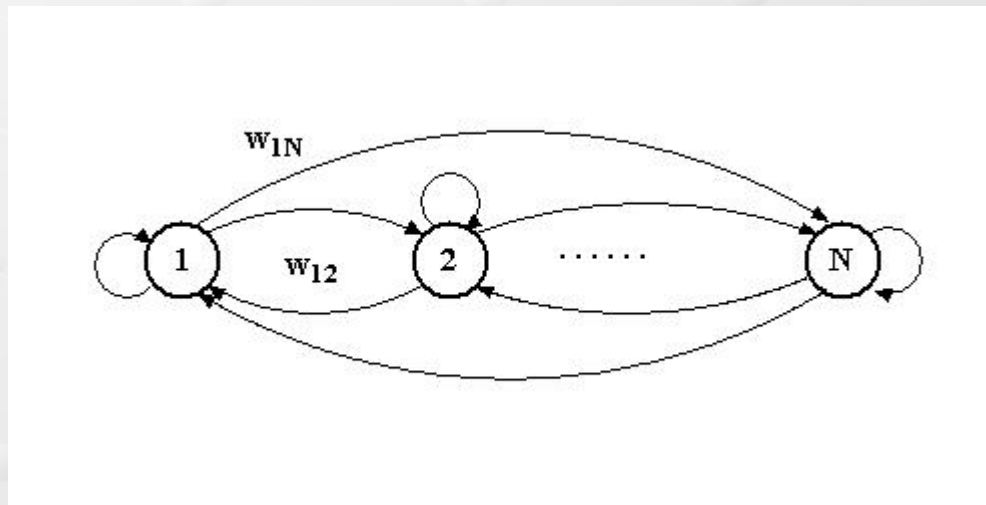


Hopfield Network

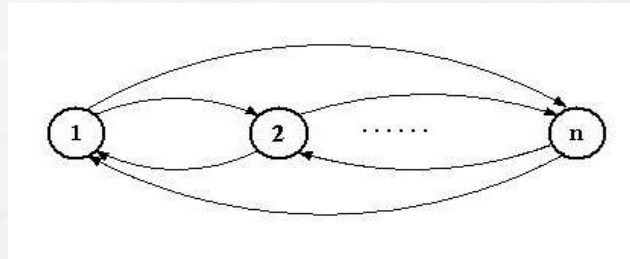
- Single Layer Recurrent Network
- Bidirectional Symmetric Connection
- Binary / Continuous Units
- Associative Memory
- Optimization Problem



Hopfield Model – Discrete Case

Recurrent neural network that uses McCulloch and Pitt's (binary) neurons.

Update rule is stochastic.



Each neuron has two “states” : V_i^L , V_i^H

$$\text{Usually : } \begin{cases} V_i^L = -1 , V_i^H = 1 \\ V_i^L = 0 , V_i^H = 1 \end{cases}$$

Input to neuron i is :

$$H_i = \sum_{j \neq i} w_{ij} V_j + I_i$$

Where:

- w_{ij} = strength of the connection from j to i
- V_j = state (or output) of neuron j
- I_i = external input to neuron i

Hopfield Model – Discrete Case

Each neuron updates its state in an *asynchronous* way, using the following rule:

$$V_i = \begin{cases} -1 & \text{if } H_i = \sum_{j \neq i} w_{ij} V_j + I_i < 0 \\ +1 & \text{if } H_i = \sum_{j \neq i} w_{ij} V_j + I_i > 0 \end{cases}$$

The updating of states is a *stochastic* process:

To select the to-be-updated neurons we can proceed in either of two ways:

- At each time step select at random a unit i to be updated (useful for simulation)
- Let each unit independently choose to update itself with some constant probability per unit time (useful for modeling and hardware implementation)

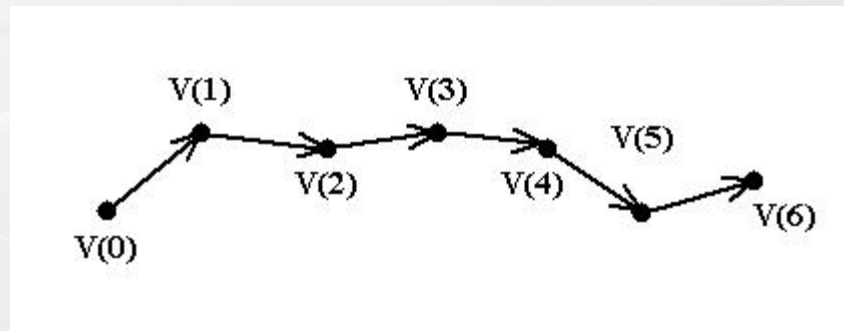
Dynamics of Hopfield Model

In contrast to feed-forward networks (which are “static”) Hopfield networks are dynamical system.

The network starts from an initial state

$$V(0) = (V_1(0), \dots, V_n(0))^T$$

and evolves in state space following a trajectory:



Until it reaches a fixed point:

$$V(t+1) = V(t)$$

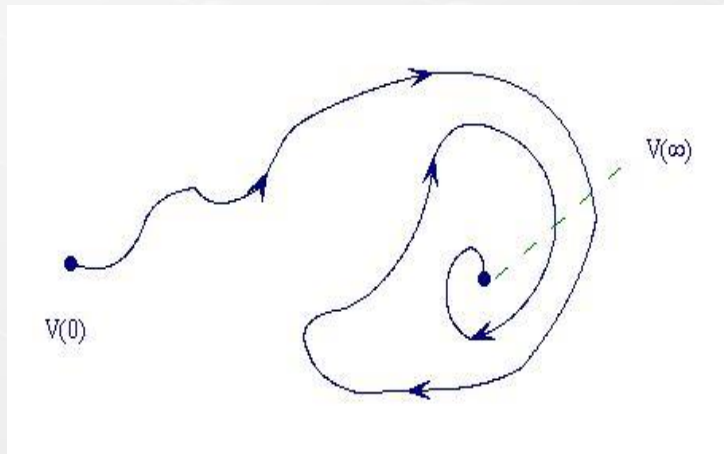
Dynamics of Hopfield Networks

What is the dynamical behavior of a Hopfield network ?

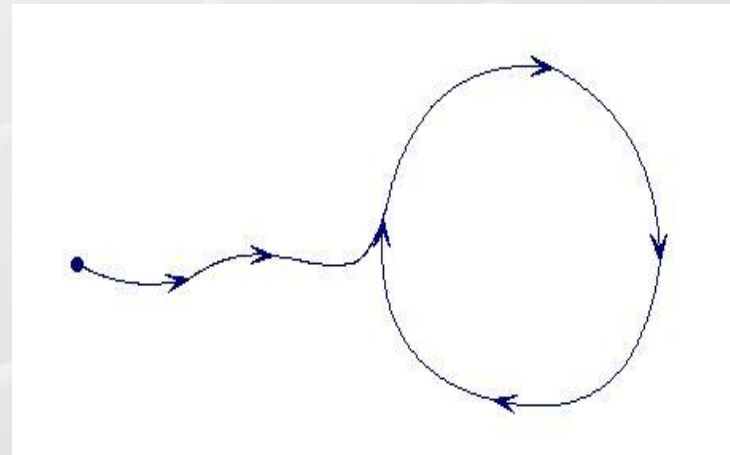
Does it converge ?

Does it produce cycles ?

Examples



(a)



(b)

Dynamics of Hopfield Networks

To study the dynamical behavior of Hopfield networks we make the following assumption:

$$w_{ij} = w_{ji} \quad \forall i, j = 1 \dots n$$

In other words, if $W = (w_{ij})$ is the weight matrix we assume:

$$W = W^T$$

In this case the network always converges to a fixed point.

In this case the system possesses a *Liapunov* (or energy) function that is minimized as the process evolves.

The Energy Function – Discrete Case

Consider the following real function:

$$E = -\frac{1}{2} \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n w_{ij} V_i V_j - \sum_{i=1}^n I_i V_i$$

and let $\Delta E = E(t+1) - E(t)$

Assuming that neuron h has changed its state, we have:

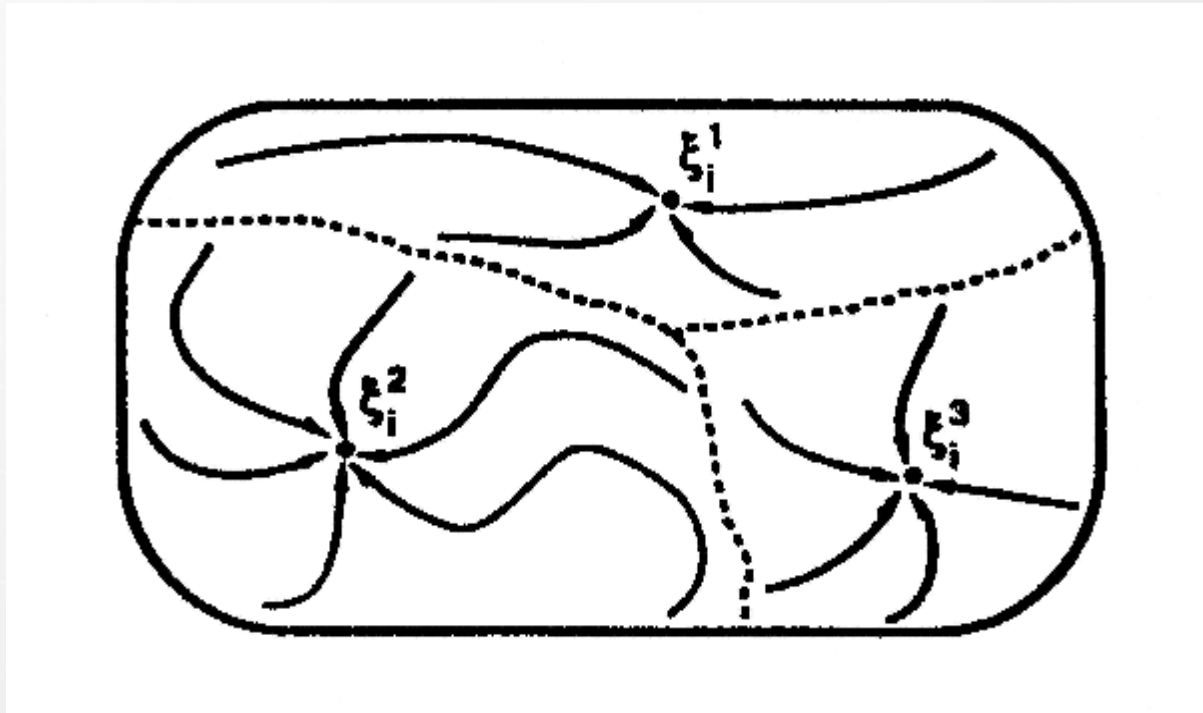
$$\Delta E = - \left[\sum_{\substack{j=1 \\ j \neq h}}^n w_{hj} V_j + I_h \right] \Delta V_h$$

But H_h and ΔV_h have the same sign.

Hence

$$\Delta E \leq 0 \quad (\text{provided that } W = W^T)$$

Schematic configuration space



model with three attractors