Maintaining Intruder Detection Capability in a Rectangular Domain with Sensors*

Evangelos Kranakis[1], Danny Krizanc[2], Flaminia L. Luccio[3], and Brett Smith[2]

[1] School of Computer Science, Carleton University, Ottawa, Ontario, Canada
[2] Dept. of Math. & Computer Science, Wesleyan University, Middletown CT, USA
[3] DAIS, Università Ca' Foscari Venezia, Venice, Italy

Abstract. In order to detect intruders that attempt to pass through a rectangular domain, sensors are placed at nodes of a regular spaced grid laid out over the rectangle. An intruder that steps within the sensing range of a sensor will be detected. It is desired that we prevent potential attacks in either one dimension or two dimensions. A one-dimensional attack succeeds when an intruder enters from the top (North) side and exits out the bottom (South) side of the domain without being detected. Preventing attacks in two dimensions requires that we simultaneously prevent the intruder from either entering North and exiting South or entering East (left side) and exiting West (right side) undetected.

Initially, all of the sensors are working properly and the domain is fully protected, i.e., attacks will be detected, in both dimensions (assuming the grid points are such that neighboring sensors have overlapping sensing ranges and include all four boundaries of the domain). Over time, the sensors may fail and we are left with a subset of working sensors. Under these conditions we wish to (1) determine if one or two-dimensional attack detection still persists and (2) if not, restore protection by adding the least number of sensors required to ensure detection in either one or two dimensions.

Ideally, the set of currently working sensors would provide some amount of fault-tolerance. In particular, it would be advantageous if for a given k, the set of sensors maintains protection (in one or two dimensions) even if up to k of the sensors fail. This leads to the problems of (1) deciding if a subset of the sensors provides protection with up to k faults and (2) if not, finding the minimum number of grid points to add sensors to in order to achieve k fault-tolerance.

In this paper, we provide algorithms for deciding if a set sensors provides k-fault tolerant protection against attacks in both one and two dimensions, for optimally restoring k-fault tolerant protection in one dimension and for restoring protection in two dimensions (optimally for $k = 0$ and approximately otherwise).

* Research supported in part by NSERC grant, and by MIUR project Security Horizons. Work partially done while the first two authors were visiting Ca' Foscari University. The final publication is available in the proceedings of ALGOSENSORS 2015. Lecture Notes in Computer Science, vol 9536. Springer, Cham. The final publication is available at https://link.springer.com/chapter/10.1007/978-3-319-28472-9_3.

Key words and phrases. Rectangular Grid, Intruder Detection, Sensors, Fault Tolerance.

1 Introduction

In order to detect intruders that attempt to pass through a rectangular domain, sensors are placed at nodes of a regular spaced grid laid out over the rectangle. An intruder that steps within the sensing range of a sensor will be detected. It is desired that we prevent potential attacks in either one dimension or two dimensions. A one-dimensional attack succeeds when an intruder enters from the top (North) side and exits out the bottom (South) side of the domain without being detected. Preventing attacks in two dimensions requires that we simultaneously prevent the intruder from either entering North and exiting South or entering East (left side) and exiting West (right side) undetected.

Initially, all of the sensors are working properly and the domain is fully protected, i.e., attacks will be detected, in both dimensions (assuming the grid points are such that neighboring sensors have overlapping sensing ranges and include all four boundaries of the domain). Over time, the sensors may fail and we are left with a subset of working sensors. Under these conditions we wish to (1) determine if one- or two-dimensional attack detection still persists and (2) if not, restore protection by adding the least number of sensors required to ensure detection in either one or two dimensions.

Ideally, the set of currently working sensors would provide some amount of fault-tolerance. In particular, it would be advantageous if for a given k, the set of sensors maintains protection (in one or two dimensions) even if up to k of the sensors fail. This leads to the problems of (1) deciding if a subset of the sensors provides protection with up to k faults and (2) if not, finding the minimum number of grid points to add sensors to in order to achieve k fault-tolerance.

In this paper, we provide algorithms for deciding if a set sensors provides k-fault tolerant protection against attacks in both one and two dimensions, for optimally restoring k-fault tolerant protection in one dimension and for restoring protection in two dimensions (optimally for $k = 0$ and approximately otherwise). The rest of this introduction provides more precise definitions for our problems, a description of our results and a discussion of related work. The following section presents our results in detail and we finish with a discussion of extensions and open problems.

1.1 Preliminaries and notation

In order to present our results we first provide some definitions. As many of our results depend upon results concerning shortest paths and network flow in directed graphs, we will, at times, represent an undirected graph, say H, with a directed graph, denoted $H_{\leftrightarrow}$, by replacing each edge with two edges of opposite orientation. Any directed path in $H_{\leftrightarrow}$ will then correspond to an undirected path in H and vice versa. Moreover, two directed paths are vertex-disjoint in $H_\leftrightarrow$ if and only if the corresponding paths in H are vertex-disjoint.

Also, we will need a binary weight function on the vertices of our graphs to indicate whether or not a vertex can be occupied by an adversary without detection by a sensor. As we will see, it is convenient to add 'dummy' vertices to our grid, G, which will not be accessible by an adversary. These vertices will be given weight zero, as will any vertex on which we place a sensor. Any grid vertex which does not contain a sensor will have weight one.

Many of the cut/flow results we would like to use depend on edge weights to determine the length of a path. As we are interested in the presence of sensors on a path, for any plane graph, H, with weighted vertices, we give edge weights to $H_\leftrightarrow$ determined by the terminal vertex of each edge. Then the weight of a directed path in $H_\leftrightarrow$ will be the sum of the edge weights plus the weight (in H) of the initial vertex on the path. This provides an equivalent weight on undirected paths in H and a distance function in both graphs.

Fix $m, n > 0$ and consider an $m \times n$-grid, G, embedded in the plane. Label the four vertex sets corresponding to the four sides of the grid, $North, South, East$ and $West$, so that $North$ and $South$ have size m; $East$ and $West$ have size n. Let $C = (c_0, ..., c_{2n+2m-5})$ be the cycle peripheral to the unbounded face of G so that

$$\begin{aligned} North &= \{c_0, ..., c_{m-1}\}, \\ East &= \{c_{m-1}, ..., c_{m+n-2}\}, \\ South &= \{c_{m+n-2}, ..., c_{2m+n-3}\}, \\ West &= \{c_{2m+n-3}, ..., c_{2m+2n-5}, c_0\}. \end{aligned}$$

A $(North, South)$**-path** in $G_\leftrightarrow$ is a directed path whose initial vertex is in $North$ and whose terminal vertex is in $South$; similarly, an $(East, West)$**-path** in $G_\leftrightarrow$ is a directed path whose initial vertex is in $East$ and whose terminal vertex is in $West$. We refer to undirected paths in G as $(North, South)$ and $(East, West)$-paths as well.

Definition 1 ((North, South)- and (East, West)-blocking set). *A* $(North,$ $South)$-blocking set*, $B \subseteq V(G)$, is a set of vertices such that there is no $(North, South)$-path in $G - B$. Similarly, an* $(East, West)$-blocking set*, $B \subseteq V(G)$, is a set of vertices such that there is no $(East, West)$-path in $G - B$.*

Placing sensors on such a blocking set will detect an adversary's movement along any path between the identified pair of sides in the grid.

We generalize these blocking sets to allow for a number, k, of faults amongst sensors.

Definition 2 (k-blocking set). *For any fixed $k \geq 0$, a $(North, South)$-k-*blocking set*, $B \subseteq V(G)$, is a set of vertices such that for any $F \subseteq B$ with $|F| \leq k$, there is no $(North, South)$-path in $G - (B \backslash F)$. Similarly, an $(East, West)$-k-*blocking set*, $B \subseteq V(G)$, is a set of vertices such that for any $F \subseteq B$ with*

$|F| \leq k$, *there is no* $(East, West)$*-path in* $G - (B \backslash F)$. *For any fixed* $k \geq 0$, *a* k-blocking set, $B \subseteq V(G)$, *is a set of vertices so that for any* $F \subseteq B$ *with* $|F| \leq k$, *there is neither a* $(North, South)$*-path nor an* $(East, West)$*-path in* $G - (B \backslash F)$.

Placing sensors on a $(North, South)$-k-blocking set (respectively, $(East, West)$-k-blocking set) will detect movement along any path between the identified pair of sides in the grid, allowing for up to k faulty sensors. Moreover, a k-blocking set detects both $(North, South)$ and $(East, West)$ movement at the same time, allowing for up to k faulty sensors.

We now define our problems.

Definition 3 (One-dimensional k-protection decision problem). *Given an* $m \times n$*-grid,* G*, a subset* $B \subseteq V(G)$*, and a non-negative integer* k*, does* B *form a* $(North, South)$*-*k*-blocking set? The case* $k = 0$ *will be referred to simply as the one-dimensional protection decision problem.*

Definition 4 (Two-dimensional k-protection decision problem). *Given an* $m \times n$*-grid,* G*, a subset* $B \subseteq V(G)$*, and a non-negative integer* k*, does* B *form a* k*-blocking set? The case* $k = 0$ *will be referred to simply as the two-dimensional protection decision problem.*

Definition 5 (One-dimensional k-protection placement problem). *Given an* $m \times n$*-grid,* G*, a subset* $B \subseteq V(G)$*, and a non-negative integer* k*, find a set* $B' \subseteq V(G) \backslash B$ *of minimum size such that* $B \cup B'$ *forms a* $(North, South)$*-*k*-blocking set. The case* $k = 0$ *will be referred to simply as the one-dimensional protection placement problem.*

Definition 6 (Two-dimensional k-protection placement problem). *Given an* $m \times n$*-grid,* G*, a subset* $B \subseteq V(G)$*, and a non-negative integer* k*, find a set* $B' \subseteq V(G) \backslash B$ *of minimum size such that* $B \cup B'$ *forms a* k*-blocking set. The case* $k = 0$ *will be referred to simply as the two-dimensional protection placement problem.*

1.2 Our results

We show the following:

1. There exist $O(mn)$ time algorithms for solving the one- and two-dimensional k-protection decision problems. See Theorems 2 and 5. (Note: the one dimensional case follows from a result in [13]. We present our own version of the proof for completeness.)
2. There exists a $O(kmn \log(mn))$ time algorithm for solving the one-dimensional k-protection placement problem. See Theorem 3.
3. There exists a $O(m^2n^2)$ time algorithm for solving the two-dimensional protection placement problem. See Theorem 6.
4. There exists a $O(kmn \log(mn))$ time 2-approximation algorithm for solving the two-dimensional k-protection placement problem. See Theorem 7.

In all of the above we assume $k < \min\{m, n\}$ as we shall see that the problems can not be solved otherwise. Further we discuss extensions of these results for solving more general versions of these problems including protecting:

1. domains containing impassable regions,
2. non-rectangular domains,
3. and against more general attacks than just North-South or East-West.

1.3 Related work

As far as we know, we are the first to study these problems as formalized above. A closely related problem concerning the placement of sensors to accomplish coverage of a given region has been studied extensively in the literature. Generally, one assumes sensors can sense a limited region defined by their sensing radius. To monitor a larger region against potential threats every point of the region must be within the sensing range of at least one of the sensors. This has been studied in several papers, and includes research on *area coverage* whereby one ensures monitoring of an entire region [11, 14], and on *perimeter* or *barrier coverage* whereby a region is monitored via its perimeter thus sensing intrusions or exits from the interior [1, 13]. The fault tolerance of such placements has also been studied [4, 13]. For the case where the sensors may be moved, the complexity of minimizing the sensor displacement has also been studied in some detail. For example, for sensors placed on a line [6] shows that there is an $O(n^2)$ algorithm for minimizing the max displacement of a sensor while the problem becomes NP-complete if there are two separate (non-overlapping) barriers (cf. also [5] for arbitrary sensor ranges). Similar research is known if one is interested in sum of sensor displacements [7], or the number of sensors moved [15]. Further, [9] considers the complexity of several natural generalizations of the barrier coverage problem with sensors of arbitrary ranges, including when the initial positions of sensors are arbitrary points in the two-dimensional plane, as well as multiple barriers that are parallel or perpendicular to each other. Perhaps the most closely related work to ours is that of [16] where the authors look at how to best randomly distribute additional sensors in order to maintain barrier coverage under the potential for faults.

2 Main Results

Our main results are based upon establishing a characterization of minimal k-blocking sets in grids. To do this, it is easier to work in the more general setting of graphs embedded in the plane and use some ideas derived from the graph theory literature. We begin by establishing some definitions and important lemmas.

2.1 Connectedness and surface graphs

Definition 7. *Let H be a simple, 2-connected graph embedded in the plane. The* **surface graph** *of H, $\widehat{H}$, is obtained from H as follows. In each face, f, of H,* *add a new node, v_f, and edges from v_f to each vertex of H which is peripheral to f. For each $X \subseteq V(H)$, the* **bounded surface set** *of X, called $\overline{X}$, is equal to $X \cup \{v_f \in V(\widehat{H}) : f \text{ is a bounded face of } H\} \subseteq V(\widehat{H})$.*

That is, we obtain a subset of vertices of $\widehat{H}$ from X by including each vertex which corresponds to a bounded face of H. The subgraph of $\widehat{H}$ induced by $\overline{X}$ is denoted by $\widehat{H}[\overline{X}]$.

Notice that $\widehat{H}$ is a maximal plane graph. We observe the following connectedness property for any maximal plane graph.

Lemma 1. *Let H be a maximal plane graph, and let $X \subseteq V(H)$. If f is a face of $H - X$, then the set,*

$$F := \{u \in V(H) : u \text{ is in the face } f\},$$

is a connected set of vertices in H.

Proof Let $u, v \in F$. Then there is some simple polygonal (u, v)-curve, say γ, contained in the face, f. Moreover, we may assume that γ does not meet any vertices from H other than its endpoints, and γ's intersection with each edge consists of an isolated point. Consider the finite multi-sequence of edges from H which intersect γ, beginning with the edge closest to u on γ, $(e_1, e_2, ..., e_n)$. For each $i \in \{1, ..., n\}$, let v_i be an endpoint of e_i contained in F (choosing arbitrarily if both endpoints of e_i lie in F). We want to show that $(u, v_1, ..., v_n, v)$ is a (u, v)-walk in H.

Suppose $1 \leq i \leq n - 1$. If we restrict γ to the curve between its identified intersection with e_i and e_{i+1}, then the interior of the resulting curve does not intersect any edges or vertices of H. Thus, its interior is contained in a single face of H. Since H is a maximal plane graph, there are three vertices incident with this face, and they form a clique in H. In particular, either $v_i = v_{i+1}$ or v_i is adjacent to v_{i+1}. Similar arguments show that u is adjacent to v_1 and v is adjacent to v_n. Therefore, $(u, v_1, ..., v_n, v)$ is a (u, v)-walk in H consisting entirely of vertices in F, so F is connected in H. ■

Our characterization of k-blocking sets in one or two dimensions will depend upon the existence of a set of paths in G with certain properties. The next lemma will be useful in establishing this correspondence.

Let H be a simple, 2-connected graph embedded in the plane and let $C = (c_0, ..., c_{t-1})$ be the cycle in H which is peripheral to the unbounded face. Following Robertson and Seymour's treatment of the DISJOINT CONNECTING SUBGRAPHS problem in [17], for any 4-tuple, (i, j, i', j'), such that

$$0 \leq i < j \leq i' \leq j' \leq t - 1$$

we say that $\{c_i, c_{i'}\}$ **crosses** $\{c_j, c_{j'}\}$ in H. Notice that we allow the degenerate cases, where $i = i'$ or $j = j'$. For convenience, we may say that $\{a, b\}$ crosses $\{c, d\}$ in H without referring to indices. In this case, given an (a, b)-path, P, in H, it is a straightforward consequence of the Jordan Curve Theorem that any (c, d)-path must contain a vertex in P.

Lemma 2. *Let H be a 2-connected plane graph, let $C = (c_0, ..., c_{t-1})$ be the cycle in H which is peripheral to the unbounded face, and let $X \subseteq V(H)$. For $c_i, c_{i'} \in V(H)\backslash X$ such that $0 \leq i \leq i' \leq t-1$, there is a $(c_i, c_{i'})$-path in $H - X$ if and only if for every j, j' such that $i \leq j \leq i' \leq j' \leq t-1$, there is no $(c_j, c_{j'})$-path in $\widehat{H}[\overline{X}]$.*

Proof The forward direction is a consequence of the Jordan Curve Theorem. For the backward direction, we prove the contrapositive. Suppose there is no $(c_i, c_{i'})$-path in $H - X$. Define j so that c_{j-1} is the last vertex on the path, $(c_i, c_{i+1}..., c_{i'-1})$, which is in the same component of $H - X$ as c_i, and define j' so that $c_{j'+1}$ is the first vertex on the path, $(c_{i'+1}, c_{i'+2}..., c_i)$, which is in the same component of $H - X$ as c_i (here, our indices are modulo t). By definition, both $c_j, c_{j'} \in X$. We want to show that c_j and $c_{j'}$ lie in the same face of $\widehat{H} - \overline{X}$.

If not, there is some pair, $c_k, c_{k'} \in V(C) \cap (V(H)\backslash X)$, with

$$c_k \in \{c_{j+1}, c_{j+2}, ..., c_{j'-1}\} \text{ and } c_{k'} \in \{c_{j'+1}, c_{j'+2}, ..., c_{j-1}\},$$

such that there is a $(c_k, c_{k'})$-path in $H - X$ and $\{c_k, c_{k'}\}$ crosses $\{c_j, c_{j'}\}$ in H. We consider three cases.

Case 1: Suppose $k \in \{j+1, ..., i'-1\}$. If $k' \in \{j'+1, ..., i\}$, then $\{c_k, c_{k'}\}$ crosses $\{c_{j'+1}, c_i\}$. Therefore c_k is in the same component of $H - X$ as c_i, contradicting the maximality of $j-1$. Otherwise, $k' \in \{i+1, ..., j-1\}$, and $\{c_k, c_{k'}\}$ crosses $\{c_{j-1}, c_i\}$. Again, c_k is in the same component of $H - X$ as c_i, contradicting the maximality of $j-1$.

Case 2: Suppose $k \in \{i'+1, ..., j'-1\}$. If $k' \in \{j'+1, ..., i\}$, then $\{c_k, c_{k'}\}$ crosses $\{c_{j'+1}, c_i\}$. Therefore c_k is in the same component of $H - X$ as c_i, contradicting the minimality of $j'+1$. Otherwise, $k' \in \{i, ..., k\}$, and $\{c_k, c_{k'}\}$ crosses $\{c_{j+1}, c_i\}$. Again, c_k is in the same component of $H - X$ as c_i, contradicting the minimality of $j'+1$.

Case 3: Suppose $k = i'$. Then $\{c_k, c_{k'}\}$ crosses $\{c_{j-1}, c_{j'+1}\}$, so c_i is in the same component of $H - X$ as $c_{i'}$. This contradicts our assumption that there is no $(c_i, c_{i'})$-path in $H - X$.

Therefore, c_j and $c_{j'}$ lie in the same face, say f, of $\widehat{H} - \overline{X}$, where $\widehat{H}$ is a maximal plane graph. By lemma 1, the collection of vertices from $\overline{X}$ which lie in f is connected in $\widehat{H}$. In particular, there is a $(c_j, c_{j'})$-path in $\widehat{H}[\overline{X}]$. This completes the proof. ■

We are now prepared to prove our main results.

2.2 One-Dimensional Blocking

For an $m \times n$-grid, G, define the plane graph, G', obtained from G by adding two new vertices, e and w, to the unbounded face so that e is adjacent to each vertex in $East$ and w is adjacent to each vertex in $West$. For each $B \subseteq V(G)$, we can think of $\widehat{G}[\overline{B}]$ as a subgraph of $\widehat{G'}[\overline{B}]$ because every bounded face of G is a bounded face of G'. Moreover, there are $k+1$ vertex-disjoint $(East, West)$-path in $\widehat{G}[\overline{B}]$ just in case there are $k+1$ internally vertex-disjoint (e, w)-path in $\widehat{G'}[B \cup \{e, w\}]$. Define $B' = B \cup \{e, w\}$.

Theorem 1. *Let $m, n > 0$, let G be the $m \times n$-grid and let B be a subset of vertices from G. For each $k \geq 0$, B is a $(North, South)$-k-blocking set if and only if $\widehat{G}[\overline{B}]$ contains $k+1$ vertex-disjoint $(East, West)$-paths.*

Proof Suppose B is a $(North, South)$-k-blocking set in G. For the sake of contradiction, suppose $\widehat{G'}[\overline{B'}]$ contains at most k vertex-disjoint (e, w)-paths. By Menger's Theorem, there is some $X \subseteq \overline{B}$ separating e and w in $\widehat{G'}[\overline{B'}]$ with $|X| \leq k$. We claim there is some $F \subseteq B$ separating e and w in $\widehat{G'}[\overline{B'}]$ with $|F| = |X|$.

We prove the claim by induction on the number of vertices in $X \backslash B$. Let X be a minimum size set of vertices separating e and w in $\widehat{G'}[\overline{B'}]$. If $|X \backslash B| = 0$, then $X \subseteq B$ and we are done. Otherwise, there is some face, f, of G' such that $v_f \in X \backslash B$. By the minimality of X, there is an (e, w)-path P in $\widehat{G'}[\overline{B'}] - (X \backslash \{v_f\})$. Moreover, it must be the case that $v_f \in V(P)$, and the two neighbors of v_f in P are not adjacent in $\widehat{G'}$.

Clearly, neither e nor w is peripheral to f since each such face is bounded by a 3-cycle. Therefore, f is a bounded face in G; let $C_f = (a, b, c, d)$ be the cycle peripheral to f. Without loss of generality, we may assume that a and c are the two neighbors of v_f in P and $b, d \notin B \backslash X$. Let $X' = X \backslash \{v_f\} \cup \{a\}$. Notice that v_f has degree 1 in $\widehat{G'}[\overline{B'}] - X'$. Therefore, no (e, w)-path in $\widehat{G'}[\overline{B'}] - X'$ uses v_f. By the choice of X, there is no (e, w)-path in $\widehat{G'}[\overline{B'}] - (X \backslash \{v_f\})$ which does not use v_f. Hence, there is no (e, w)-path in $\widehat{G'}[\overline{B'}] - X'$. Finally, $|X| = |X'|$ and $|X \backslash B| - 1 = |X' \backslash B|$. This completes the induction.

We have shown the existence of some $F \subseteq B$ separating e and w in $\widehat{G'}[\overline{B'}]$ with $|F| \leq k$. That is, there is no (e, w)-path in $\widehat{G'}[\overline{B' \backslash F}]$. But, since B is a $(North, South)$-k-blocking set, there is no $(North, South)$-path in $G' - (B' \backslash F)$. This contradicts Lemma 2.

For the backward direction, suppose $\widehat{G'}[\overline{B'}]$ contains $k+1$ vertex disjoint (e, w)-paths. Then for any $F \subseteq B$ with $|F| \leq k$, there is an (e, w)-path in $\widehat{G'}[\overline{B' \backslash F}]$. By Lemma 2, there is no $(North, South)$-path in $G' - (B' \backslash F)$. Therefore, B is a $(North, South)$-k-blocking set. ■

Theorem 2. *Let $m, n > 0$, let G be the $m \times n$-grid and let B be a subset of vertices from G. For each $k \geq 0$, one can decide whether B is a $(North, South)$-k-blocking set in $O(mn)$ time.*

Proof Theorem 1 implies that B is a $(North, South)$-k-blocking set if and only if there are $k+1$ internally vertex-disjoint (e, w)-paths in $\widehat{G'}[\overline{B'}]$. Letting ℓ equal the number of vertices in $\widehat{G'}[\overline{B'}]$, we can compute the vertex connectivity between two vertices in a planar graph in $O(\ell)$ time ([2, 10]). Moreover, Euler's formula tells us that

$$\begin{aligned}\ell &\leq |V(\widehat{G'})\backslash\{v_{f_\infty}\}| \\ &= |V(G')| + |F(G')| = |E(G')| + 2 \\ &= n(m-1) + m(n-1) + 2n + 2 \\ &= 2mn + n - m + 2.\end{aligned}$$

Here, $F(G')$ is the set of faces in G' and f_∞ is the unbounded face of G'. ■

Define a weight function, σ, on $V(\widehat{G})$ so that

$$\sigma(v) = \begin{cases} 1, \, v \in V(G) \\ 0, \, v \in V(\widehat{G})\backslash V(G). \end{cases}$$

From σ, we obtain a weight function, $\sigma_{\leftrightarrow}$, on the vertices of $\widehat{G}_{\leftrightarrow}$ and a distance function δ on the edges of $\widehat{G}_{\leftrightarrow}$, as described in the preliminaries. These functions are easily extended to $\widehat{G'}$ and $\widehat{G'}_{\leftrightarrow}$ by giving each new vertex weight zero.

Now suppose we are given some initial set of sensors in the grid. We are interested in placing additional sensors to obtain a $(North, South)$-k-blocking set. Moreover, we would like to minimize the number of sensors used to obtain this result. Let $B_0 \subseteq V(G)$ be a set of sensors initially placed on the grid. We define a new weight function, σ_0, by altering σ so that $\sigma_0(v) = 0$ for each $v \in B_0$. The functions $\sigma_{0\leftrightarrow}$ and δ_0 are defined naturally from these new weights.

Theorem 3. *Let $m, n > 0$, let G be the $m \times n$-grid and let $B_0 \subseteq V(G)$ be given weight zero. For each $k \geq 0$, there is an $O(kmn \log(mn))$ algorithm to find a minimum size set, $B_1 \subseteq V(G)\backslash B_0$, such that $B_0 \cup B_1$ is a $(North, South)$-k-blocking set.*

Proof By Theorem 1, $B \subseteq V(G)$ is a $(North, South)$-k-blocking set if and only if there exist $k+1$ internally-disjoint (e, w)-paths in $\widehat{G'}_{\leftrightarrow}$. Given the weight function, $\sigma_{0\leftrightarrow}$, on the edges of $\widehat{G'}_{\leftrightarrow}$, we can use Suurballe's algorithm [19] to find $k+1$ internally vertex-disjoint (e, w)-paths of minimum total length. Since $\sigma(e), \sigma(w) = 0$, the total length of these $k+1$ paths will be equal to the number of vertices used which are in G and do not have a sensor placed on them. Let B be the set of vertices in these $k+1$ internally vertex-disjoint paths, and let B_1 be the set of vertices in B which have weight 1. Then $\overline{B_0 \cup B_1} \supseteq B$ and $\widehat{G}[\overline{B_0 \cup B_1}]$ contains $k+1$ vertex-disjoint $(East, West)$-paths. Therefore, $B_0 \cup B_1$ is a $(North, South)$-k-blocking set. By construction, no set smaller than B_1 has this property.

Since $\widehat{G}_{\leftrightarrow}$ is a plane graph, we can use Borradaile and Klein's shortest directed path algorithm from [2] in the implementation of Suurballe's algorithm. Borradaile and Klein's algorithm runs in $O(mn \log(mn))$ time, and Suurballe requires $k+1$ iterations. ■

2.3 Two-Dimensional Blocking

It is straightforward to extend Theorems 1 and 2 to two dimensions. First we characterize the two-dimensional solution in terms of disjoint paths.

Theorem 4. *Let $m, n > 0$, let G be the $m \times n$-grid and let B be a subset of vertices from G. For each $k \geq 0$, B is k-blocking set if and only if $\widehat{G}[\overline{B}]$ contains $k+1$ vertex-disjoint $(North, South)$-paths and $k+1$ vertex-disjoint $(East, West)$-paths.*

From this, the solution to the decision version follows:

Theorem 5. *Let $m, n > 0$, let G be the $m \times n$-grid and let B be a subset of vertices from G. For each $k \geq 0$, one can decide whether B is a k-blocking set in $O(mn)$ time.*

Using the distance function, δ, defined on $\widehat{G}$, we now describe a property of a minimum weight 0-blocking set.

Lemma 3. *Let $m, n > 0$, let G be the $m \times n$-grid and let $B \subseteq V(G)$ be given weight zero. If B is a minimum weight 0-blocking set, then $\widehat{G}[\overline{B}]$ contains a tree, T, such that $B \subseteq V(T)$, and there exist special vertices $u, v \in V(T)$ (possibly $u = v$) such that T is the union of five shortest paths, a (u, v)-path, a $(North, \{u\})$-path, a $(South, \{v\})$-path, and either a $(East, \{u\})$-path and a $(West, \{v\})$-path or an $(East, \{v\})$-path and a $(West, \{u\})$-path.*

Proof If B is a 0-blocking set, $\widehat{G}[\overline{B}]$ contains a $(North, South)$-path, say P, and an $(East, West)$-path, say Q. The endpoints of P cross the endpoints of Q; therefore, $S = P \cup Q$ is a connected graph. Moreover, S is a 0-blocking set, so by minimality, $B \subseteq V(S)$. Let P_1 be the subpath of P beginning with the initial vertex in $North$ and ending with the first vertex in $V(Q)$, say u. Let P_2 be the subpath of P beginning with the last vertex of P in $V(Q)$, say v, and ending with the terminal vertex in $South$. Either u occurs before v in Q, u occurs after v in Q or $u = v$.

In the first case, let Q_1 be the subpath of Q beginning with the initial vertex in $East$ and ending with u and let Q_2 be the subpath of Q beginning with v and ending with the terminal vertex in $West$. Let R be the subpath of P beginning with u and ending with v. Then $S' = P_1 \cup P_2 \cup Q_1 \cup Q_2 \cup R$ is a subgraph of S and is also a 0-blocking set. By minimality, $V(S') \cap B = V(S) \cap B$. Moreover, their definition ensures that these subpaths are internally disjoint. The minimality of B implies that P_1 is a shortest $(North, \{u\})$-path, P_2 is a shortest $(\{v\}, South)$-path, Q_1 is a shortest $(East, \{u\})$-path, Q_2 is a shortest $(\{v\}, West)$-path and R is a shortest (u, v)-path. The second and third cases follow similarly. ■

We use the above characterization to describe an algorithm for finding special vertices and a minimum 0-blocking set.

Algorithm: $\mathcal{A}_1$, Minimum sensor 0-blocking in G.

Input: Fixed integers, $m, n > 0$, the plane graph $\widehat{G}$ obtained from the $m \times n$-grid, G, and a set of vertices $B_0 \subseteq V(G)$.

Initialization: Order the vertices of $\widehat{G}$, $\{u_1, ..., u_s\}$. Here $s = |V(\widehat{G})|$. Define the distance function, δ_0, on $\widehat{G}$.

1. For $i = 1, ..., s$:

(a) Run the single-source shortest path algorithm from [10] on $\widehat{G}$ with source, u_i, obtaining a weighted distance, $\delta_0(u_i, v)$, for each $v \in V(\widehat{G})$.

(b) Set

$$\begin{aligned} r_N(u_i) &= \min\{\delta_0(u_i, v) : v \in North\}, \\ r_S(u_i) &= \min\{\delta_0(u_i, v) : v \in South\}, \\ r_E(u_i) &= \min\{\delta_0(u_i, v) : v \in East\}, \\ r_W(u_i) &= \min\{\delta_0(u_i, v) : v \in West\}. \end{aligned}$$

2. For $i = 1, ..., s$ and $j = i, ..., s$:

(a) Set

$$\begin{aligned} R_{NE}(i,j) &= r_N(u_i) + r_E(u_i) + r_S(u_j) + r_W(u_j) + \delta_0(u_i, u_j), \\ R_{NW}(i,j) &= r_N(u_i) + r_W(u_i) + r_S(u_j) + r_E(u_j) + \delta_0(u_i, u_j), \\ R_{SE}(i,j) &= r_S(u_i) + r_E(u_i) + r_N(u_j) + r_W(u_j) + \delta_0(u_i, u_j), \\ R_{SW}(i,j) &= r_S(u_i) + r_W(u_i) + r_N(u_j) + r_E(u_j) + \delta_0(u_i, u_j). \end{aligned}$$

(b) Set

$$\begin{aligned} R(i,j) = &\min\{R_{NE}(i,j), R_{NW}(i,j), R_{SE}(i,j), R_{SW}(i,j)\} \\ &- 2\sigma_0(u_i) - 2\sigma_0(u_j) \end{aligned}$$

and set $D(i,j) = (X, Y) \in \{North, South\} \times \{East, West\}$ such that $R(i,j) = R_{XY}(i,j)$.

3. Set $\rho(G) = \min\{R(i,j) : 1 \leq i \leq j \leq s\}$.
4. Set $(\alpha, \beta) = \min\{(i,j) : R(i,j) = \rho(G)\}$, given the lexicographic ordering on tuples.
5. Run the single-source shortest path algorithm from [10] on $\widehat{G}$ to find shortest paths for $(u_\alpha, X), (u_\alpha, Y), (u_\beta, X^c), (u_\beta, Y^c)$ and (u_α, u_β), where $D(\alpha, \beta) = (X, Y)$. Here, if $X = North$, then $X^c = South$; if $Y = West$, $Y^c = East$, etc. Stop.

Output: T, the graph obtained from the union of the five shortest paths found in step 5. $\rho(G)$, which gives the number of vertices in $(T \cap V(G)) \backslash (B_0)$.

Theorem 6. *Let $m, n > 0$, let G be the $m \times n$-grid and let $B_0 \subseteq V(G)$ contain sensors. There is an $O(m^2n^2)$ time algorithm for finding a minimum size set that extends B_0 to a 0-blocking set.*

Proof By Theorem 1, the set, $V(T)$, output by algorithm, $\mathcal{A}_1$ is a 0-blocking set. By Lemma 3, no 0-blocking set has smaller size. Step 1 of $\mathcal{A}_1$ consists of $s = 2mn + n - m + 2$ iterations of the shortest-path algorithm in [10], which runs in $O(mn)$ time. Thus, step 1 runs in $O(m^2n^2)$ time. Step 2 is iterated $\binom{s}{2} + s$ times, running in $O(m^2n^2)$ time. Step 5 is completed in $O(mn)$ time. Therefore, algorithm $\mathcal{A}_1$ runs in $O(m^2n^2)$ time. ∎

While a characterization similar to that Lemma 3 for k-blocking sets ($k > 0$) is easily derived, unfortunately it does not immediately lead to a polynomial time algorithm for finding the optimal placement.

Instead we describe an efficient 2-approximation algorithm for this case. The graph used is $\widehat{G}$ with weights on the vertices are as above.

1. Using Suurballe's algorithm (with the optimization by Borradaile and Klein) find $k+1$ disjoint paths of minimum total weight connecting $East$ to $West$ (adding a start node with weight 0 attached to all of the nodes in $East$ and a finish node with weight 0 attached to all of the nodes in $West$). Let those paths be $EW_0, \ldots, EW_k$ with total cost ew.
2. Using Suurballe's algorithm find $k+1$ disjoint paths of minimum total weight connecting $North$ to $South$. Let those paths be $NS_0, \ldots, NS_k$ with cost ns.
3. Return the combination of paths $EW_0, \ldots, EW_k$ and $NS_0, \ldots, NS_k$ with total weight at most $ew + ns$.

Theorem 7. *Let $m, n > 0$ and let G be the $m \times n$-grid and let $B_0 \subseteq V(G)$ contain sensors. There is an $O(kmn \log(mn)))$ algorithm for finding a set of vertices that extends B_0 to a k-blocking set and that is within a factor of 2 of optimal in size.*

Proof By Theorem 4 and by construction the set output by the algorithm above is a k-blocking set and it clearly runs in $O(kmn \log(mn))$ time. Let the value of the optimal solution by OPT. Observe that $OPT \geq ns$. This follows from the fact that the optimal solution must contain $k+1$ disjoint paths from $North$ to $South$ and therefore must have cost at least ns (which is optimal). Similarly, we have $OPT \geq ew$. If follows that the value of our solution is at most $ew + ns \leq 2 \cdot OPT$. ■

3 Extensions and open problems

Lemmas 1 and 2 are written in such generality as to allow us to easily extend our results to other domains and problems. For example, the original (planar) domain need not be rectangular and may contain "holes" representing impassable regions. The attacks detected need not be North-South or East-West paths but an intruder may enter at any contiguous portion of the border of the region and exit any other (disjoint) contiguous region. Multiple such disjoint attacks may be tested for simultaneously generalizing the results of Theorems 5, 6 and 7.

Two major open problems come to mind. The first is extending the result of Theorem 6 to $k > 0$ faults. While it is easy to generalize the characterization given in Lemma 3 for $k = 0$ faults to the case $k > 0$, it is not obvious that this results in a polynomial time algorithm. To make it effective, it seems that one must solve the minimum sum t vertex disjoints paths problem. In particular, a polynomial time solution to that problem would be sufficient to solve the two-dimensional k-protection placement problem in time $O((mn)^{2(k+1)^2})$ using an algorithm analogous to $\mathcal{A}_1$. It is known that for variable t this problem is NP-complete [12]. On the other hand, for fixed t the problem of deciding if the paths exist is in P [18]. While some progress has been made on this question [3], it remains open.

The second problem involves the case of movable sensors. Instead of replacing faulty sensors with new sensors, what if one was allowed to move non-faulty sensors to new points in the grid. The question of moving the least number of sensors the least total distance or the minimum maximum distance may be of interest. Related problems concerning coverage appear to be NP-hard [8]. An experimental study of a greedy strategy for this problem appears in [20].

References

1. P. Balister, B. Bollobas, A. Sarkar, and S. Kumar. Reliable density estimates for coverage and connectivity in thin strips of finite length. In *Proceedings of MobiCom '07*, pages 75–86. ACM, 2007.
2. G. Borradaile and P. Klein. An $O(n \log n)$ algorithm for maximum st-flow in a directed planar graph. *J. ACM*, 56(2):Art. 9, 30, 2009.
3. G. Borradaile, A. Nayyeri, and F. Zafarani. Towards single face shortest vertex-disjoint paths in undirected planar graphs. In *European Symposium on Algorithms*. Springer, 2015.
4. A. Chen, T. H. Lai, and D. Xuan. Measuring and guaranteeing quality of barrier-coverage in wireless sensor networks. In *Proceedings of the 9th ACM international symposium on Mobile ad hoc networking and computing*, pages 421–430. ACM, 2008.
5. D. Z. Chen, Y. Gu, J. Li, and H. Wang. Algorithms on minimizing the maximum sensor movement for barrier coverage of a linear domain. In *Proceedings of SWAT'12*, pages 177–188, 2012.
6. J. Czyzowicz, E. Kranakis, D. Krizanc, I. Lambadaris, L. Narayanan, J. Opatrny, L. Stacho, J. Urrutia, and M. Yazdani. On minimizing the maximum sensor movement for barrier coverage of a line segment. In *Proceedings of ADHOC-NOW, LNCS v. 5793*, pages 194–212, 2009.
7. J. Czyzowicz, E. Kranakis, D. Krizanc, I. Lambadaris, L. Narayanan, J. Opatrny, L. Stacho, J. Urrutia, and M. Yazdani. On minimizing the sum of sensor movements for barrier coverage of a line segment. In *Proceedings of ADHOC-NOW, LNCS v. 6288*, pages 29–42, 2010.
8. S. Dobrev. personal communication.
9. S. Dobrev, S. Durocher, M. Eftekhari, K. Georgiou, E. Kranakis, D. Krizanc, L. Narayanan, J. Opatrny, S. Shende, and J. Urrutia. Complexity of barrier coverage with relocatable sensors in the plane. In *CIAC*, pages 170–182. Springer, 2013.
10. M. R. Henzinger, P. Klein, S. Rao, and S. Subramanian. Faster shortest-path algorithms for planar graphs. *J. Comput. System Sci.*, 55(1, part 1):3–23, 1997. 26th Annual ACM Symposium on the Theory of Computing (STOC '94) (Montreal, PQ, 1994).
11. C. F. Huang and Y. C. Tseng. The coverage problem in a wireless sensor network. In *WSNA '03: Proceedings of the 2nd ACM International Conference on Wireless Sensor Networks and Applications*, pages 115–121. ACM, 2003.

12. R. Karp. On the complexity of combinatorial problems. *Networks*, 5:45–68, 1975.
13. S. Kumar, T. H. Lai, and A. Arora. Barrier coverage with wireless sensors. In *Proceedings of MobiCom '05*, pages 284–298. ACM, 2005.
14. S. Meguerdichian, F. Koushanfar, M. Potkonjak, and M.B. Srivastava. Coverage problems in wireless ad-hoc sensor networks. In *Proceedings of INFOCOM, vol, 3*, pages 1380–1387, 2001.
15. M. Mehrandish, L. Narayanan, and J. Opatrny. Minimizing the number of sensors moved on line barriers. In *Proceedings of IEEE WCNC'11*, pages 1464–1469, 2011.
16. T. Park and H. Shi. Extending the lifetime of barrier coverage by adding sensors to a bottleneck region. In *12th IEEE Consumer Communications and Networking Conference (CCNC)*. IEEE, 2015.
17. N. Robertson and P. D. Seymour. Graph minors. VI. Disjoint paths across a disc. *J. Combin. Theory Ser. B*, 41(1):115–138, 1986.
18. N. Robertson and P. D. Seymour. Graph minors. XIII. The disjoint paths problem. *J. Combin. Theory Ser. B*, 63:65–110, 1995.
19. J. W. Suurballe. Disjoint paths in a network. *Networks*, 4:125–145, 1974.
20. Huan Xie, Menglong Li, Weidong Wang, Chaowei Wang, Xiuhua Li, and Yinghai Zhang. Minimal patching barrier healing strategy for barrier coverage in hybrid wsns. In *International Symposium on Personal, Indoor, and Mobile Radio Communication (PIMRC)*, pages 1558–1563. IEEE, 2014.