

A Continuous Characterization of Maximal Cliques in k -uniform Hypergraphs

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Abstract. In 1965 Motzkin and Straus established a remarkable connection between the local/global maximizers of the Lagrangian of a graph G over the standard simplex Δ and the maximal/maximum cliques of G . In this work we generalize the Motzkin-Straus theorem to k -uniform hypergraphs, establishing an isomorphism between local/global minimizers of a particular function over Δ and the maximal/maximum cliques of a k -uniform hypergraph. This theoretical result opens the door to a wide range of further both practical and theoretical applications, concerning continuous-based heuristics for the maximum clique problem on hypergraphs, as well as the discover of new bounds on the clique number of hypergraphs. Moreover we show how the continuous optimization task related to our theorem, can be easily locally solved by mean of a dynamical system.

1 Introduction

Many problems of practical interest are inherently intractable, in the sense that it is not possible to find fast (i.e., polynomial time) algorithms to solve them exactly, unless the classes P and NP coincide. The Maximum Clique Problem (MCP) is one of the most famous intractable combinatorial optimization problems, that asks for the largest complete subgraph of a given graph. This problem is even hard to approximate within a factor of $n/2^{(\log n)^{1-\epsilon}}$ for any $\epsilon > 0$ where n is the number of nodes in the graph [15]. Although this pessimistic state of affairs and because of its important applications in different fields such as computer vision, experimental design, information retrieval and fault tolerance, much attention has gone into developing efficient heuristics for the MCP, even if no formal guarantee of performance may be provided, but are nevertheless useful in practical applications. Moreover many important problems can be easily reduced to maximum clique problem e.g. boolean satisfiability problem, subgraph isomorphism problem, vertex cover problem etc.

Plenty of heuristics have been proposed over the last 50 years and we refer to [7] for a complete survey about complexity issues and applications of the MCP. In this introduction, we will focus our attention in particular on the continuous-based class of heuristics, since they are strongly related to the topics addressed in this paper. The heuristics of this class are mostly based on a result

of Motzkin and Straus [17] that establishes a remarkable connection between the maximum clique problem and the extrema of the Lagrangian of a graph (1). In Section 2 we will see in deeper details the Motzkin-Straus theorem, but briefly it states, especially in its regularized version, an isomorphism between the set of maximal/maximum cliques of an undirected graph G and the set of local/global maximizers of the Lagrangian of G . This continuous formulation of the MCP suggests a fundamental new way of solving this problem, by allowing a shift from the discrete domain to the continuous one in an elegant manner. As pointed out in [20] continuous formulations of discrete problems are particularly attractive, because they not only allow us to exploit the full arsenal of continuous optimization techniques, thereby leading to the development of new algorithms, but may also reveal unexpected theoretical properties.

From an applicative point of view the Motzkin-Straus result led to the development of several MCP heuristics [6, 8, 13, 21, 23], but very interesting are also its theoretical implications. This result in fact was originally achieved by Motzkin and Straus to support an alternative proof of a slightly weaker version of the fundamental Turán theorem [27], moreover it was successfully used to achieve several bounds for the clique number of graphs [9, 28, 29]. The Motzkin-Straus theorem was also successfully generalized to vertex-weighted graphs [14] and edge-weighted graphs [22].

Recently the interest of researchers in many fields is focusing on hypergraphs, i.e. generalizations of graphs where edges are subsets of vertices, because of their greater expressiveness in representing higher-order relations. Just as graphs naturally represent many kinds of information in mathematical and computer science problems, hypergraphs also arise naturally in important practical problems [10, 19, 30]. Moreover, many theorems involving graphs, as for example the Ramsey's theorem or the Szemerédi lemma, also hold for hypergraphs, in particular for k -uniform hypergraphs (or more simply k -graphs), i.e. hypergraphs whose edges have all cardinality k . Nevertheless, all known intractable problems on graphs can be reformulated on hypergraphs and in particular the maximum clique problem.

Even if clique problems on hypergraphs are gaining increasing popularity in several scientific communities, a bridge from these discrete structures to the continuous domain is still missing. With our work we will fill up this gap, in the same way as the Motzkin-Straus theorem filled it up in the context of graphs. Hence the contribution of this paper is purely theoretical and basically consists in a generalization of the Motzkin-Straus theorem to k -uniform hypergraphs. However, as happened for the Motzkin-Straus theorem, our hope is to open the door to a wide range of further both practical and theoretical applications. First of all, we furnish a continuous characterization of maximal cliques in k -graphs, allowing the development of continuous-based heuristics for the maximum clique problem over hypergraphs based on it. Thereby, in Section 5 we provide a discrete dynamical system to elegantly find maximal cliques in k -graphs, that turns out to include the heuristic for MCP developed by Pelillo [23] on graphs as a special case (in fact graphs are 2-uniform hypergraphs). Moreover our theorem can be

used to achieve new bounds for the clique number on k -graphs, a very popular problem in the extremal graph theory field, however we leave this topic as a future development of this work.

2 The Motzkin-Straus theorem

Let $G = (V, E)$ be an (undirected) graph, where $V = \{1, \dots, n\}$ is the vertex set and $E \subseteq \binom{V}{2}$ is the edge set, with $\binom{V}{k}$ denoting the set of all k -element subsets of V . A *clique* of G is a subset of mutually adjacent vertices in V . A clique is called *maximal* if it is not contained in any other clique. A clique is called *maximum* if it has maximum cardinality. The maximum size of a clique in G is called the *clique number* of G and is denoted by $\omega(G)$.

Consider the following function $L_G : \Delta \mapsto \mathbb{R}$, sometimes called the *Lagrangian* of graph G

$$L_G(\mathbf{x}) = \sum_{\{i,j\} \in E} x_i x_j \quad (1)$$

where

$$\Delta = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq 0, \sum_{i=1}^n x_i = 1\}$$

is the *standard simplex*.

In 1965, Motzkin and Straus [17] established a remarkable connection between the maxima of the Lagrangian of a graph and its clique number.

Theorem 1 (Motzkin-Straus). *Let G be a graph with clique number $\omega(G)$, and \mathbf{x}^* a maximizer of L_G then*

$$L_G(\mathbf{x}^*) = \frac{1}{2} \left[1 - \frac{1}{\omega(G)} \right]$$

Additionally Motzkin and Straus proved that a subset of vertices S is a maximum clique of G if and only if its *characteristic vector* \mathbf{x}^S is a global maximizer of L_G .¹ The characteristic vector of a set S is the vector in Δ defined as:

$$\mathbf{x}_i^S = \frac{1_{i \in S}}{|S|}$$

where $|S|$ indicates the cardinality of the set S and 1_P is an indicator function giving 1 if property P is satisfied and 0 otherwise. With $\sigma(\mathbf{x})$ we will denote the *support* of a vector $\mathbf{x} \in \Delta$, i.e. the set of positive components in \mathbf{x} . For example, the support of the characteristic vector of a set S is S .

Gibbons, Hearn, Pardalos and Ramana [14], and Pelillo and Jagota [24], extended the theorem of Motzkin and Straus, providing a characterization of maximal cliques in terms of local maximizers of L_G , however not all local maximizers

¹ Actually, Motzkin and Straus provided just the “only if” part of this theorem, even if the converse direction is a direct consequence of their results [24].

were in the form of a characteristic vector. Finally Bomze et al. [6] introduced a regularizing term in the graph Lagrangian obtaining $L_G^\tau : \Delta \mapsto \mathbb{R}$ defined as

$$L_G^\tau(\mathbf{x}) = L_G(\mathbf{x}) + \tau \sum_{i \in V} x_i^2$$

and proved that all local maximizers of L_G^τ are strict and in one-to-one relation with the characteristic vector of the maximal cliques of G , provided that $0 < \tau < \frac{1}{2}$.

Theorem 2 (Bomze). *Let G be a graph and $0 < \tau < \frac{1}{2}$. A vector $\mathbf{x} \in \Delta$ is a global/local maximizer of L_G^τ over Δ if and only if it is the characteristic vector of a maximum/maximal clique of G .*

The Motzkin-Straus theorem was successfully extended also to vertex-weighted graphs by Gibbons et al. [14] and edge-weighted graphs by Pavan and Pelillo [22].

In this paper we provide a further generalization of the Motzkin-Straus as well as the Bomze theorems to k -uniform hypergraphs, but firstly, we will introduce hypergraphs and review another generalization of the Motzkin-Straus theorem due to Sós and Straus [26].

3 k -uniform hypergraphs

Let $\mathcal{P}(A)$ be the power set of A . A *hypergraph* G is a pair (V, E) where $V = \{1, \dots, n\}$ is a set of vertices and $E \subseteq \mathcal{P}(V)$ is a set of hyperedges. If all hyperedges have cardinality k , then the hypergraph is *k -uniform* (or more easily it is called *k -graph*). A *clique* C of G is a set of vertices such that every subset of C of order k forms an hyperedge of G . A clique is *maximal* if it is not contained in any other clique. It is *maximum* if it has maximum cardinality. The *clique number* $\omega(G)$ of a k -graph G is the cardinality of a maximum clique.

The Lagrangian of a k -graph $G = (V, E)$ is denoted by $L_G : \Delta \mapsto \mathbb{R}$ and defined as

$$L_G(\mathbf{x}) = \sum_{e \in E} \prod_{j \in e} x_j. \quad (2)$$

Unfortunately L_G cannot be directly used to extend the Motzkin-Straus theorem to k -graphs. Frankl and Rödl [12] proved that by taking a maximizer \mathbf{x}^* of L_G with support as small as possible, the subhypergraph induced by the support of \mathbf{x}^* is a 2-cover, i.e. a hypergraph such that every pair of vertices is contained in some hyperedge. Since 2-covers in graphs are basically cliques, we could expect a possible generalization of the Motzkin-Straus theorem where the clique number is replaced by the size l of the maximum 2-cover in the hypergraph. However \mathbf{x}^* is not necessarily in the form of a characteristic vector, and it is not in general possible to express l as a function of $L_G(\mathbf{x}^*)$. Nevertheless, this result was used by Mubay [18] to achieve a bound for $L_G(\mathbf{x}^*)$ in terms of l on k -graphs, obtaining

$$L_G(\mathbf{x}^*) \leq \binom{l}{k} l^{-k}.$$

and he used it to provide an hypergraph extension of the Turán's theorem.

The only real case of generalization of the Motzkin-Straus theorem to k -uniform hypergraphs, although not explicit and not general, is due to Sós and Straus [26]. They attach a nonnegative weight $x(H_l)$ to every complete l -subgraph H_l of a graph G , normalized by the condition

$$\sum_{H_l \subseteq G} x(H_l)^l = 1$$

and to every complete $(l+1)$ -subgraph H_{l+1} of G they attach the weight

$$x(H_{l+1}) = \prod_{H_l \subset H_{l+1}} x(H_l)$$

and they define

$$f_G(x) = \sum_{H_{l+1} \subseteq G} x(H_{l+1}).$$

Then they get the following.

Theorem 3. $\max_x f_G(x) = \binom{k}{l+1} / \binom{k}{l}^{(l+1)/l}$, where k is the order of a maximum clique K of G . This maximum is attained by attaching weights $\binom{k}{l}^{-1/l}$ to the l -subgraphs of K and weight 0 to all other complete l -subgraphs.

Note that, the case $l = 1$ is exactly the Motzkin-Straus theorem.

Even if this result does not explicitly apply to hypergraphs, actually this theorem could be extended to k -graphs by attaching weights to subsets of hyperedges. However in order this theorem to work, the k -graph should satisfy a strong property that it to be a complete-subgraph graph of an ordinary graph (also said to be *conformal* [4]). This restricts the applicability of this theorem to a class of hypergraphs isomorphic to a subclass of 2-graphs having cliques of cardinality $\geq k$. We will see in the subsequent sections that our generalization applies to all k -uniform hypergraphs.

4 Characterization of maximal cliques on k -graphs

Let $G = (V, E)$ be a k -graph. The *complement* of G is given by $\bar{G} = (V, \bar{E})$ where $\bar{E} = \binom{V}{k} \setminus E$.

Consider the following function $h_G(\mathbf{x}) : \Delta \mapsto \mathbb{R}$ defined as

$$h_G(\mathbf{x}) = L_G(\mathbf{x}) + \tau \sum_{i=1}^n x_i^k \tag{3}$$

where $\tau \in \mathbb{R}$ and L_G is the Lagrangian of the hypergraph G defined in (2).

We will indicate with $h_G^j(\mathbf{x})$ the partial derivative of h_G with respect to x_j , i.e.

$$h_G^j(\mathbf{x}) = \frac{\partial h_G(\mathbf{x})}{\partial x_j} = \sum_{e \in E} 1_{j \in e} \prod_{i \in e \setminus \{j\}} x_i + \tau k x_j^{k-1}.$$

Lemma 1. *Let G be a k -graph and let \mathbf{x} be a global/local minimizer of $h_{\bar{G}}$ with $\tau > 0$. If $C = \sigma(\mathbf{x})$ is a clique of G then it is a maximum/maximal clique and \mathbf{x} is the characteristic vector of C .*

Proof. Since \mathbf{x} is a local minimizer of $h_{\bar{G}}$ over the simplex, it satisfies the first order necessary Karush-Kuhn-Tucker (KKT) condition. Therefore for all $j \in C$ we have that $\lambda = \tau k x_j^{k-1}$ and this implies that \mathbf{x} is the characteristic vector of C . Moreover if there exists a larger clique D that contains C , then there exists a vertex $j \in D \setminus C$ such that $h_{\bar{G}}^j(\mathbf{x}) = 0 < \lambda$. This contradicts the KKT condition and hence C is a maximal clique of G .

Finally, $h_{\bar{G}}(\mathbf{x}) = \tau |\sigma(\mathbf{x})|^{1-k}$ attains its global minimum when \mathbf{x} is the characteristic vector of a maximum clique. \square

Lemma 2. *Let G be a k -graph and \mathbf{x}^C the characteristic vector of a maximum/maximal clique C of G . Then \mathbf{x}^C is a strict global/local minimizer of $h_{\bar{G}}$, provided that $0 < \tau < \frac{1}{k}$.*

Proof (Sketch). For simplicity let $\mathbf{x} = \mathbf{x}^C$. We can prove [25] that \mathbf{x} is a strict local minimizer of $h_{\bar{G}}$ by checking that the second order sufficiency conditions for local minimum are satisfied.

Finally, $h_{\bar{G}}(\mathbf{x}^C) = \tau |C|^{1-k}$ attains its global minimum where C is as large as possible, i.e. a maximum clique. \square

Lemma 3. *Let G be a k -graph. If \mathbf{x} is a global/local minimizer of $h_{\bar{G}}$ then it is the characteristic vector of a maximum/maximal clique of G , provided that $\tau < \frac{1}{2^k - 2}$.*

Proof (Sketch). Let \mathbf{x} be a local minimizer of $h_{\bar{G}}$. We claim that its support forms a clique of G . Otherwise suppose that an edge $\tilde{e} \subseteq \sigma(\mathbf{x})$ is missing. Let $w, j \in \tilde{e}$ and take $\mathbf{y} = \mathbf{x} + \varepsilon(\mathbf{e}^j - \mathbf{e}^w)$, where \mathbf{e}^j denotes a zero vector except for the j -th element set to 1 and where $0 < \varepsilon < x_w$ and assume without loss of generality that $x_w \leq x_j \leq \min_{z \in \tilde{e} \setminus \{j, w\}} x_z$.

We can prove [25] that $h_{\bar{G}}(\mathbf{y}) - h_{\bar{G}}(\mathbf{x}) < 0$, contradicting the minimality of \mathbf{x} and hence $C = \sigma(\mathbf{x})$ is a clique of G . Finally by applying Lemma 1 we conclude the proof. \square

The following theorem generalizes the Bomze's Theorem (2) on k -graphs.

Theorem 4. *Let G be a k -graph and $0 < \tau < \frac{1}{2^k - 2}$. A vector $\mathbf{x} \in \Delta$ is a global/local minimizer of $h_{\bar{G}}(\mathbf{x})$ if and only if it is the characteristic vector of a maximum/maximal clique of G .*

Proof. It follows from Lemmas 2 and 3. \square

Note that if we take $k = 2$ and $0 < \tau < \frac{1}{2}$ then global/local minimizers of h correspond to global/local maximizers of $L_G^{\frac{1}{2}-\tau}$. In fact

$$\begin{aligned} h(\mathbf{x}) &= \sum_{\{i,j\} \in \bar{E}} x_i x_j + \tau \sum_{i=1}^n x_i^2 = \frac{1}{2} - \sum_{\{i,j\} \in E} x_i x_j + \left(\tau - \frac{1}{2}\right) \sum_{i=1}^n x_i^2 = \\ &= \frac{1}{2} - \left[\sum_{\{i,j\} \in E} x_i x_j + \left(\frac{1}{2} - \tau\right) \sum_{i=1}^n x_i^2 \right] = \frac{1}{2} - L_G^{\frac{1}{2}-\tau}(\mathbf{x}) \end{aligned}$$

Since $0 < \frac{1}{2} - \tau < \frac{1}{2}$, what we obtain is an equivalent formulation of the Bomze Theorem on graphs in terms of a minimization task.

The following corollary is our generalization of the Motzkin-Straus Theorem (1) to k -graphs, with the only difference that we deal for convenience with minimizers of a function instead of maximizers.

Corollary 1. *Let G be a k -graph with clique number $\omega(G)$. Then $h_{\bar{G}}$ attains its minimum at $\tau \omega(G)^{1-k}$ provided that $0 < \tau \leq \frac{1}{2k-2}$.*

Proof (Sketch). Let \mathbf{x} be a global minimizer of $h_{\bar{G}}$ with support as small as possible. We claim that its support forms a clique of G . Otherwise suppose that an edge $\tilde{e} \subseteq \sigma(\mathbf{x})$ is missing. Let $w, j \in \tilde{e}$ and take $\mathbf{y} = \mathbf{x} + x_w(\mathbf{e}^j - \mathbf{e}^w)$ and assume without loss of generality that $x_w \leq x_j \leq \min_{z \in \tilde{e} \setminus \{j,w\}} x_z$. Then we can prove [25] that $h_{\bar{G}}(\mathbf{y}) \leq h_{\bar{G}}(\mathbf{x})$. If the inequality is strict, it clearly contradicts the minimality of \mathbf{x} , and if $h_{\bar{G}}(\mathbf{y}) = h_{\bar{G}}(\mathbf{x})$ it contradicts the minimality of the support size of \mathbf{x} .

Hence $\sigma(\mathbf{x})$ is a clique of G . By Lemma 1 follows that \mathbf{x} is the characteristic vector of a maximum clique of G and thereby $h_{\bar{G}}(\mathbf{x}) = \tau |C|^{1-k} = \tau \omega(G)^{1-k}$. \square

Note that this result is equivalent to the original Motzkin-Straus Theorem (1) for graphs, if we take $k = 2$ and $\tau = \frac{1}{2}$. In fact, in this case we obtain

$$L_G(\mathbf{x}) = \sum_{\{i,j\} \in E} x_i x_j = \frac{1}{2} - \sum_{\{i,j\} \in \bar{E}} x_i x_j - \frac{1}{2} \sum_{i=1}^n x_i^2 = \frac{1}{2} - h(\mathbf{x})$$

and it follows that

$$\max_{\mathbf{x} \in \Delta} L_G(\mathbf{x}) = \frac{1}{2} - \min_{\mathbf{x} \in \Delta} h(\mathbf{x}) = \frac{1}{2} - \frac{1}{2\omega(G)} = \frac{1}{2} \left[1 - \frac{1}{\omega(G)} \right].$$

5 Finding maximal cliques of k -graphs

Summarizing our results, we propose a generalization of a well-known theorem in the extremal graph theory field to k -graphs that turns out to provide a continuous characterization of a purely discrete problem, i.e. finding maximal cliques

in k -graphs. More precisely, we implicitly provide an isomorphism between the set of maximal/maximum cliques of a k -graph G and the set of local/global minimizers of a particular function $h_{\bar{C}}$ over Δ , that permits to perform local optimization on $h_{\bar{C}}$ in order to extract, through the isomorphism, a maximal clique of the k -graph G . In this section we will see that the optimization of $h_{\bar{C}}$ may be easily carried out thanks to a theorem due to Baum and Eagon [1].

In the late 1960s, Baum and Eagon [1] introduced a class of nonlinear transformations in probability domain and proved a fundamental result which turns out to be very useful for the optimization task at hand. Their result generalizes an earlier one by Blakley [5] who discovered similar properties for certain homogeneous quadratic transformations. The next theorem introduces what is known as the Baum-Eagon inequality.

Theorem 5 (Baum-Eagon). *Let $P(\mathbf{x})$ be a homogeneous polynomial in the variables x_i with nonnegative coefficients, and let $\mathbf{x} \in \Delta$. Define the mapping $\mathbf{z} = \mathcal{M}(\mathbf{x})$ as follows:*

$$z_i = x_i \frac{\partial P(\mathbf{x})}{\partial x_i} \bigg/ \sum_{j=1}^n x_j \frac{\partial P(\mathbf{x})}{\partial x_j}, \quad i = 1, \dots, n. \quad (4)$$

Then $P(\mathcal{M}(\mathbf{x})) > P(\mathbf{x})$, unless $\mathcal{M}(\mathbf{x}) = \mathbf{x}$. In other words \mathcal{M} is a growth transformation for the polynomial P .

This result applies to homogeneous polynomials, however in a subsequent paper, Baum and Sell [3] proved that Theorem 5 still holds in the case of arbitrary polynomials with nonnegative coefficients, and further extended the result by proving that \mathcal{M} increases P homotopically, which means that

$$P(\eta\mathcal{M}(\mathbf{x}) + (1 - \eta)\mathbf{x}) \geq P(\mathbf{x}), \quad 0 \leq \eta \leq 1$$

with equality if and only if $\mathcal{M}(\mathbf{x}) = \mathbf{x}$.

The Baum-Eagon inequality provides an effective iterative means for maximizing polynomial functions in probability domains, and in fact it has served as the basis for various statistical estimation techniques developed within the theory of probabilistic functions of Markov chains [2]. As noted in [3], the mapping \mathcal{M} defined in Theorem 5 makes use of the first derivative only and yet is able to take finite steps while increasing P . This contrasts sharply with classical gradient methods, for which an increase in the objective function is guaranteed only when infinitesimal steps are taken, and determining the optimal step size entails computing higher-order derivatives. Additionally, performing gradient ascent in Δ requires some projection operator to ensure that the constraints not be violated, and this causes some problems for points lying on the boundary [11, 16]. In (4), instead, a computationally simple vector normalization is required.

It is worth noting that not all stationary points of the mapping \mathcal{M} correspond to local maxima of the polynomial P ; consider the vertices of Δ as an example. However all local maxima are the only stationary states that are stable, or even asymptotically stable if they are strict. Therefore if the dynamics gets trapped

in non optimal stationary states, it suffices a small perturbation to get rid of the problem. We will see an example of this fact in Section 6.

Moving a step back to our function $h_{\bar{G}}$, it satisfies the hypothesis of Theorem 5 since it is a homogeneous polynomial of degree k with nonnegative coefficients in the variables x_i with $\mathbf{x} \in \Delta$. However our targets are not local maxima but local minima. Fortunately, it turns out that we can transform our minimization problem into an equivalent maximization one, by keeping the conditions of Theorem 5 still satisfied.

Note that all coefficients of $h_{\bar{G}}(\mathbf{x})$ are positive and upper bounded by 1. Furthermore, let $\xi = \max[\tau, \frac{1}{k!}]$ and note that ξ can be expressed as a complete homogeneous polynomial $\pi(\mathbf{x})$ of degree k in the variables x_i as follows

$$\xi = \xi \left(\sum_{i=1}^n x_i \right)^k = \pi(\mathbf{x}), \quad \forall \mathbf{x} \in \Delta.$$

It is trivial to verify that the polynomial $\pi(\mathbf{x}) - h_{\bar{G}}(\mathbf{x})$ is a homogeneous polynomial of degree k with nonnegative coefficients. Moreover

$$\arg \min_{\mathbf{x} \in \Delta} h_{\bar{G}}(\mathbf{x}) = \arg \max_{\mathbf{x} \in \Delta} [\xi - h_{\bar{G}}(\mathbf{x})] = \arg \max_{\mathbf{x} \in \Delta} [\pi(\mathbf{x}) - h_{\bar{G}}(\mathbf{x})].$$

Therefore, in order to minimize $h_{\bar{G}}$ we can apply Theorem 5 considering $P(\mathbf{x}) = \pi(\mathbf{x}) - h_{\bar{G}}(\mathbf{x})$, and since

$$\frac{\partial \pi(\mathbf{x})}{\partial x_i} = k\xi \left(\sum_{i=1}^n x_i \right)^{k-1} = k\xi$$

we end up with the following dynamics for the minimization of $h_{\bar{G}}$ over Δ

$$x_i^{(t+1)} = \frac{x_i^{(t)} [k\xi - h_{\bar{G}}^i(\mathbf{x}^{(t)})]}{k\xi - \sum_{j=1}^n x_j^{(t)} h_{\bar{G}}^j(\mathbf{x}^{(t)})}, \quad (5)$$

that will converge to a local minima of $h_{\bar{G}}$ starting from any state \mathbf{x} in the interior of Δ , which corresponds by Theorem 4 to a maximal clique of the k -graph G .

6 A toy example

This section is not intended to provide experimental evidence that the dynamics (5) works, since we have a proof that guarantees it. Indeed, we provide a very simple toy example.

Figure 1 represents a 3-graph T , and the two sets that seem to be 4-edges are actually complete 3-subgraphs on the respective vertex sets. Hence T contains all possible 3-edges on the 5 vertices except for $\{0, 3, 4\}$ and $\{1, 3, 4\}$. T is a small example of a non conformal graph, i.e. it is not a complete-subgraphs graph of an ordinary graph. In other words, there exists no ordinary graph that has the

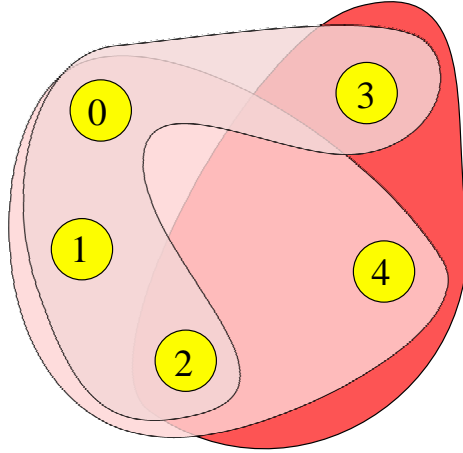


Fig. 1. A non conformal 3-graph T . Note that the sets $\{0, 1, 2, 3\}$ and $\{0, 1, 2, 4\}$ should be interpreted as complete 3-subgraphs on the respective vertex set. We draw them as 4-edges only for graphical clarity. In other words T contains all possible 3-edges on the 5 vertices except for $\{0, 3, 4\}$ and $\{1, 3, 4\}$.

same maximal cliques and therefore the generalization of the Motzkin-Straus theorem due to Sós and Straus [26] does not hold on this 3-graph. The set of maximal cliques of T is $\{\{0, 1, 2, 3\}, \{0, 1, 2, 4\}, \{2, 3, 4\}\}$.

We illustrate the behaviour of the dynamics (5) when applied to our toy example. Our parameters choice in this test is $\tau = \frac{1}{12}$, but no matter what is chosen as long as $0 < \tau < \frac{1}{6}$ as stated in Theorem 4, and therefore in the dynamics we have $k\xi = \frac{1}{2}$. The initial state encodes the hypothesis we make about the likelihood of a vertex to be part of a maximal clique, in fact if we set for example the i -th component of the initial state vector to zero then the i -th vertex will never be considered in a solution. Figure 2 presents three plots of the evolution of the state vector of the dynamics (5) for the 3-graph T over time. The initial states are respectively set to the simplex barycenter in the first two plots in order to have full uncertainty, and to $\mathbf{x}^{(0)} = (0.1, 0.1, 0.1, 0.35, 0.35)'$ in the last one in order to provide an initial stronger preference on the vertices 3 and 4.

Analysing our toy graph, we see that vertex 2 belongs to every maximal clique of T , while vertices 0 and 1 are shared between the two maximal 4-cliques and finally vertices 3 and 4 belong individually to a different maximal 4-clique, but together to the maximal 3-clique. Considering the first 114 iterations of the first two plots in Figure 2, we see that without advancing preferences of vertices, i.e. we start from the barycenter of the simplex, the dynamics converges to a stationary state, that is not optimal and hence not stable, but very informative. In fact, vertex 2 that certainly belongs to a maximal clique, has the highest likelihood, followed by vertices 0 and 1, that are shared between the two biggest

maximal cliques in T and finally we find vertices 3 and 4. By inducing a small perturbation at that point, we introduce some random preference on vertices that leads the dynamics to a certain solution; in the first case we end up with the maximal 4-clique $\{0, 1, 2, 3\}$, while in the second one we end up with the maximal 4-clique $\{0, 1, 2, 4\}$. Let us move now our attention on the last plot in Figure 2. In order to extract the smallest maximal clique, that has a smaller basin of attraction we have to put stronger preferences on some vertices; for example we put stronger hypothesis on the vertices 3 and 4, since the only maximal clique they share is the smallest one. As we can see, the dynamics is able to extract also the maximal clique $\{2, 3, 4\}$. The solutions we found for this small example are the only stable ones for the dynamics (5) when applied to T . Hence randomly choosing the initial state we will certainly end up with a maximal clique, but clearly the maximal cliques with a larger basin of attraction are more likely to be extracted.

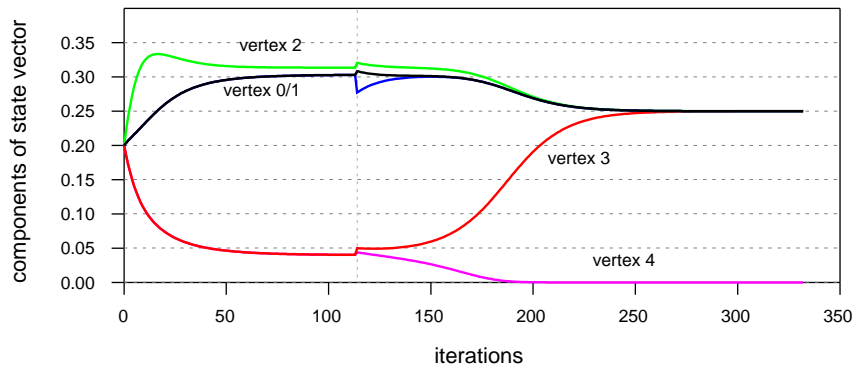
7 Conclusions and future work

In this paper we provide a generalization of a well-known extremal graph theory result, i.e. the Motzkin-Straus theorem, to k -uniform hypergraphs, and through it, we are able to provide a bridge between the purely discrete problem of finding maximal cliques in k -graphs and a minimization task of a continuous function. More precisely we introduce an isomorphism from the set of maximal/maximum cliques of a k -graph G and the set of local/global minima of the function h_G . In this way we can focus our attention on minimizing h_G in order to find maximal cliques in G . Nevertheless, in the last section we provide also a dynamical system, derived from a result due to Baum and Eagon, to easily solve the optimization problem at hand. This basically furnishes an heuristic for the maximum clique problem on k -graphs.

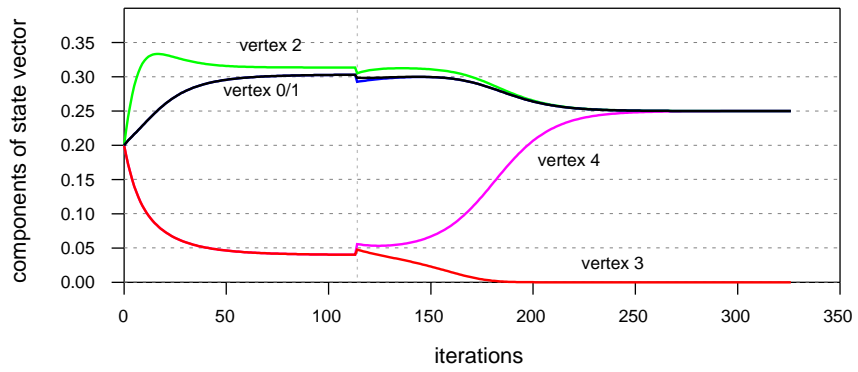
This result opens a wide range of possible future works. First of all, we may conduct experiments on the effectiveness of our heuristic for the maximum clique problem on k -graphs, however we expect in general performances on hypergraphs comparable with those obtained by Pelillo [23] on simple graphs. Even more interesting could be the theoretical applications carrying on with our work, such as finding new bounds on the clique number of k -uniform hypergraphs, or further generalizing the Motzkin-Straus theorem to vertex-weighted and edge-weighted hypergraphs.

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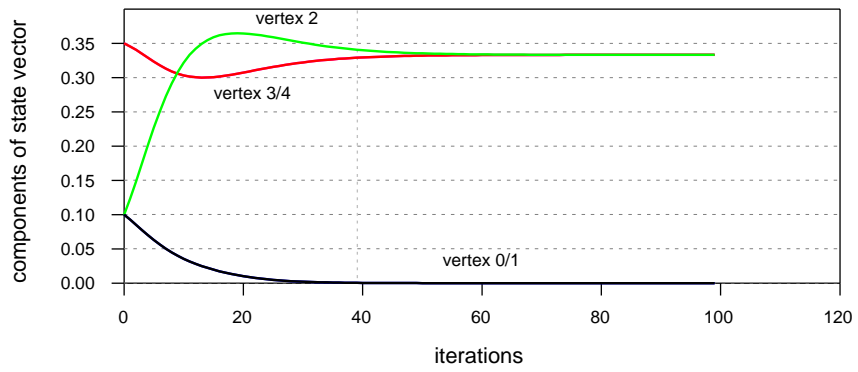
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(a) Extraction of the maximal clique $\{0, 1, 2, 3\}$



(b) Extraction of the maximal clique $\{0, 1, 2, 4\}$



(c) Extraction of the maximal clique $\{2, 3, 4\}$

Fig. 2. Evolution of the components of the state vector $\mathbf{x}^{(t)}$ for the k -graph in Figure 1, using the dynamics (5)

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