*

Example:



Mapping MCP's onto Relaxation Nets

The system starting from u(0) will maximize the Motzkin-Straus function and will converge to a fixed point u^* which corresponds to a (local) maximum of f.

The value

$$k^* = \frac{1}{1-2f\left(u^*\right)}$$

can be regarded as an approximation of the maximum clique size.

Con Q–measure si misura la qualità

$$Q = \frac{f_{ave} - f_{RE}}{f_{ave} - a}$$

dove f_{ave} è il termine di confronto rispetto alla media, f_{RE} è la replicator equation e a è il valore ottimale. Quando $Q \rightarrow 1$ il risultato è buono.

Experimental Setup

Experiments were conducted over random graphs having:

- size: n = 10, 25, 50, 75, 100
- density: d = 0.10, 0.25, 0.50, 0.75, 0.90

Comparison with Bron-Kerbosch (BK) clique-finding algorithm (1974).

For each pair (n , d) 100 graphs generated randomly with size n and density $\approx d$.

The case n = 100 and d = 0.90 was excluded due to the high cost of BK algorithm.

Total number of graphs = 2400.

d \ n	10	25	50	75	100
0.10	0.99 (54)	0.99 (36)	0.99 (53)	0.97 (59)	0.92 (82)
0.25	0.99 (54)	0.99 (64)	0.99 (84)	1.00 (98)	0.97 (112)
0.50	1.00 (56)	0.99 (118)	0.99 (153)	0.96 (160)	0.90 (187)
0.75	1.00 (99)	1.00 (175)	1.00 (268)	1.00 (284)	1.00 (369)
0.90	1.00 (119)	1.00 (224)	1.00 (367)	0.99 (513)	----

Values of Q-measure for various sizes and densities