A new Interpolation approach for Sea Temperature and Salinity Enforcing Hydrostatic Equilibrium

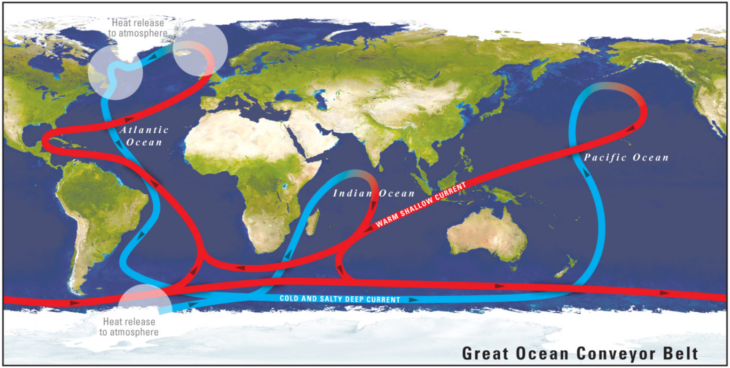
Filippo Bergamasco, PhD

Department of Environmental Sciences, Informatics and Statistics
University Ca'Foscari of Venice (Italy)

14 Oct 2015

Introduction

One of the core topics of phisical oceanography is to study the movement of sea water masses around the globe.



Università Ca' Foscari Venezia

What causes the water to move?

Surface current is, intuitively, caused by the wind. What about the bottom layers?

Thermohaline circulation

In the deep ocean, sea water movements are driven by **temperature** and **salinity** variations which, in turn, cause differences in **density**.

- Lighter water masses float over denser ones

The measurement of sea water density is one of the basic tools to study the ocean circulation that affects the earth climate.

How do we measure sea water density?

Direct measurement is highly impractical to be performed on the field.

Most of the time it is calculated from in situ **sparse** measurements of Temperature and Salinity.

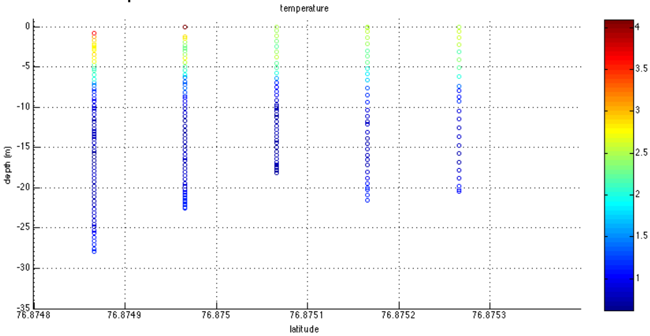
$$
\begin{aligned}
D(t,s) &= As + Bs^{3/2} + Cs^2 \\
A &= 8.24 \cdot 10^{-1} - 4.08 \cdot 10^{-3} t + 7.64 \cdot 10^{-5} t^2 \\
&\quad -8.24 \cdot 10^{-7} t^3 + 5.38 \cdot 10^{-9} t^4 \\
B &= -5.72 \cdot 10^{-3} + 1.022 \cdot 10^{-4} t 1.654 \cdot 10^{-6} t^2 \\
C &= 4.8314 \cdot 10^{-4}
\end{aligned}
$$

An instrument called CTD is deployed on water given a sparse set of measurements within an area

Università Ca' Foscari Venezia

How do we interpolate the data samples?

To study the water stratification, the collected sparse samples must be interpolated to a 2D or 3D field



Two common approaches:

1. Statistical methods interpolate T and S independently, exploiting spatial properties of the data **(No phisical constraints!)**
2. Model based approaches based on accurate physical simulations **(Difficult initialization and boundary conditions!)**

Università Ca' Foscari Venezia

Our general goal

Interpolate temperature and salinity field in a simple manner without using an accurate phisical model

- But... enforce some basic physical constraint to improve the interpolation

What we suppose?

- All the samples are taken "at the same time"
- Sea water field is stationary (not changing over time)

What must be satisfied?

- Less-dense water must be above denser water (**hydrostatic equilibrium**)

Problem formulation

We suppose to have:

- A discrete vertical 2D temperature $T(i,j)$ and salinity field $S(i,j)$, defined over a regular grid of $M \times N$ points.
- A sparse set of $N_m \ll M \times N$ temperature and salinity measurements taken at certain grid points.

Specifically, let $T_d(1) \dots T_d(N_m)$ be the temperature measurements taken at grid coordinates $(i_1^t, j_1^t) \dots (i_{N_m}^t, j_{N_m}^t)$ and $S_d(1) \dots S_d(N_m)$ be the salinity measurements taken at grid coordinates $(i_1^s, j_1^s) \dots (i_{N_m}^s, j_{N_m}^s)$.

- A function $D(T_{ij}, S_{ij})$ mapping $T(i,j)$ and $S(i,j)$ to the empirical sea water density at 1 Atm.

Università
Ca' Foscari
Venezia

Problem formulation

We pose the temperature and salinity interpolation problem as the following constrained minimization:

$$
\begin{aligned}
\underset{T,S}{\text{argmin}} \quad & \alpha \sum_{k=1}^{N_m} \left(T(i_k^t, j_k^t) - T_d(k) \right)^2 + \\
& \beta \sum_{k=1}^{N_m} \left(S(i_k^s, j_k^s) - S_d(k) \right)^2 + \\
& \rho_T \sum_i \sum_j \left(\Delta T(i,j) \right)^2 + \\
& \rho_S \sum_i \sum_j \left(\Delta S(i,j) \right)^2
\end{aligned}
$$

$$
\begin{aligned}
\text{subject to} \quad & D(T_{ij}, S_{ij}) \geq D(T_{i-1\,j}, S_{i-1\,j}), \\
& \forall \; 1 < i \leq M, \; 1 \leq j \leq N
\end{aligned}
$$

Università Ca' Foscari Venezia

Problem formulation

Our goal is to recover T and S given the sparse measurements T_d and S_d by simultaneously:

- Minimizing the fitting error at the data points. Intuitively, $T(i,j)$ should be almost equal to $T_d(i,j)$ for each $(i,j) = (i_k^t, j_k^t)$. (The same principle is applied to salinity as well)
- Enforcing the hydrostatic equilibrium so that the associated density field gradient is orented downward (ie. the higher grid row, higher the density)
- Minimizing the total squared curvature of T and S

Let's see the energy function again...

$$\underset{T,S}{\operatorname{argmin}} \quad \alpha \sum_{k=1}^{N_m} \left(T(i_k^t, j_k^t) - T_d(k) \right)^2 + \beta \sum_{k=1}^{N_m} \left(S(i_k^s, j_k^s) - S_d(k) \right)^2 + \rho_T \sum_i \sum_j (\Delta T(i,j))^2 + \rho_S \sum_i \sum_j (\Delta S(i,j))^2$$

$$\text{subject to} \quad D(T_{ij}, S_{ij}) \geq D(T_{i-1\,j}, S_{i-1\,j}), \quad \forall \ 1 < i \leq M, \ 1 \leq j \leq N$$

- Essentially a non linear least squares
- Energy constraints let the optimization difficult to optimize in practice

Università Ca' Foscari Venezia

Convex relaxation

We introduce a new scalar field D_n, and solve the new problem:

$$\begin{aligned} \underset{T,S,D_n}{\operatorname{argmin}} \quad & \alpha \textstyle\sum_{k=1}^{N_m} \left(T(i^t_k, j^t_k) - T_d(k) \right)^2 + \\ & \beta \textstyle\sum_{k=1}^{N_m} \left(S(i^s_k, j^s_k) - S_d(k) \right)^2 + \\ & \rho_T \textstyle\sum_i \sum_j \left(\Delta T(i,j) \right)^2 + \\ & \rho_S \textstyle\sum_i \sum_j \left(\Delta S(i,j) \right)^2 \\ & \rho_D \textstyle\sum_i \sum_j \left(D(T_{ij}, S_{ij}) - D_n(i,j) \right)^2 \\ \text{subject to} \quad & \\ & D_n(i,j) \geq D_n(i-1,j), \\ & \forall \ 1 < i \leq M, \ 1 \leq j \leq N \end{aligned}$$

Università Ca' Foscari Venezia

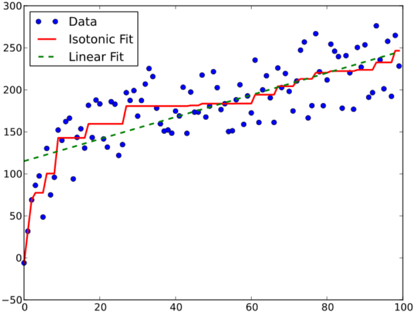
Isotonic Regression

Why we did that simplification?

There exists an efficient $O(n)$ solution for the problem

$$\begin{aligned} \underset{D_n}{\text{argmin}} \quad & \rho_D \sum_i \sum_j \left(D(i,j) - D_n(i,j)\right)^2 \\ \text{subject to} \quad & D_n(i,j) \geq D_n(i-1,j), \\ & \forall \; 1 < i \leq M, \; 1 \leq j \leq N \end{aligned}$$

via the so-called **Isotonic Regression**



Università
Ca' Foscari
Venezia

Numerical solution

To numerically solve the optimization, we iterate between the following two minimizations:

$$\underset{T,S}{\text{argmin}} \quad \alpha \sum_{k=1}^{N_m} \left(T(i_k^t, j_k^t) - T_d(k)\right)^2 + \beta \sum_{k=1}^{N_m} \left(S(i_k^s, j_k^s) - S_d(k)\right)^2 + \rho_T \sum_i \sum_j (\Delta T(i,j))^2 + \rho_S \sum_i \sum_j (\Delta S(i,j))^2 \; \rho_D \sum_i \sum_j (D(T_{ij}, S_{ij}) - D_n(i,j))^2 \qquad (1)$$

$$\underset{D_n}{\text{argmin}} \quad \rho_D \sum_i \sum_j (D(T_{ij}, S_{ij}) - D_n(i,j))^2 \qquad (2)$$

$$\text{subject to} \quad D_n(i,j) \geq D_n(i-1,j), \quad \forall \; 1 < i \leq M, \; 1 \leq j \leq N$$

Density linearization

Problem (1) is still non-linear due to the function D.
Two ways to overcome the problem:

- Directly optimize (1) via Levenberg-Marquardt (slow)
- Linearize D and take an iterative approach (very fast and effective in this case)

$$
\begin{aligned}
\hat{D}(T^n, S^n) &= D(T^{n-1}, S^{n-1}) + \\
&+ (T^n - T^{n-1})\frac{\delta}{\delta T} D(T^{n-1}, S^{n-1}) + \\
&+ (S^n - S^{n-1})\frac{\delta}{\delta S} D(T^{n-1}, S^{n-1})
\end{aligned}
$$

Solving problem (1)

- Start from an initial interpolation of temperature and salinity
- Iteratively solve:

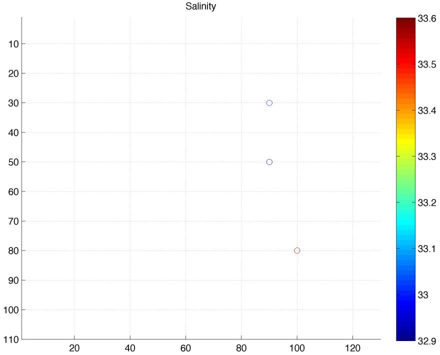
$$
\begin{aligned}
\underset{T^n, S^n}{\operatorname{argmin}} \quad & \alpha \sum_{k=1}^{N_m} \left(T^n(i_k^t, j_k^t) - T_d(k) \right)^2 + \\
& \beta \sum_{k=1}^{N_m} \left(S^n(i_k^s, j_k^s) - S_d(k) \right)^2 + \\
& \rho_T \sum_i \sum_j (\Delta T^n(i,j))^2 + \\
& \rho_S \sum_i \sum_j (\Delta S^n(i,j))^2 \\
& \rho_D \sum_i \sum_j (\hat{D}(T^n, S^n) - D_n(i,j))^2
\end{aligned}
$$

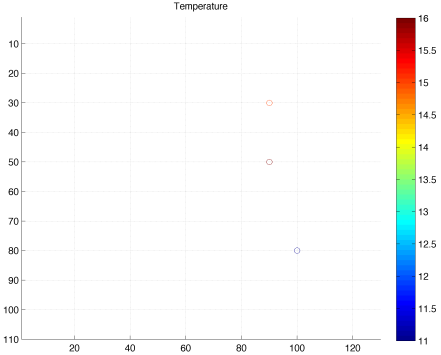
Until $max(|T^n - T^{n-1}|)$ and $max(|S^n - S^{n-1}|)$ are below a threshold

Minimizing the whole problem

1. Compute an initial estimate of T and S (with any interpolation method)
2. Compute $D_n = D(T_{ij}, S_{ij})$
3. Solve problem (1) to obtain a new estimate of T and S
4. Solve problem (2) via isotonic regression to obtain a new estimate of D_n
5. Return to step 3 until convergence

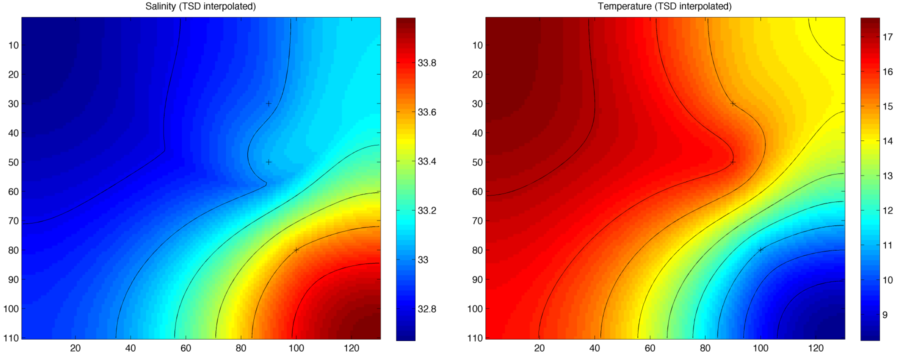
Some toy examples





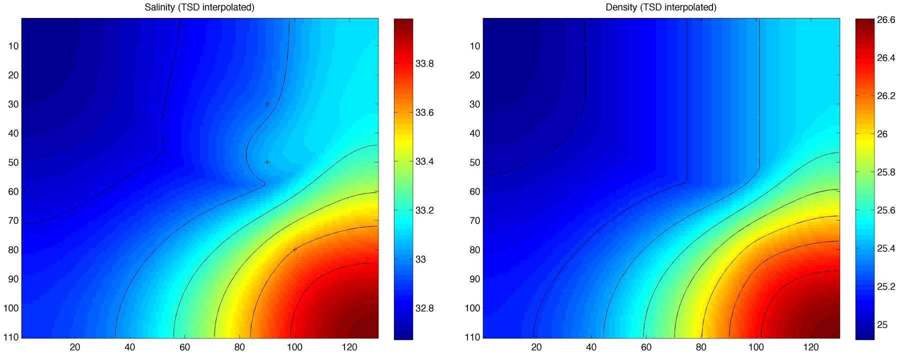
A simple test with just 3 points

Some toy examples



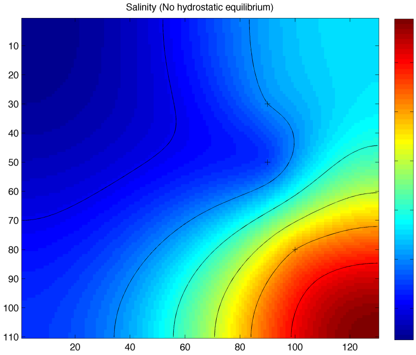
Interpolated salinity and temperature fields

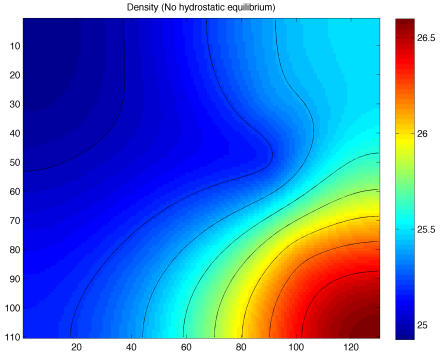
Some toy examples



Interpolated salinity and density fields

Some toy examples



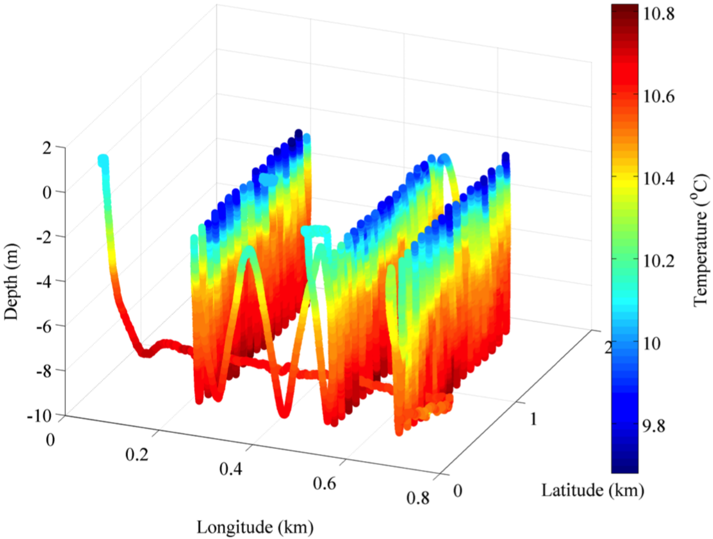


Interpolated result without hydrostatic constraint

Case study: Data from an UAV

- REMUS was deployed (Feb 2014) near Isonzo river.
- It acquired data spanning different lat/lon/depth planes

Case study: Data from an UAV



Università Ca' Foscari Venezia

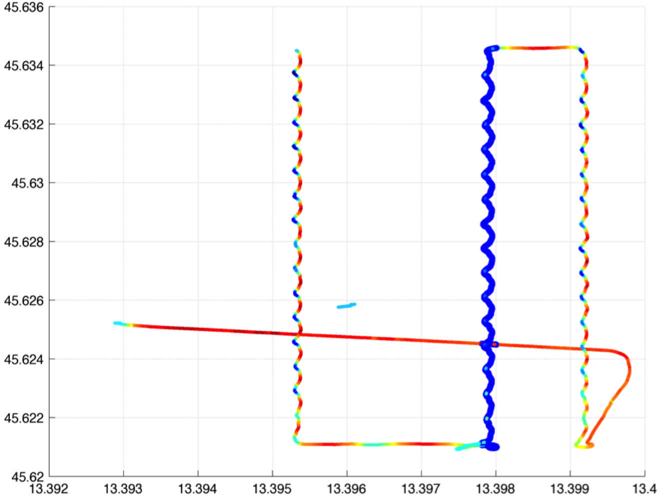
Introduction

Problem formulation

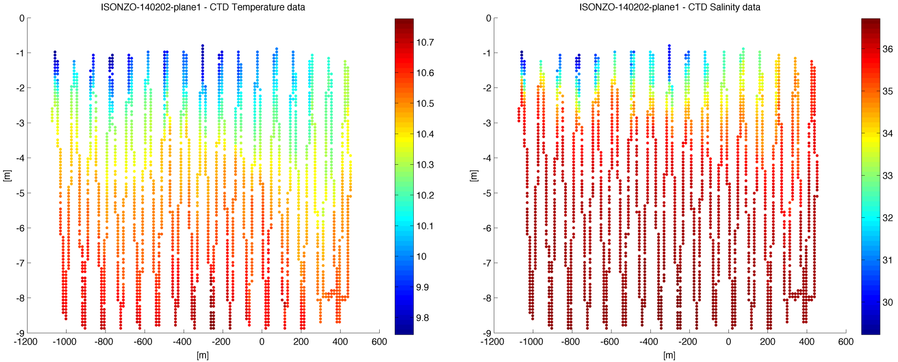
Optimization

> Experiments

Case study: Data from an UAV

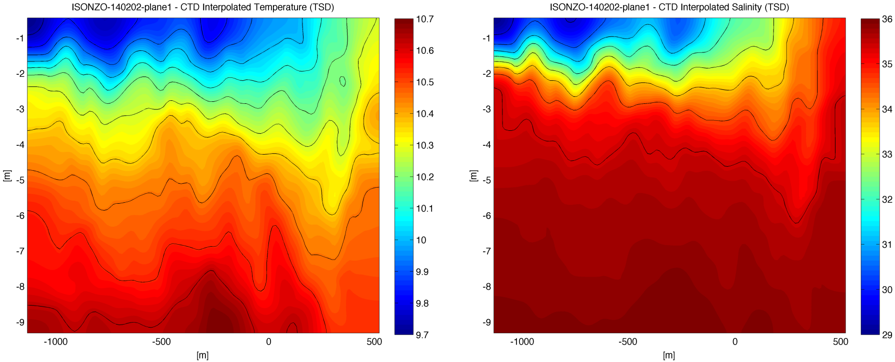


Case study: Data from an UAV



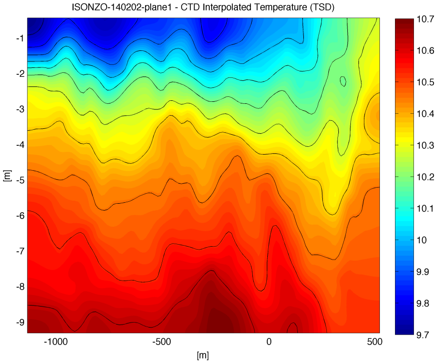
Temperature and salinity

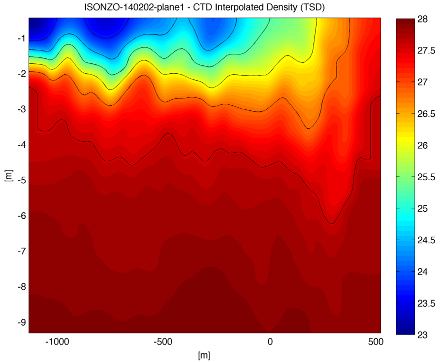
Case study: Data from an UAV



Interpolated Temperature and salinity

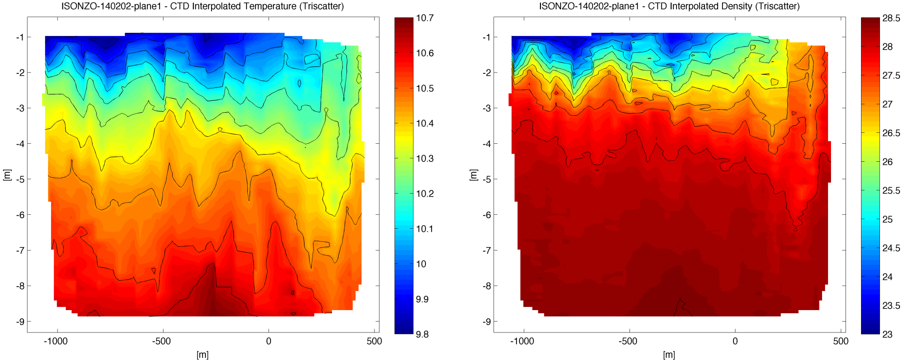
Case study: Data from an UAV





Interpolated Temperature and Density

Case study: Data from an UAV



Temperature and Density (Matlab Triscatter interp)

Conclusions

- We developed a simple yet powerful interpolation method for sea temperature and salinity
- By enforcing hydrostatic equilibrium we both ensure some physical properties of the field and improve the interpolation even with few data
- Preliminary synthetic tests demonstrate the potentials of such approach

For the future?

- Give an estimate of the interpolation error over the field
- 3D interpolation
- Consider the temporal extent of the data to introduce additional constraints on the velocity fields

Università
Ca' Foscari
Venezia

Thank you for your attention

http://dsi.unive.it/~bergamasco
filippo.bergamasco@unive.it