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Computer Vision

Projective geometry and 3D transformations

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Projective 3-space

Similarly to the 2D projective space, a point **x** in $\mathbb{R}^3$ is represented in homogeneous coordinates as a 4 dimensional vector:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \rightarrow \begin{pmatrix} wx \\ wy \\ wz \\ w \end{pmatrix} \in \mathbb{P}^3 \,, w \in \mathbb{R} - \{0\}$$

The points at infinity have 0 in the last component and cannot be transformed back to inhomogeneous coordinates

$$\begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix}$$

3D Planes

In $\mathbb{P}^3$ points and planes are dual, similarly of what happens with points and lines in $\mathbb{P}^2$.

A plane in 3D Euclidean space can be represented as the locus of points $\mathbf{p} = (x, y, z)^T \in \mathbb{R}^3$ such that:

$$\pi_1 x + \pi_2 y + \pi_3 z + \pi_4 = 0$$

In homogeneous coordinates, the same relation can be expressed as:

$$\pi^T \mathbf{x} = 0 \quad \text{with} \quad \pi \in \mathbb{P}^3, \mathbf{x} \in \mathbb{P}^3$$

3D Planes

$$\pi = (\pi_1, \pi_2, \pi_3, \pi_4)^T$$

The first 3 components $N = (\pi_1, \pi_2, \pi_3)$ define the plane normal. If the vector is normalized such that $||N|| = 1$ then π_4 is the plane distance to the origin.

3 non-collinear points are needed to define a plane. The best way to describe it is by stacking the points in a 3x4 matrix such that:

$$\begin{pmatrix} X_1^T \\ X_2^T \\ X_3^T \end{pmatrix} \pi = 0$$

π is obtained (up to scale, since we are in homogeneous coordinates) as the 1-dimensional right null-space of the matrix.

3D Planes

Since planes and points are dual, it is also true that 3 (non parallel) planes π_1, π_2, π_3 define a point.

The intersection point of the 3 planes can be obtained in a similar manner by computing the right null-space of the 3x4 matrix composed by stacking the planes:

$$\begin{pmatrix} \pi_1^T \\ \pi_2^T \\ \pi_3^T \end{pmatrix} X = 0$$

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Projective transformations

A projective transformation of 3-space is a linear transformation in $\mathbb{P}^3$ that can be represented by any non-singular 4x4 matrix:

$$\begin{pmatrix} \mathbf{A} & \mathbf{t} \\ V^T & v \end{pmatrix}$$

Where $\mathbf{A}$ is a 3x3 invertible matrix, V^T and $\mathbf{t}$ are 3D vectors and v is a scalar. Since the transformation is up to scale is subject to **15 dof**.

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Rigid motion

The Euclidean transfomation is a projective transformation composed by a rotation around an axis and a translation:

$$\begin{pmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix}$$

It is subject to 6 dof. (3 for rotation, 3 for the translation part). It is very important because it preserve the distances, parallelism of planes and lines and the volume. For this reason it is also called **rigid motion**.

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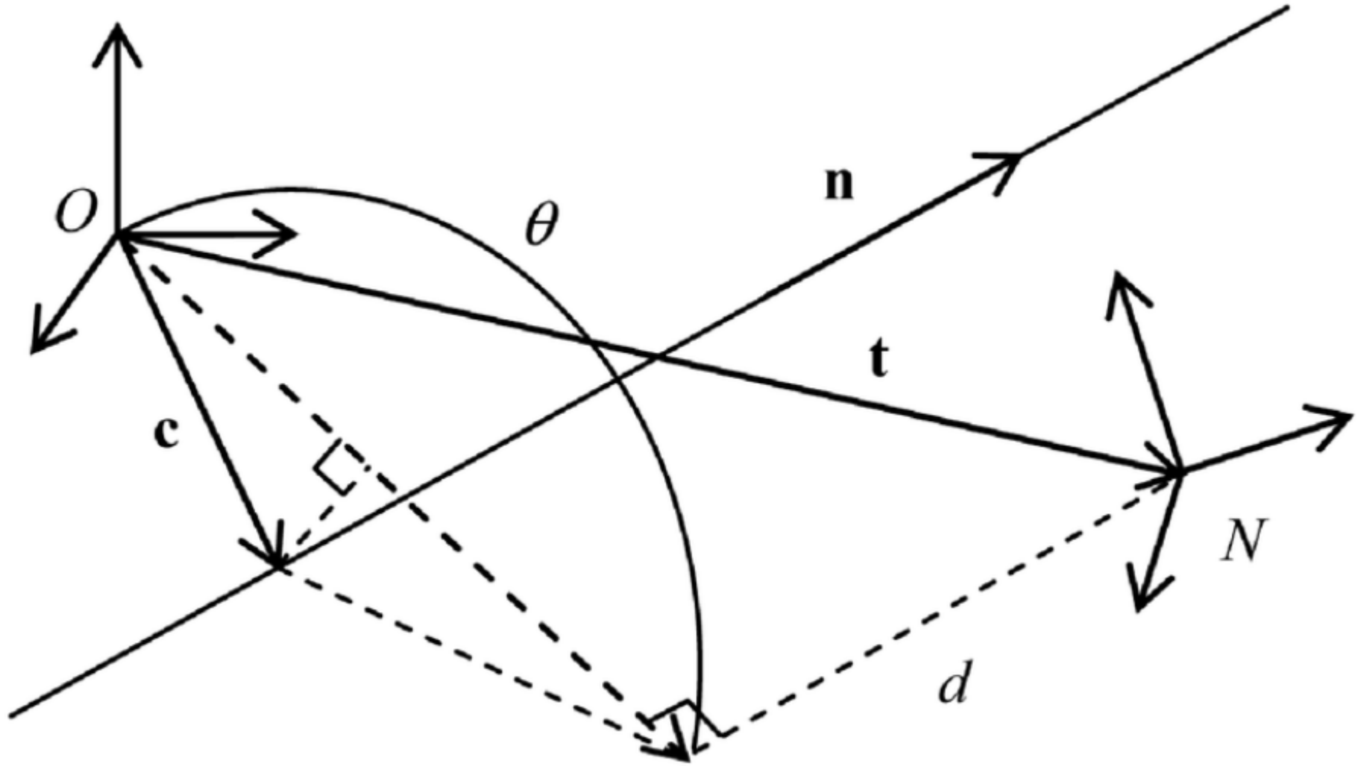
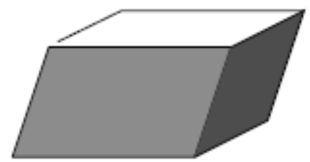
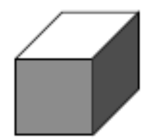
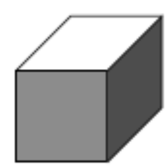
Chasles' Theorem



Any particular translation and rotation is equivalent to a rotation about a screw axis together with a translation along the screw axis. The screw axis is parallel to the rotation axis.

Hierarchy of transformations

Group	Matrix	Distortion	Invariant properties
Projective 15 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^{\mathsf{T}} & v \end{bmatrix}$		Intersection and tangency of surfaces in contact. Sign of Gaussian curvature.
Affine 12 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		Parallelism of planes, volume ratios, centroids. The plane at infinity, π_∞, (see section 3.5).
Similarity 7 dof	$\begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		The absolute conic, Ω_∞, (see section 3.6).
Euclidean 6 dof	$\begin{bmatrix} \mathtt{R} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		Volume.

Plane at infinity

In $\mathbb{P}^3$ we can define the so-called plane at infinity with canonical position $\pi_\infty = (0,0,0,1)^T$.

π_∞ contains all the directions (ideal points) $(D_1, D_2, D_3, 0)^T$

- Two planes are parallel if, and only if, their line of intersection is on π_∞
- A line is parallel to another line, or to a plane, if the point of intersection is on π_∞

Plane at infinity

The plane at infinity is important because it remains fixed under an affine transformation but not by a general projective transformation (the behavior is similar to the line at infinity).

The plane at infinity is a fixed plane under the projective transformation H if and only if H is an affinity. Note that:

- The plane is not fixed "point-wise" but it is just mapped to the same plane
- π_∞ may not be the only fixed plane (for example a plane orthogonal to the rotation axis in a rigid motion is fixed)