Algebra and Topology in Lambda Calculus

Antonino Salibra
Dipartimento di Informatica
Università Ca'Foscari di Venezia
Via Torino 155, 30172 Venezia, Italy
salibra@dsi.unive.it

The untyped lambda calculus was introduced around 1930 by Church [11] as part of an investigation in the formal foundations of mathematics and logic. Although lambda calculus is a very basic language, it is sufficient to express all the computable functions. The process of application and evaluation reflects the computational behavior of many modern functional programming languages, which explains the interest in the lambda calculus among computer scientists.

The lambda calculus, although its axioms are all in the form of equations, is not a true equational theory since the variable-binding properties of lambda abstraction prevent variables in lambda calculus from operating as real algebraic variables. Consequently the general methods that have been developed in universal algebra, for defining the semantics of an arbitrary algebraic theory for instance, are not directly applicable. There have been several attempts to reformulate the lambda calculus as a purely algebraic theory. The earliest, and best known, algebraic models are the combinatory algebras of Curry and Schönfinkel (see [12]). Combinatory algebras have a simple purely equational characterization and were used to provide an intrinsic first-order, but not equational, characterization of the models of lambda calculus, as a special class of combinatory algebras called λ-*models* [1, Def. 5.2.7].

Lambda theories are equational extensions of the untyped lambda calculus closed under derivation. They arise by syntactical or semantic considerations. Indeed, a λ-theory may correspond to a possible operational semantics of lambda calculus, as well as it may be induced by a model of lambda calculus through the kernel congruence relation of the interpretation function. The set of lambda theories is naturally equipped with a structure of complete lattice (see [1, Chapter 4]), where the meet of a family of lambda theories is their intersection, and the join is the least lambda theory containing their union. The bottom element of this lattice is the minimal lambda theory $\lambda\beta$, while the top element is the inconsistent lambda theory, hereafter denoted by ∇. Although researchers have mainly focused their interest on a limited number of them, the lattice of lambda theories, hereafter denoted by λT, has a very rich and complex structure (see e.g. [1, 3, 4]).

The lattice of lambda theories is isomorphic to the congruence lattice of the term algebra of the minimal lambda theory $\lambda\beta$. This remark is the starting point for studying the structure of λT by universal algebraic methods. In [20] Salibra has shown that the variety generated by the term algebra of $\lambda\beta$ is axiomatized by the finite schema of identities characterizing *lambda abstraction algebras* (LAA's). The equational theory of lambda abstraction algebras, introduced by Pigozzi and Salibra (see [18] and the references there), constitutes a purely algebraic theory of the untyped lambda calculus in the same spirit that cylindric and polyadic (Boolean) algebras constitute an algebraic theory of the first-order predicate logic. The variety of LAA's is intended as an alternative to the variety of combinatory algebras in this regard since it is a first-order algebraic description of lambda calculus, which keeps the lambda notation and hence all the functional intuitions. In [20] Salibra has shown that, for every variety of LAA's, there exists exactly one lambda theory whose term algebra generates the variety. Thus, the properties of a lambda theory can be studied by means of the variety of LAA's generated by its term algebra.

Many longstanding open problems of lambda calculus can be restated in terms of algebraic properties of varieties of LAA's. For example, the open problem of the order-incompleteness of lambda calculus, raised by Selinger (see [25]), asks for the existence of a lambda theory not arising as the equational theory of a non-trivially partially ordered model of lambda calculus. A partial answer to the order-incompleteness problem was obtained by Salibra in [23], where it is shown the existence of a lambda theory not arising as the equational theory of a non-trivially partially ordered model with a *finite* number of connected components. The order-incompleteness of lambda calculus is equivalent to the existence of an n-permutable variety of LAA's for some natural number $n \geq 2$ (see the remark after Theorem 3.4 in [25]). Plotkin, Selinger and Simpson (see [25]) have shown that 2-permutability and 3-permutability are inconsistent with lambda calculus. The problem of n-permutability remains open for $n \geq 4$. Berline and Salibra [6] have recently shown that there is a finitely axiomatizable λ-theory T such the variety of LAA's generated by the term algebra of T is congruence distributive. The existence of a variety of LAA's satisfying strong algebraic properties, such as n-permutability or congruence distributivity, was an open problem since Salibra [21] proved that the variety LAA is not congruence modular. The existence of a congruence distributive variety of LAA's shows, against a common belief, that the lambda calculus satisfies interesting algebraic properties.

At the end of the nineties, Salibra proposed the conjecture that the lattice λT satisfies no (non-trivial) lattice identity. There is a good reason to be also interested in large intervals of the form $[T, \nabla]$, where T is a lambda theory, because this interval is isomorphic to the congruence lattice of the term algebra of T, which is a bridge to universal algebra. The following results have been shown:

(i) (Visser [28]) Every countable partially ordered set embeds into λT by an order-preserving map.

(ii) (Visser [28]) Every interval of λT, whose bounds are recursively enumerable lambda theories, has a continuum of elements.

(iii) (Salibra [21]) λT is not modular.

(iv) (Statman [26]) The meet of all coatoms is $\neq \lambda\beta$.

(v) (Salibra [20]) λT is isomorphic to the lattice of the equational theories of LAA's.

(vi) (Lusin-Salibra [16]) λT satisfies the Zipper condition and the ET condition (this follows from (v) and from Lampe's results [15] on the lattices of equational theories).

(vii) (Lusin-Salibra [16]) For any nontrivial lattice identity e, there exists a natural number n such that the identity e fails in the lattice of the λ-theories over a language of lambda calculus extended with n constants.

(viii) (Berline-Salibra [6]) There is a finitely axiomatizable theory T such the interval $[T, \nabla]$ is distributive.

(ix) (Salibra 2006, unpublished) For every natural number n there exists a lambda theory T_n such that the interval sublattice $[T_n, \nabla]$ is isomorphic to the finite Boolean lattice with 2^n elements.

In [16] Lusin and Salibra have shown that a lattice identity is satisfied by all congruence lattices of combinatory algebras (lambda abstraction algebras, respectively) iff it is true in all lattices. As a consequence, it is not possible to apply to combinatory algebras the nice results developed in universal algebra in the last thirty years, which essentially connect lattice identities satisfied by all congruence lattices of algebras in a variety, and Mal'cev conditions (that characterize properties in varieties by the existence of suitable terms involved in certain identities). Thus there is a common belief that lambda calculus and combinatory logic are algebraically pathological. On the contrary, in [17] Manzonetto and Salibra have recently shown that combinatory algebras satisfy interesting algebraic properties. One of the milestones of modern algebra is the Stone representation theorem for Boolean algebras, which was generalized by Pierce to commutative rings with unit and next by Comer to the class of algebras with Boolean factor congruences. By applying a theorem by Vaggione [27], it was shown in [17] that Comer's generalization of Stone representation theorem holds also for combinatory algebras: any combinatory algebra is isomorphic to a "weak" Boolean product of directly indecomposable combinatory algebras (i.e., algebras which cannot be decomposed as the Cartesian product of two other nontrivial algebras). The proof of the representation theorem for combinatory algebras is based on the fact that every combinatory algebra has central elements (introduced by Vaggione [27] in universal algebra), i.e., elements which define a direct decomposition of the algebra as the Cartesian product of two other combinatory algebras, just like idempotent elements in rings. The central elements of a combinatory algebra constitute a Boolean algebra, whose Boolean operations can be defined by suitable combinators.

Topology is at the center of the known approaches to give models of the untyped lambda calculus. The first model, found by Scott in 1969 in the category of complete lattices and Scott continuous functions, was successfully used to show that all the unsolvable λ-terms can be consistently equated. After Scott, a large number of mathematical models for lambda calculus, arising from syntax-free constructions, have been introduced in various categories of domains and were classified into semantics according to the nature of their representable functions, see e.g. [1, 3, 19]. Scott continuous semantics [24] is given in the category whose objects are complete partial orders and morphisms are Scott continuous functions. The stable semantics (Berry [7]) and the strongly stable semantics (Bucciarelli-Ehrhard [8]) are refinements of the continuous semantics, introduced to capture the notion of "sequential" Scott continuous function. The continuous, stable and strongly stable semantics are structurally and equationally rich [14] in the sense that, in each of them, it is possible to build up $2^{\aleph_0}$ models inducing pairwise distinct λ-theories. Nevertheless, the above denotational semantics are equationally incomplete, where a semantics is *(equationally) incomplete* if there exists a lambda theory which is not induced by any model in the semantics. The problem of the equational incompleteness was positively solved by Honsell-Ronchi della Rocca [13] for the continuous semantics and by Bastonero-Gouy [2] for the stable semantics. In [22, 23] Salibra has shown in a uniform way that all semantics (including the strongly stable semantics), which involve monotonicity with respect to some partial order and have a bottom element, fail to induce a continuum of λ-theories. Further results in [22, 23] are: (i) an incompleteness theorem for partially ordered models with finitely many connected components; (ii) an incompleteness theorem for topological models whose topology satisfies a suitable property of connectedness; (iii) a completeness theorem for topological models whose topology is non-trivial and metrizable. Manzonetto and Salibra [17] have recently shown an algebraic incompleteness theorem for lambda calculus: the semantics of lambda calculus given in terms of models, which are directly indecomposable as combinatory algebras (i.e., they cannot be decomposed as the Cartesian product of two other non-trivial combinatory algebras), is incomplete, although it strictly includes the continuous semantics, its stable and strongly stable refinements and the term models of all semi-sensible λ-theories.

The notion of an *effective model* of lambda calculus has been introduced in [5], where the Visser topology has been used for investigating the question of whether the equational theory of a model of lambda calculus can be recursively enumerable (r.e. for brevity). The following results have been obtained:

(i) The equational theory of an effective model cannot be $\lambda\beta$, $\lambda\beta\eta$, where $\lambda\beta\eta$ is the least extensional lambda theory;
(ii) The order theory of an effective model cannot be r.e.;
(iii) No effective model living in the stable or strongly stable semantics has an r.e. equational theory.

Concerning Scott's semantics, the class of graph models has been investigated in a series of papers [9, 10, 4, 6, 5] and the following results were obtained, where "graph theory" is a shortcut for "theory of a graph model":

(iv) There exists a minimum equational graph theory different from $\lambda\beta$ [9];
(v) There exists a maximum sensible equational graph theory, that is characterized as the theory of Böhm trees [9];
(vi) There exists a minimum order graph theory [5];
(vii) The minimum equational/order graph theory is the theory of an effective graph model [5];
(viii) No order graph theory can be r.e. [5].

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