Product-form solutions for models with joint-state dependent transition rates

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Presentation outline

Notation

- We consider Labelled Markovian Automata (LMA) defined as follows:

$$S_i =< \mathcal{S}_i, \mathcal{L}_i, \mathcal{T}_i, q_i >$$

- Let S_i be the i-th model
- $\mathcal{S}_i = \{n_i, n_i', n_i'', \ldots\}$: denumerable set of states of S_i
- $\mathcal{L}_i$: finite set of labels of S_i
- $\mathcal{T}_i = \{n_i \xrightarrow{a_i} n_i'\}$: transition from state n_i to n_i' labelled by $a_i \in \mathcal{L}$
- $q_i : \mathcal{T}_i \to \mathbb{R}^+$ is a partial function which associates a positive real number with each *active* transition (e.g., $q(n_i \xrightarrow{a_i} n_i') = \lambda$)
- Transitions without rates are *passive*
- Transitions with the same label must be all active or all passive
- $\mathcal{P}_i, \mathcal{A}_i$: sets of passive and active labels of S_i. $\mathcal{L}_i = \mathcal{P}_i \cup \mathcal{A}_i$

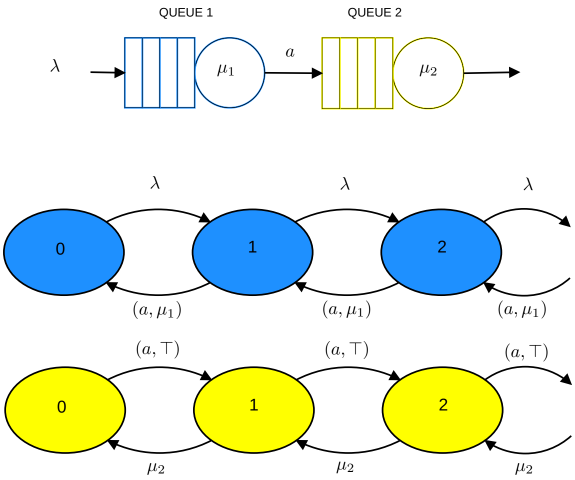
Closed automaton

- An automaton S_i is closed if $\mathcal{P}_i = \emptyset$
  - $\mathcal{L}_i = \mathcal{A}_i$
  - All the transitions have an associated rate
- The transition rates are the parameters of the exponential distributed time needed to carry a transition on
- The process underlying a closed automaton is a Continuous Time Markov Chain (CTMC)
- If S_i is an open LMA and $a \in \mathcal{P}_i$, then $S_i a \leftarrow \lambda$ is the automaton S_i in which each transition labelled by a takes λ as a rate (closure)

Specifying the cooperation

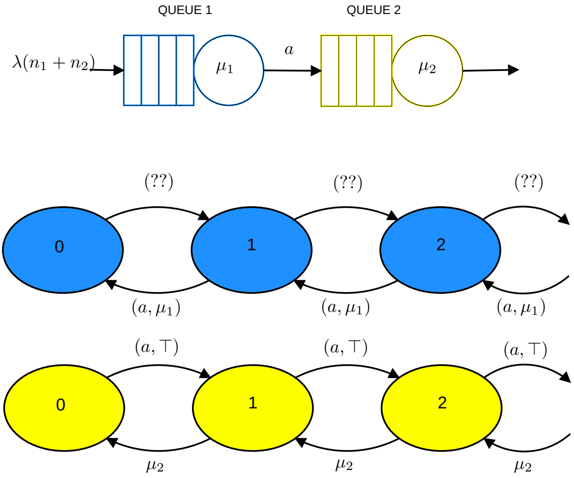
- $S_1, \ldots, S_N$ is the set of cooperating models
- The state space is $S_1 \times S_2 \times \ldots \times S_N$
- For each label $a \in \cup_{i=1}^{N} \mathcal{L}_i$ we have one of the following:
  - No-cooperating label: $a \in \mathcal{A}_i$ for some $i = 1 \ldots N$ and $a \notin L_j$ with $j \neq i$
  - Cooperating label: $a \in \mathcal{A}_i \cap \mathcal{P}_j$ and a$\notin L_k$ with $k \neq i, j$
- If $a \in \mathcal{A}_i \cap \mathcal{P}_j$ transitions labelled by a in S_i and S_j can be performed only jointly. The rate of the joint transition is given by the rate of the active transition in S_i
- The automaton resulting from a cooperation has still an underlying CTMC
- We can specify only pairwise cooperations!

An example



- Tandem of exponential queues
- Arrivals according to a Poisson process
- Independent service times

An example with joint-state dependent transition rates



- Tandem of exponential queues
- Arrivals according to a Poisson process whose rate depends on the total number of customers in the system
- Independent service times

RCAT product-form

Let $S_1, \ldots, S_N$ a cooperation of LMAs Assume that the following conditions are satisfied:

- for each synchronising label a:
  - if $a \in \mathcal{P}_i$ then $\forall n \in \mathcal{S}_i\ \exists!\ n' \in \mathcal{S}_i$ s.t. $n \xrightarrow{a} n' \in \mathcal{T}_i$
  - if $a \in \mathcal{A}_i$ then $\forall n \in \mathcal{S}_i\ \exists!\ n' \in \mathcal{S}_i$ s.t. $n' \xrightarrow{a} n \in \mathcal{T}_i$
- There exists a set of positive real value $\mathcal{K} = \{K_1, \ldots K_T\}$ for each synchronising label $a_1, \ldots, a_T$ such that $S_i^C = S_i\{a_t \leftarrow K_t,\ \forall a_t \in \mathcal{P}_i\}$ satisfies the following condition:

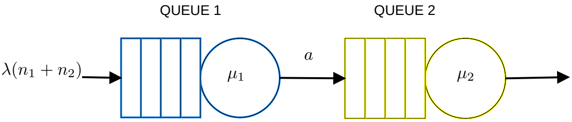
$$\forall a_u \in \mathcal{A}_i, \forall n \in \mathcal{S}_i\ \frac{\pi_i(n')}{\pi_i(n)} q_i(n' \xrightarrow{a_u} n) = K_u$$

Then the steady-state distribution of π of the joint automata is in product-form:

$$\pi(\mathbf{n}) \propto \prod_{i=1}^{N} \pi_i(n_i) \quad \mathbf{n} = (n_1, \ldots, n_N)$$

Product-form solutions for model with joint-state dependent rates

- Values in $\mathcal{K}$ represent the reversed rates of the active transitions
- RCAT requires them to be constant
- How to check this condition with models in isolation?
- Is this a necessary condition for joint-state dependent transition rates?



$$\pi(n_1, n_2) = \prod_{w=0}^{n_1+n_2-1} \lambda(w) \frac{1}{\mu_1^{n_1}} \frac{1}{\mu_2^{n_2}}$$

Solution for queueing networks and stochastic Petri nets in product-form

- We take inspiration from earlier works of Coleman, Henderson, Taylor, Lucic for Stochastic Petri nets, and Serfozo for queueing networks
- We explain their technique for the tandem of exponential queues with joint-state dependent arrival rate
- Define the joint-state dependent rates of station i as follows:

$$q_i(\mathbf{n} - \mathbf{1}_i + \mathbf{1}_j) = \frac{\psi(\mathbf{n} - \mathbf{1}_i)}{\phi(\mathbf{n})}\chi_i$$

Why does it work?

Restrictions on the model class

Product-form theorem

Intuition of the conditions

The theorem applied to HT models

An example

Conclusion