From Systems to Components: Constructive Methods for Product-Form Solutions: other product-forms

Andrea Marin [1] Maria Grazia Vigliotti[2]

[1]Dipartimento di Informatica
Università Ca' Foscari di Venezia

[2]Department of Computing
Imperial College London

Second part: sketch

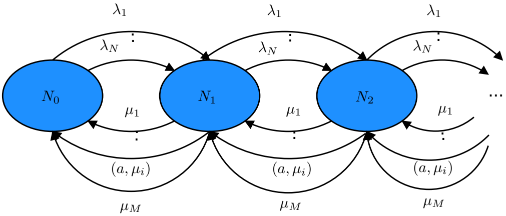
1. **Multiple application of (G)RCAT** A class of non-pairwise cooperations are considered. We show how multiple applications of (G)RCAT can still derive the product-form solution when it exists. Case studies: finite capacity queues with *skipping* [Pittel '79, Balsamo et al. '10], G-networks with signals [Harrison '04b].
2. **Extended Reversed Compound Agent Theorem (ERCAT).** The Extended Reversed Compound Agent Theorem [Harrison '04a] is introduced. Applications for cooperations of pairs of automata which do not yield structural conditions of RCAT are shown.

Part I

Multiple applications of RCAT

Outline

Preliminary



- Value K_a may be interpreted as the sum of the reversed rates of the active transitions labelled by a incoming into each state
- In case of Birth and Death processes this may be easily computed, i.e.:

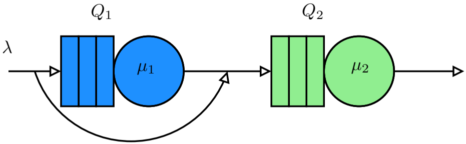
$$K_a = \frac{\sum_{j=1}^{N} \lambda_j}{\sum_{j=1}^{M} \mu_j} \mu_i$$

RCAT or GRCAT?

- The Reversed Compound Agent Theorem (RCAT) [Harrison '03] requires each state to have one incoming active transition for each synchronising label. Value K_a may be interpreted as the (constant) reversed rate of this unique transition.
- The Generalisation (GRCAT) proposed in [Marin et al. '10] requires each state to have at least one incoming active transition for each synchronising label. Value K_a may be interpreted as the (constant) sum of the reversed rates of these transitions.

Skipping mechanism for queues with finite capacity

- Consider a tandem of exponential queues, Q_1 and Q_2
- Q_1 has a finite capacity $B_1 > 0$
- Customers arrive according to a homogeneous Poisson process at Q_1
- If at the arrival epoch Q_1 is saturated, the customer immediately enters in Q_2
- After service completion in Q_1 customers go to Q_2



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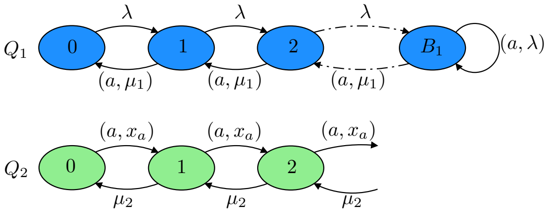
Standard RCAT analysis

- Processes:



- Clearly, the reversed rates of *a*-transitions are constant, hence $K_a = \lambda$
- Structural (G)RCAT conditions are satisfied
- Steady-state distribution:

$$\pi(n_1, n_2) \propto \left(\frac{\lambda}{\mu_1}\right)^{n_1} \left(\frac{\lambda}{\mu_2}\right)^{n_2} \quad \text{with } 0 \le n_1 \le B_1, n_2 \ge 0$$

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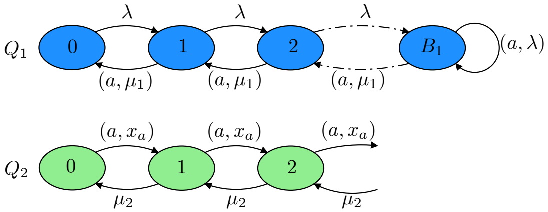

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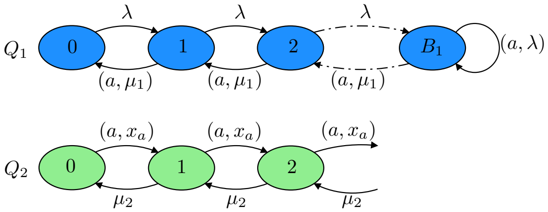

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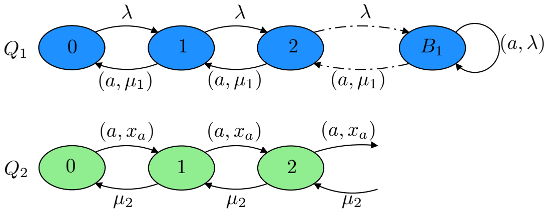

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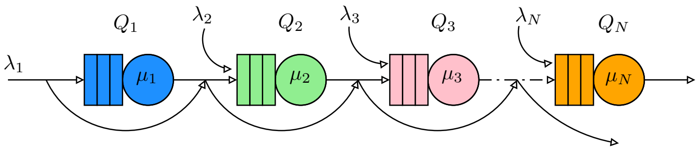
Possible generalisation?

- Consider a sequence of N exponential stations $Q_1, \ldots, Q_N$ with finite capacities $B_1, \ldots, B_N$
- Customers arrive at Q_i according to a homogeneous Poisson process with rate λ_i, $1 \leq i \leq N$
- At a job completion at queue Q_i, the customer tries to enter queue Q_{i+1}, $1 \leq i < N$
- A customer is allowed to enter Q_i if this is not saturated, or must try to enter Q_{i+1} otherwise, $1 \leq i < N$
- After a job completion at queue Q_N or if this is saturated, customers leave the system
- Note the system in unconditionally stable

Are these pairwise cooperations?



- Each transition in the system may change the state of only two components but. . .
- Consider the cooperation between Q_1 and Q_3: an arrival or a job completion at Q_1 may generate an arrival at Q_3 depending on the state of Q_2!
- The cooperation cannot be described only in terms of pairs of queues in isolation
- These cases may still be studied by RCAT with multiple applications

Outline

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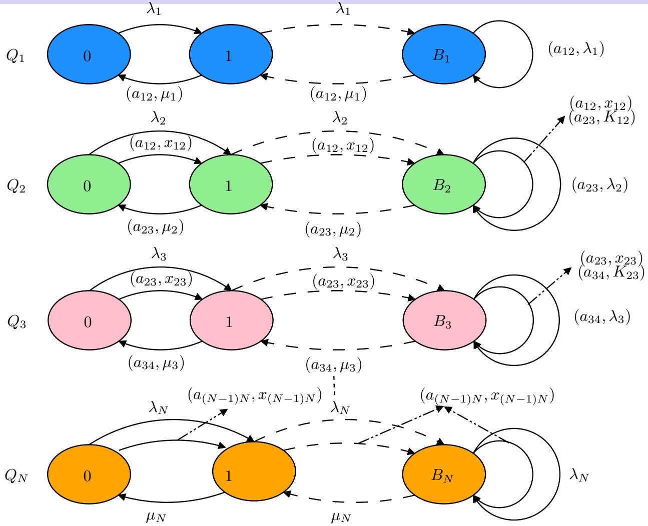
Introduction
Skipping queues
G-networks and triggers

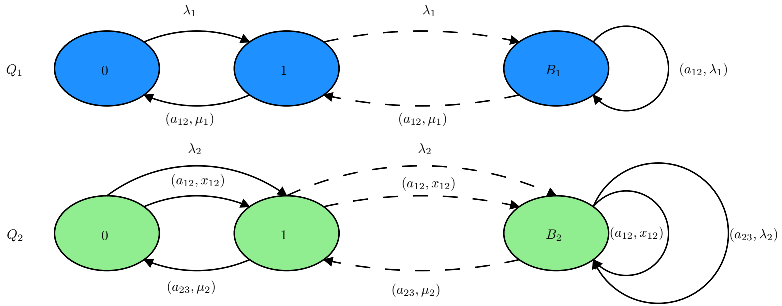
Cooperating processes



Peculiarity of the model

- For $1 < i < N$ a self-loop of state B_i has two *roles*:
  - it is passive with respect to cooperation label $a_{(i-1)i}$
  - it is active with respect to cooperation label $a_{i(i+1)}$ and has $K_{(i-1)i}$ as a forward rate
- We apply (G)RCAT multiple times adding at each time a new queue

Application of RCAT to the first two queues



RCAT can be applied because:

- Structural conditions on passive transitions are satisfied
- Structural conditions on active transitions are satisfied
- We have $K_{12} = \lambda_1$

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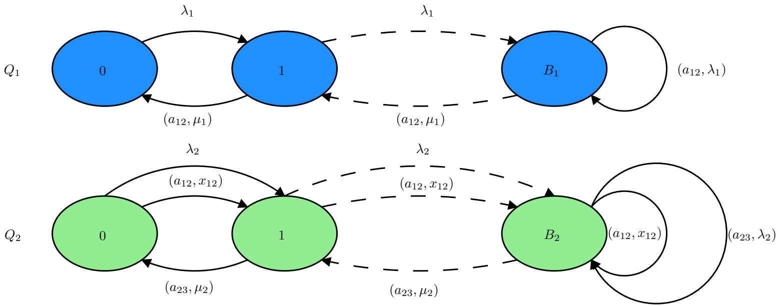

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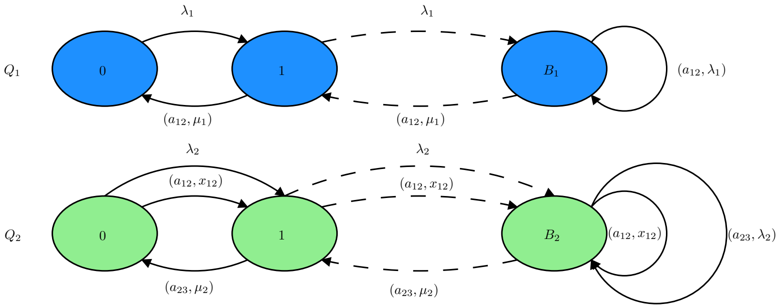

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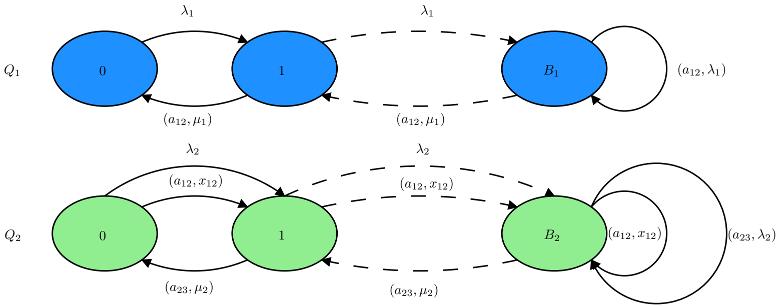

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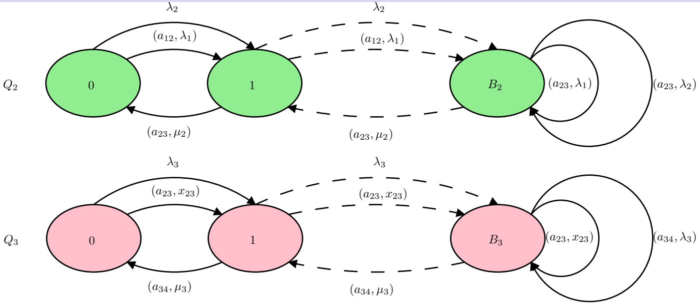
Application of GRCAT to Q_2 and Q_3



- Structurally, the situation is analogue to the previous case
- Note that state B_2 has two transitions incoming with the same label $\Rightarrow$ We apply GRCAT and sum the reversed rates obtaining $\lambda_2 + \lambda_1$
- The reversed rate of the death transitions is $\lambda_2 + \lambda_1$ which is the value of x_{23}

Steady-state distribution

- Multiple applications of (G)RCAT lead to the following values of the reversed rates:

$$K_{i(i+1)} = \sum_{\ell=1}^{i} \lambda_i \quad 1 \leq i < N$$

- The steady-state distribution is in product-form:

$$\pi(n_1, \ldots, n_N) \propto \prod_{\ell=1}^{N} \rho_\ell^{n_\ell},$$

with $0 \leq n_\ell \leq B_\ell$ and

$$\rho_\ell = \frac{\sum_{j=1}^{\ell} \lambda_j}{\mu_\ell}$$

Some notes

- The result may be easily extended to more general topologies
- Does the product-form yield in case of multiple server stations?
  - Yes! ⇒ the reversed rates do not change!
- Does the product-form yield in case of negative customers?
  - No! ⇒ the reversed rates of the “death” transitions are different (smaller) from those of the self-loops
  - But if we properly slow-down the arrival rates to saturated queues we may still obtain a product-form solution!

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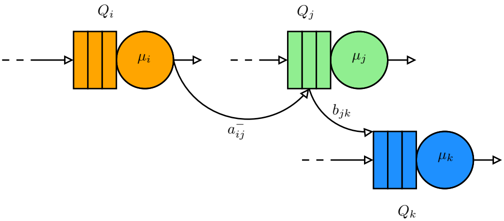
Outline

1. A mild introduction
2. Finite capacity queues with skipping: the RCAT solution
3. Product-form solution for G-networks with positive

Model description

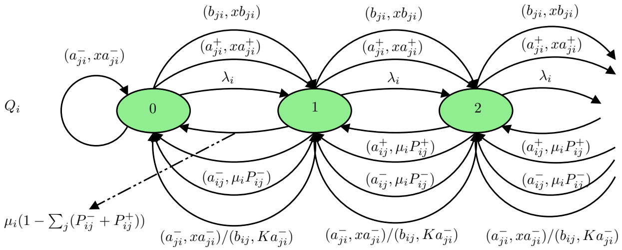
- Network of N exponential queues $Q_1, \ldots, Q_N$ with external Poisson customer arrivals with rate λ_i and service rate μ_i
- At a job completion at Q_i a customer can:
  - go to queue Q_j, $j \neq i$, with probability P^+_{ij} as a standard customer
  - go to queue Q_j, $j \neq i$, with probability P^-_{ij} as a trigger
  - leave the system with probability $1 - \sum_j (P^+_{ij} + P^-_{ij})$
- At a trigger arrival at Q_j it:
  - vanishes if Q_j is empty
  - removes a customer from Q_j and add a customer to Q_k, $k \neq j$, with probability R_{jk}, if Q_j is non-empty
  - removes a customer from Q_j with probability $1 - \sum_k R_{jk}$, if Q_j is non-empty

Model picture



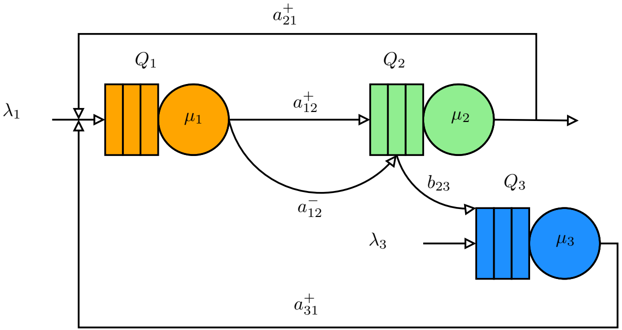
- The picture shows just the cooperation among three queues Q_i, Q_j, Q_k embedded in a general networks
- We focus on the analysis of the trigger behaviours
- Positive customer analysis is the same of Jackson's networks
- A job completion in Q_i may change the state of three queues simultaneously: Q_i, Q_j, Q_k

Process underlying a generic queue Q_i



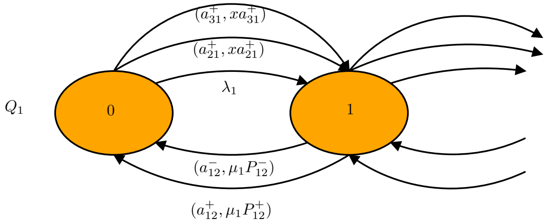
- $1 \leq j \leq N$, $j \neq i$
- a^+_{ij}: positive customer from Q_i to Q_j
- a^-_{ij}: trigger from Q_i to Q_j
- b_{ij}: customer arrival at Q_j caused by a trigger arrival at queue Q_i

Example



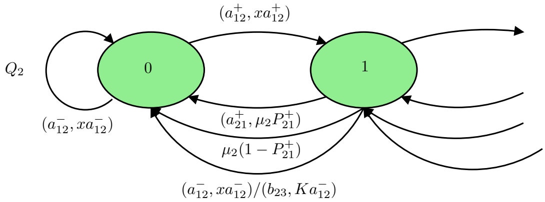
- We set up the RCAT traffic equations by the analysis of each queue in isolation
- This operation can be done algorithmically

Queue 1



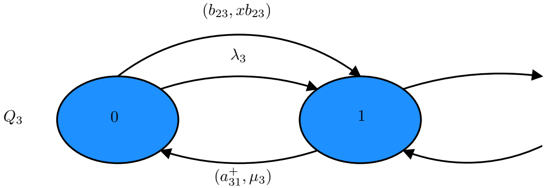
- $$Ka_{12}^- = (\lambda_1 + Ka_{31}^+ + Ka_{21}^+)P_{12}^-$$
- $$Ka_{12}^+ = (\lambda_1 + Ka_{31}^+ + Ka_{21}^+)P_{12}^+$$

Queue 2



- $$Ka^+_{21} = \frac{Ka^+_{12}}{\mu_2 + Ka^-_{12}} \mu_2 P^+_{21}$$
- $$Kb_{23} = \frac{Ka^+_{12}}{\mu_2 + Ka^-_{12}} Ka^-_{12}$$

Queue 3



- $$Ka^+_{31} = \lambda_3 + Kb_{23}$$

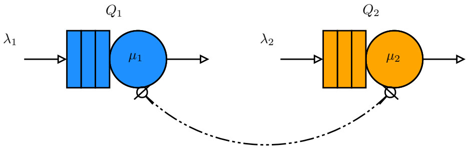
Concluding the example

- The solution of the traffic equations straightforwardly gives the product-form solution
- The traffic equations may be solved either symbolically or numerically
- The algorithm presented in [Marin et al. '09] applies an iterative schema to efficiently solve such networks of queues
- The approach may be extended to deal with negative triggers (at a trigger arrival the receiving non-empty queue may send a trigger to another queue)

Part II

Extended Reversed Compound Agent Theorem (ERCAT)

A system in Boucherie's product-form



- Two exponential queues Q_1 and Q_2 with independent Poisson arrival streams with rate λ_1 and λ_2
- Service rates are μ_1 and μ_2
- If one of the queues enters in state 0 the other one is blocked (i.e. no arrivals or service completions occur)
- The model is known to be in Boucherie's product-form

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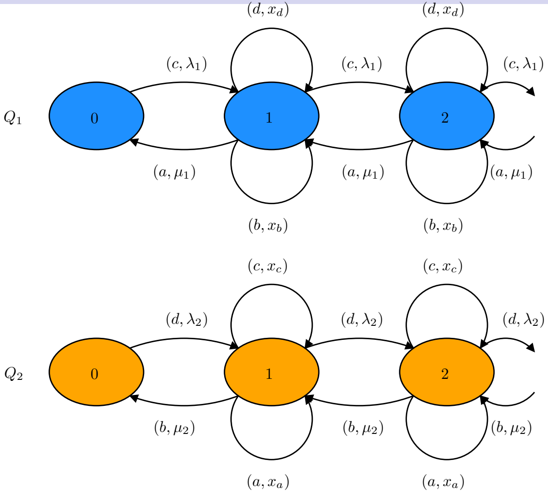
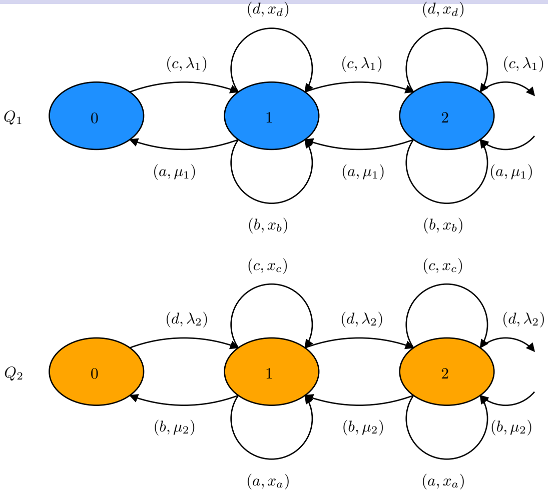
Process representation



Are (G)RCAT structural conditions satisfied? NO!

Process representation



Are (G)RCAT structural conditions satisfied? NO!

Joint state space

- ERCAT requires to check a rate equation for each state of the irreducible subset of the joint process
- Often, states can be opportunely clustered and hence the computation becomes feasible
- The computational complexity is higher than the standard (G)RCAT
- Let (s_1, s_2) be a state of the irreducible subset of the joint process

Fundamental definitions

- $\mathcal{P}^{(s_1,s_2)\rightarrow}$: outgoing labels from s_1 or s_2
- $\mathcal{P}^{(s_1,s_2)\leftarrow}$: incoming passive labels into s_1 or s_2
- $\mathcal{A}^{(s_1,s_2)\rightarrow}$: outgoing active labels from s_1 or s_2
- $\mathcal{A}^{(s_1,s_2)\leftarrow}$: incoming active labels into s_1 or s_2
- $\alpha^{(s_1,s_2)}(a)$: rate of transition labelled by a outgoing from (s_1, s_2)
- $\overline{\beta}^{(s_1,s_2)}(a)$: reversed rate of the passive transition labelled by a incoming into (s_1, s_2)

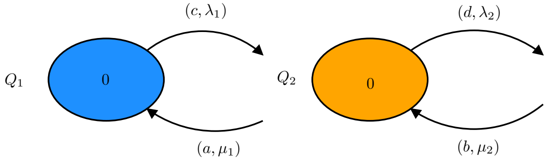
ERCAT formulation

Theorem (ERCAT)

Given two models Q_1 and Q_2 in which RCAT structural conditions are not satisfied but the reversed rates of the active transitions are constant, their cooperation is in product-form if the following rate equation is satisfied for each state (s_1, s_2) of the irreducible subset of states of the joint process:

$$\sum_{a \in \mathcal{P}^{(s_1,s_2)\rightarrow}} x_a - \sum_{a \in \mathcal{A}^{(s_1,s_2)\leftarrow}} x_a = \sum_{a \in \mathcal{P}^{(s_1,s_2)\leftarrow} \setminus \mathcal{A}^{(s_1,s_2)\leftarrow}} \overline{\beta}_a^{(s_1,s_2)} - \sum_{a \in \mathcal{A}^{(s_1,s_2)\rightarrow} \setminus \mathcal{P}^{(s_1,s_2)\rightarrow}} \alpha_a^{(s_1,s_2)}$$

State (0,0)



$$\mathcal{P}^{(0,0)\rightarrow} = \{\} \quad \mathcal{A}^{(0,0)\leftarrow} = \{a, b\}$$

$$\mathcal{P}^{(0,0)\leftarrow} \setminus \mathcal{A}^{(0,0)\leftarrow} = \{\} \quad \mathcal{A}^{(0,0)\rightarrow} \setminus \mathcal{P}^{(0,0)\rightarrow} = \{c, d\}$$

$$-x_a - x_b = -\alpha_c^{(0,0)} - \alpha_d^{(0,0)} \quad \text{Ok}$$

Note that:

$$x_a = \lambda_1,\ \alpha_c^{(0,0)} = \lambda_1,\ x_b = \lambda_2, \alpha_d^{(0,0)} = \lambda_2$$

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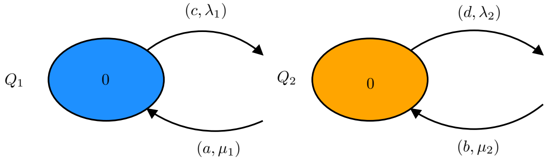

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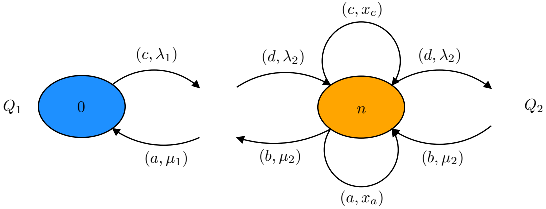
State (0,n), n>0



$$\mathcal{P}^{(0,n)\rightarrow} = \{a, c\} \quad \mathcal{A}^{(0,n)\leftarrow} = \{a, b, d\}$$

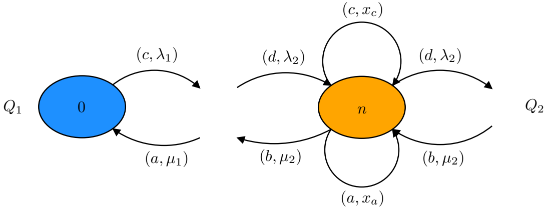
$$\mathcal{P}^{(0,n)\leftarrow} \setminus \mathcal{A}^{(0,n)\leftarrow} = \{c\} \quad \mathcal{A}^{(0,n)\rightarrow} \setminus \mathcal{P}^{(0,n)\rightarrow} = \{b, d\}$$

$$x_a + x_c - x_a - x_b - x_d = \overline{\beta}_c^{(0,n)} - \alpha_b^{(0,n)} - \alpha_d^{(0,n)}$$ Ok!

Note that:

$$x_b = \lambda_2,\ x_c = \mu_1,\ x_d = \mu_2,\ \overline{\beta}_c^{(0,n)} = \mu_1,\ \alpha_b^{(0,n)} = \mu_2,\ \alpha_d^{(0,n)} = \lambda_2$$

State (0,n), n>0



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$$x_a + x_c - x_a - x_b - x_d = \overline{\beta}_c^{(0,n)} - \alpha_b^{(0,n)} - \alpha_d^{(0,n)} \text{ Ok!}$$

Note that:

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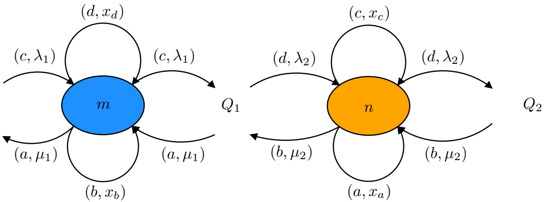
State (m,n), m,n>0



$$\mathcal{P}^{(m,n)\rightarrow} = \{a, b, c, d\} \quad \mathcal{A}^{(m,n)\leftarrow} = \{a, b, c, d\}$$

$$\mathcal{P}^{(m,n)\leftarrow} \setminus \mathcal{A}^{(m,n)\leftarrow} = \{\} \quad \mathcal{A}^{(m,n)\rightarrow} \setminus \mathcal{P}^{(m,n)\rightarrow} = \{\}$$

0=0

Note that states $(m, 0)$ with $m > 0$ are similar to $(0, n)$, $n > 0$.

Conclusion of the running example

- The model, as expected, is in product-form:

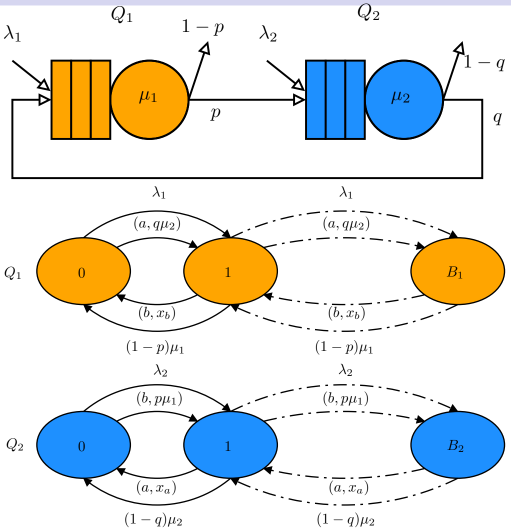
$$\pi(m, n) \propto \left(\frac{\lambda_1}{\mu_1}\right)^m \left(\frac{\lambda_2}{\mu_2}\right)^n$$

- Note that state $(0, 0)$ is either the only ergodic state or does not belong to the irreducible subset
- Hence, the normalising constant distinguishes this solution from the case of independent queues
- Every Boucherie's product-form with full blocking can be studied by ERCAT [Harrison '04a]

Queues with finite capacity and Repetitive Service (RS) blocking

- We consider a network of queues, $Q_1, \ldots, Q_N$ with finite capacity B_i and service rate μ_i
- At a job completion at Q_i the customer goes to Q_j with probability P_{ij}. If Q_j if saturated the customer service is restarted and a new target station is selected at job completion
- In open networks λ_i is the arrival rate at Q_i and customers leave the system with probability $1 - \sum_j P_{ij}$. Arrivals at saturated queues are not allowed

Example

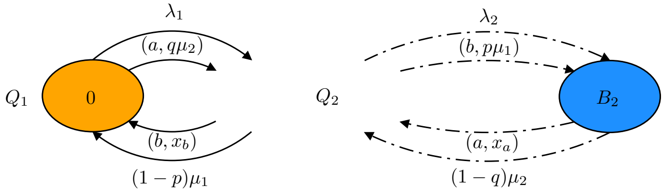


Q_1
Q_2
λ_1
λ_2
$1-p$
$1-q$
μ_1
μ_2
p
q
λ_1
$(a, q\mu_2)$
Q_1
0
1
B_1
(b, x_b)
$(1-p)\mu_1$
λ_2
$(b, p\mu_1)$
Q_2
0
1
B_2
(a, x_a)
$(1-q)\mu_2$

Notes

- Differently from ordinary queueing networks we use active transitions to model synchronised arrivals and passive to model synchronised departures
- Which states shall we consider?
  1. $(0, 0)$
  2. $(0, K)$ with $0 < K < B_2$ (and symmetrically we obtain $(K, 0)$ with $0 < K < B_1$)
  3. $(0, B_2)$
  4. (K, B_2) with $0 < K < B_1$ (and symmetrically we obtain $(0, K)$ with $0 < K < B_2$)
  5. (B_1, B_2)
- Note that $\alpha_a^{(\cdot,\cdot)} = q\mu_2$, $\alpha_b^{(\cdot,\cdot)} = p\mu_1$ and also $\overline{\beta}_a^{(\cdot,\cdot)} = \overline{\beta}_a$ and $\overline{\beta}_b^{(\cdot,\cdot)} = \overline{\beta}_b$

State $(0, B_2)$

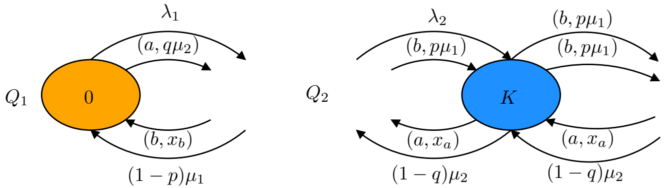


$$\mathcal{P}^{(0,B_2)\rightarrow} = \{a\} \quad \mathcal{A}^{(0,B_2)\leftarrow} = \{b\}$$

$$\mathcal{A}^{(0,B_2)\rightarrow} \setminus \mathcal{P}^{(0,B_2)\rightarrow} = \{\} \quad \mathcal{P}^{(0,B_2)\leftarrow} \setminus \mathcal{A}^{(0,B_2)\leftarrow} = \{\}$$

$$x_a - x_b = 0 \Rightarrow x_a = x_b \tag{1}$$

State $(0, K)$



$$\mathcal{P}^{(0,K)\rightarrow} = \{a\} \quad \mathcal{A}^{(0,K)\leftarrow} = \{b\}$$

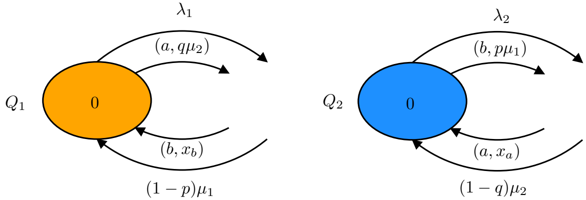
$$\mathcal{A}^{(0,K)\rightarrow} \setminus \mathcal{P}^{(0,K)\rightarrow} = \{b\} \quad \mathcal{P}^{(0,K)\leftarrow} \setminus \mathcal{A}^{(0,K)\leftarrow} = \{a\}$$

i.e.:

$$x_a - x_b = \overline{\beta}_b^{(0,K)} - \alpha_a^{(0,K)} \xrightarrow{(1)} \overline{\beta}_b = \alpha_a \tag{2}$$

By symmetry, state $(K, 0)$ gives $\overline{\beta}_a = \alpha_b$

State $(0, 0)$



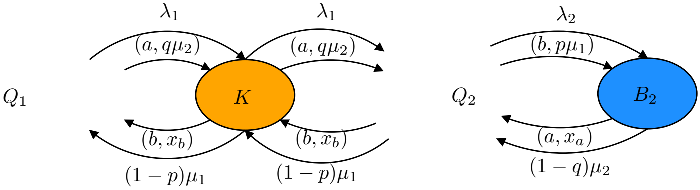
$$\mathcal{P}^{(0,0)\rightarrow} = \{\} \quad \mathcal{A}^{(0,0)\leftarrow} = \{\}$$

$$\mathcal{A}^{(0,0)\rightarrow} \setminus \mathcal{P}^{(0,0)\rightarrow} = \{a, b\} \quad \mathcal{P}^{(0,0)\leftarrow} \setminus \mathcal{A}^{(0,0)\leftarrow} = \{a, b\}$$

$$\overline{\beta}_a^{(0,0)} + \overline{\beta}_b^{(0,0)} = \alpha_a^{(0,0)} + \alpha_b^{(0,0)}$$

which is a consequence of (2)

States (K, B_2) and (B_1, K), (B_1, B_2)



- For these states we have:
  - $\mathcal{P}^{(\cdot,\cdot)\rightarrow} = \{a, b\}$
  - $\mathcal{A}^{(\cdot,\cdot)\leftarrow} = \{a, b\}$
- Since all the synchronising labels are present in both these sets, the rate equation for these states is an identity.

Conditions derived from the ERCAT rate equations

$$\begin{cases} x_a = x_b \\ \overline{\beta}_b = \alpha_a = q\mu_2 \\ \overline{\beta}_a = \alpha_b = p\mu_1 \end{cases}$$

The process analysis gives:

$$\overline{\beta}_b = \frac{x_b(\lambda_1 + q\mu_2)}{x_b + (1-p)\mu_1} \quad \overline{\beta}_a = \frac{x_a(\lambda_2 + p\mu_1)}{x_a + (1-q)\mu_2}$$

From which we straightforwardly derive:

$$x_a = \frac{(1-q)p\mu_1\mu_2}{\lambda_2} \quad x_b = \frac{(1-p)q\mu_1\mu_2}{\lambda_1} \tag{3}$$

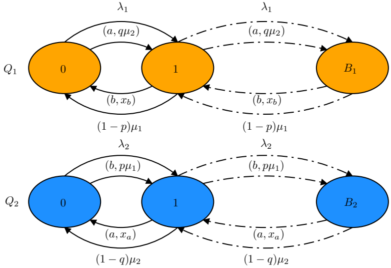
Product-form rate condition

Since $x_a = x_b$ by (1) we have the product-form rate condition:

$$(1-p)q\lambda_2 = (1-q)p\lambda_1$$

Under this assumption expressions (3) for x_a, x_b satisfies:

$$x_a = \frac{(x_b + (1-p)\mu_1)q\mu_2}{\lambda_1 + q\mu_2} \quad x_b = \frac{(x_a + (1-q)\mu_2)p\mu_1}{\lambda_2 + p\mu_1}$$



Generalisation

- ERCAT may be applied to a set of agent with pairwise cooperations (this is also known a MARCAT)
- In case of QN with RS blocking and general topology in [Balsamo et al. '10] is proved that:

Theorem

A QN (open or closed) with finite capacity stations and RS blocking policy with reversible routing matrix always satisfies ERCAT rate equations.

- Product-form for reversible routing has been proved in [Akyildiz '87]

Closed QN with RS blocking

- Consider a closed QN with RS blocking policy
- Note that the ERCAT rate equation is an identity for state **n** when none of the stations is empty in **n**
- We immediately have the following result:

Theorem (QN with strict non-empty condition)

A closed QN with finite capacity stations and RS blocking is in product-form if the number of customers is such that none of the station can be empty (strict non-empty condition)

- In [Balsamo et al. '10] we prove that the same result for QN in which at most one station can be empty (non-empty condition)

Final remarks

- In the second part of the tutorial we have shown how to overcome some limitations of original RCAT and GRCAT formulation
  - In case of some non-pairwise cooperations we can apply (G)RCAT iteratively to obtain the product-form
  - In case structural conditions of (G)RCAT are not satisfy we may apply ERCAT
- Application of (G)RCAT or ERCAT may be done algorithmically, however the computational cost of ERCAT is higher than that of (G)RCAT

Applications

- Other models than those presented here may be studied by RCAT and its extensions (e.g. product-form Stochastic Petri Nets)
- New product-form may be derived
- The solution of the traffic equations may be efficiently computed by means of the algorithm presented in [Marin et al. '09]
    - Numerical and iterative algorithm
- Product-form of models expressed in terms of different formalisms may be derived.

Appendix: Reversible routing matrix

- Consider a queueing network with N stations and fixed routing probability matrix $\mathbf{P} = [p_{ij}]$, $1 \leq i, j \leq N$
- p_{i0} is the probability of leaving the network after a job completion at station i
- e_i is the (relative) visit ratio to station i
- λ_i is the arrival rate at station i

Definition (Reversible routing matrix)

The routing matrix **P** is said reversible if:

$$\begin{cases} e_i p_{ij} = e_j p_{ji} & \text{for } 1 \leq i, j \leq N \\ \lambda_i = e_i p_{i0} & \text{for } 1 \leq i \leq N \end{cases}$$

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