Lectures on Order and Topology

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1 Orders: main definitions and notation

Let D be a poset and $X \subseteq D$. Then

$$X{\downarrow} = \{z : (\exists x \in X)\ z \leq x\}; \quad X{\uparrow} = \{z : (\exists x \in X)\ z \geq x\}.$$

A set X is an *upper set* (*lower set*, respectively) if $X = X{\uparrow}$ ($X = X{\downarrow}$). In particular, $\emptyset{\downarrow} = \emptyset$.

We define:

$$X{\uparrow}^{\forall} = \cap_{x \in X} x{\uparrow} = \{z : (\forall x \in X)\ z \geq x\}; \quad X{\downarrow}^{\forall} = \cap_{x \in X} x{\downarrow} = \{z : (\forall x \in X)\ z \leq x\}.$$

In particular, $\emptyset{\uparrow}^{\forall} = L$.

A set $X \subseteq D$ is *upward-directed (downward-directed)* if $a, b \in X \ \Rightarrow\ (\exists c)\ c \geq (\leq)\ a, b$.

An *ideal (filter)* of a poset D is an upward-directed lower subset of D (a downward-directed upper subset of D). An ideal I is principal if there exists $x \in D$ such that $I = x{\downarrow}$. If X is an upward-directed subset of D then $X{\downarrow}$ is an ideal.

1.1 Prime and co-prime elements

An element p of a poset D is:

1. *irreducible* if $p{\uparrow} - \{p\}$ is a possibly empty filter.
2. *prime* if $D - p{\downarrow}$ is a filter.
3. *co-prime* if it is a prime element of D^{op} (i.e., $D - p{\uparrow}$ is an ideal in the poset D).

If D is a meet semilattice, then

1. p is irreducible iff $p = ab \ \Rightarrow\ p = a$ or $p = b$.
2. p is prime iff $ab \leq p \ \Rightarrow\ a \leq p$ or $b \leq p$.
3. Prime implies irreducible (if $p = ab$ then, without loss of generality, we may assume that $a \leq p$. Then from $p = ab \leq a \leq p$ it follows that $p = a$).

If D is a lattice, then

1. p is co-prime iff $p \leq a + b \Rightarrow\ p \leq a$ or $p \leq b$.

A meet semilattice is *distributive* if $ab \leq p$ implies the existence of c, d such that $a \leq c$, $b \leq d$ and $p = cd$.
If D is a distributive meet semilattice, then

1. Prime = irreducible (Let p be irreducible. We show that p is prime. Let $ab \leq p$. Then by hypothesis there are $a \leq c$, $b \leq d$ such that $p = cd$. Then, $p = c$ or $p = d$ and we get the conclusion).

A lattice is distributive as a meet semilattice iff it satisfies the usual distributivity law: $a + (bc) = (a+b)(a+c)$ (We have that $a + (bc) \leq (a+b)(a+c)$ holds in every lattice. Consider $p = a + (bc)$. Then $bc \leq p$. By hypothesis there are $b' \geq b$ and $c' \geq c$ such that $p = b'c'$. Then $a + b \leq a + b'$ and $a + c \leq a + c'$. But $a \leq a + (bc) = p \leq b'$, so that $a + b' = b'$. In conclusion $(a+b)(a+c) \leq (a+b')(a+c') = b'c' = p = a + (bc)$).

1.2 Directed complete posets

Definition 1.1 *A poset D is called* directed complete *(dcpo, for short) if every upward directed subset X of D has a least upper bound (lub) denoted by sup X. A dcpo with least element (written $\bot$) is called a cpo.*

Example 1 (Power Set)*: The power set of a set A partially ordered by inclusion.*
(Partial Functions)*: The set of partial functions from A to B (denoted by $[A \rightarrow_p B]$) is a cpo according to the following partial ordering. For every partial map $h : A \rightarrow_p B$, let graph$(h) = \{(x, h(x)) : x \in dom(h)\}$. Then $f \leq g$ iff graph$(f) \subseteq$ graph(g). The least element is the empty function. Every partial function is the lub of its finite approximants: $f =$ sup $\{g \leq f :$ dom(g) finite$\}$. The total functions are the maximal elements.*
(Natural Numbers)*: The set N of natural numbers with the usual ordering is not a cpo. The set $N \cup \{\infty\}$ with $n \leq \infty$ for all $n \in N$ is a cpo.*

Ideal Completion: There is a standard way to get dcpos from partial orderings.

Proposition 2 *The poset $(Ide(D), \subseteq)$ of ideals of a poset D is a dcpo, where the lub of a directed set U of ideals is the union of all ideals in U. The map $x \in D \rightarrow x{\downarrow} \in Ide(D)$ embeds D into $Ide(D)$.*

Example 3 *Let $Sb_f(N)$ be the poset of the finite subsets of the set N of natural numbers. The ideal completion of $Sb_f(N)$ is isomorphic to the power set $Sb(N)$. The ideal $\{\emptyset, \{0\}, \{2\}, \{4\}, \dots, \{0,2\}, \{0,4\}, \dots$ represents the set of even natural numbers.*

Example 4 *(Trees): The set F of finite trees w.r.t. an algebraic similarity type is defined as follows:*

- *$\bot \in F$;*
- *If $P_1, \dots, P_r \in F$ and f is a function symbol of arity r then $f(P_1, \dots, P_r) \in F$.*

Consider on F the least compatible partial order including

$$\bot \leq P, \quad \text{for all } P \in F.$$

For example, $f(\bot, g(\bot)) \leq f(4, g(h(\bot)))$. The ideal completion of F defines a cpo of possibly infinite trees. The ideal $\{\bot, f\bot, f(f\bot), f(f(f\bot)), \dots\}$ is equal to the tree $f(f(f(....f(....)...)))$.

1.3 Algebraic dcpo

What is a finite element in a poset? How is it possible to characterize the properties of finite elements (for example, finite sets, finite functions and finite Bohm trees) in a poset? An element d of a poset D is finite (or compact) if, for every directed set X such that sup X exists, we have

$$d \leq \text{sup } X \Rightarrow\ d \leq e,\ \text{for some } e \in X.$$

The set of all compact elements of a poset D will be denoted by $C(D)$.

Intuition: a complete data z can be obtained as union $z =$ sup X of all the compatible pieces $e \in X$. If a compact element d is contained within the complete data sup X, then there exists an piece $e \in X$ of sup X containing d.

Definition 1.2 *A poset D is algebraic if for all $x \in D$ the set $\{d \in C(D) : d \leq x\}$ is directed and has lub x.*

The set of the Bohm trees, the set of the partial functions from A to B, the set of all subsets of a set A are examples of algebraic cpos.

Theorem 5 *Every algebraic dcpo is isomorphic to the ideal completion of the poset of its compact elements. Vice versa, if D is a poset, then D is the set of all compact elements of its ideal completion $Ide(D)$.*

Example 6 *: The ideal completion of the set N of natural numbers is isomorphic to $N \cup \{\infty\}$ with $n \leq \infty$ for all $n \in N$. N is the set of compact elements of $N \cup \{\infty\}$.*

1.4 Continuous posets

The real line with the Scott topology does not admit compact elements, so that it is not an algebraic dcpo. However, every real a is the lub of the reals less than a.

Let L be a poset. We say that x *is way below* y, in symbols $x \ll y$, if, for all directed subsets $D \subseteq L$ for which sup D exists,

$$y \leq \text{sup } D\ \Rightarrow\ x \leq d, \quad \text{for some } d \in D.$$

Then x is compact iff $x \ll x$.

Intuitive meaning: $x \ll y$ if x is an important (possibly infinite) piece of y such that, whenever y is less than an infinite data sup D, then sup D is constructed by means a suitable $d \geq x$. In other words, the building blocks of sup D must contain x.

In the real line $x \ll y$ iff $x < y$.

The way below relation satisfies the following conditions:

Proposition 7
1. *$x \ll y\ \Rightarrow\ x \leq y$.*
2. *$x \ll y$ iff x belongs to every ideal I such that $y \leq$ sup I.*
3. *$x \leq y \ll z \leq u\ \Rightarrow\ x \ll u$.*
4. *$x \ll z$ and $y \ll z$ implies $x \vee y \ll z$ whenever $x \vee y$ exists.*
5. *$\perp \ll x$ whenever there exists a smallest element.*

Definition 1.3 *A poset is* continuous *if* $x\Downarrow = \{d : d \ll x\}$ *is directed (and then an ideal) and* $x =$ *sup* $x\Downarrow$.

As a matter of terminology,

- *Continuous lattice* stands for complete lattice that is continuous;
- *Continuous Scott domain* stands for continuous dcpo having meets of non-empty subsets (= joins of upper bound subsets).

Example 8 *The extended real line* $[-\infty, +\infty]$ *is a continuous lattice.*

A continuous Scott domain becomes a continuous lattice by adding a top element $\top$ (in such a case, $\top \ll \top$, because every directed set having $\top$ as a lub must contain $\top$; otherwise without top we would not have a dcpo). If we remove the top element from a continuous lattice we get a continuous Scott domain only if $\top \ll \top$. Consider, for example the extended line $[-\infty, +\infty)$ without the top.

Proposition 9 *Let L be a continuous poset.*

1. *If* $x \ll y$ *and* $y \leq$ *sup D for a directed D, then* $x \ll d$ *for a suitable* $d \in D$.
2. $x \ll y \ \Rightarrow\ (\exists z)\, x \ll z \ll y$.

Proof. (1) The set $I = \cup_{d\in D} d\Downarrow$ is an ideal such that sup $I =$ sup D. Then by Prop. 7(2) $x \in I$.
(2) By (1) applied to the directed set $y\Downarrow$. ■

Proposition 10 *Let L be a poset. Then the following conditions are equivalent:*

1. *L is continuous;*
2. *For each* $x \in L$, *the set* $x\Downarrow$ *is non-empty and it is the smallest ideal I satisfying* $x \leq$ *sup I;*
3. *For each* $x \in L$, *there exists a smallest nonempty ideal I satisfying* $x \leq$ *sup I.*

Proposition 11 *Let L be a dcpo. Then the three conditions of Prop. 10 are equivalent to:*

4. *The sup map from IdL into L, defined by* $I \mapsto$ *sup I, has a lower adjoint.*

All these equivalent conditions imply that

5. *The sup map preserves all existing infs.*

and, if L is bounded complete or a complete lattice, all the five conditions are equivalent.

Example 12 (Topology) *If U and V are open sets then* $U \ll V$ *if every open cover of V has a finite subcover of U (this does not mean that U is compact in the subspace V with the induced topology). A space is called* core-compact *if the open sets constitute a continuous lattice.*

A space is locally compact *if every element has a neighborhood basis of compact (not necessarily open) sets. In a locally compact space* $U \ll V$ *iff there exists a compact set Q of the space such that* $U \subseteq Q \subseteq V$ *(Consider for every* $x \in V$ *a compact* $Q_x \subseteq V$. *Then the family of open sets constituted by the interior of* Q_x $(x \in V)$ *is an open cover of V admitting a finite subcover of U.) Locally compact spaces are core-compact.*

For sober spaces and Hausdorff spaces we have core-compact = locally compact. Moreover, a space is core-compact iff its sobrification is locally compact.

Example 13 (The real plane) *The plane is a continuous poset under its coordinatewise order, with $x \ll y$ iff $x_1 < y_1$ and $x_2 < y_2$.*

Example 14 (The interval domain) *The closed intervals of the extended Euclidean line form a continuous lattice under the reverse-inclusion order, with $x \ll y$ iff the interior of the interval x contains the interval y. If the empty interval (the top element) is removed, a continuous Scott domain is obtained.*

Example 15 (Algebraic dcpo) *Every algebraic dcpo A is continuous satisfying: $x \ll y$ iff there exists a compact $d \leq y$ such that $x \leq d$. Since $y = sup\,\{d : d \leq y, d \in C(A)\}$, then it follows that $x \leq d$ for a suitable compact d.*

2 Topology: main definitions and notation

Let X be a topological space. We denote the open sets of X by $\mathcal{O}(X)$. A *closed* set is the complement of an open set, while a *clopen* set is a set which is both open and closed. The clopen subsets of a space constitutes a Boolean set algebra.

Example 16 *Let Z be the set of all integers. We provide a topological proof that there exist infinite prime numbers. For $b > 0$ and $0 \leq a < b$ let*

$$N_{a,b} = \{x \in Z : x \equiv a\ (mod\, b)\}.$$

The family of sets $N_{a,b}$ ($0 \leq a < b$) is closed under complementation and finite intersection:

$$N_{a,b} \cap N_{a,c} = N_{a,lcm(b,c)} \quad Z - N_{a,b} = \cup_{0 \leq i \neq a < b} N_{i,b}.$$

Then it generates a T_1-topology on Z with a base of clopen sets (= zero-dimensional) such that any non-empty open set is infinite. Except for -1, $+1$ and 0, all integers have prime factors. Therefore each is contained in one or more $N_{0,p}$, where p is prime. We thus arrive at the following identity, where P is the set of prime numbers:

$$Z - \{-1, +1\} = \cup_{p \in P} N_{0,p}.$$

If the set P were finite, then the right hand side would be closed as the union of a finite number of closed sets. Then the set $\{-1, +1\}$ would then be open as a complement of a closed set. This would contradict that every open set is infinite.

2.1 Base of a topology

A family $\mathcal{B}$ of open sets is a *base* (*subbase*) if every other open set is union of elements of $\mathcal{B}$ (is union of finite intersections of sets in $\mathcal{B}$).

A family $\mathcal{B}$ of subsets of a set X is a base for a topology on X if it satisfies the following two conditions: (1) For every $x \in X$, there is $B \in \mathcal{B}$ such that $x \in B$; (2) If $A, B \in \mathcal{B}$ and $x \in A \cap B$, then there is $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.

2.2 Interior

The *interior* $int(A)$ of a set A is the largest open subset of A.

2.3 Neighborhoods

A *neighborhood* of a point x is any set U such that $x \in int(U)$. The family of neighborhoods of a point x constitutes a filter of sets.

2.4 Closure of a set, adherent points, accumulation points

The *closure* A^c of a subset A of X is the intersection of all closed sets containing A. The closure of a singleton set $\{x\}$ will be denoted by x^c.

The closure of A is the set of all points *adherent to* A, where a point $a \in X$ is adherent to A if we have $U \cap A \neq \emptyset$ for every neighborhood U of a.

We distinguish two kinds of points adherent to A: the *accumulation points*, i.e., all the points a such that $(U \cap A) - \{a\} \neq \emptyset$ for every neighborhood U of a (these include all the points in $A^c - A$), and the points of A which are isolated in the subspace A (isolated point = the singleton of a is open in A).

The closure of A is the union of A and its accumulation points.

The notion of closure can be characterized in terms of convergence of nets (see below).

2.5 Boundary

The *boundary* (or *frontier*) A^b of A is $A^c - int(A)$. The boundary is a closed set because $A^b = A^c \cap (X - A)^c$. The closure of A is the union of $int(A)$ and A^b.

Example 17 *Let* $A = \{(x, y) : x^2 + y^2 < 1\} \cup \{(2, 0)\}$ *in the plane with the Euclidean topology. Then* $A^c = \{(x, y) : x^2 + y^2 \leq 1\} \cup \{(2, 0)\}$, $A^b = \{(x, y) : x^2 + y^2 = 1\} \cup \{(2, 0)\}$, *while the set of accumulation points is equal to* $\{(x, y) : x^2 + y^2 \leq 1\}$.

2.6 Dense in itself set

A set $A \subseteq X$ is called *dense in itself* if the subspace A does not contain isolated points. A closed set A is dense in itself iff it agrees with the set of its accumulation points.

2.7 Dense and nowhere dense sets

A set A is *dense* in a space X if $A^c = X$, while it is *nowhere dense* in X if $int(A) = \emptyset$. The set of (ir)rational numbers is both dense and nowhere dense in the set of real numbers.

An open set A is dense iff $X - A$ is nowhere dense.

A set is *meager* in X if it is the union of a countable collection of nowhere dense subsets of X. Any set of real numbers is meager ($A = (A \cap \text{Rational}) \cup (A \cap \text{Irrational})$).

2.8 Separable, first countable, second countable, Baire spaces

A space is

1. *separable* if it has a countable dense subset.
2. *second countable* if it has a countable basis.
3. *first countable* if every point has a countable local basis.

4. *a Baire space* if any countable intersection of dense open sets is dense.

Example 18 *Let X be an uncountable set with the topology consisting of the empty set and the cofinite sets. Then every countable set of X is dense.*

Example 19 *The Euclidean line is separable (because of the rational numbers) and first countable (the family of intervals $[x - 1/n, x + 1/n]$ ($n \in \mathbb{N}$) constitutes a base of neighborhoods of the real number x).*

Example 20 *The Scott topology over the powerset of $\mathbb{N}$ is second countable.*

2.9 Specialization preorder

The *specialization preorder* $\leq_X$ on a space X is defined by one of the following equivalent conditions:

1. $(\forall U \in \mathcal{O}(X))\ x \in U \Rightarrow y \in U$;
2. $x \in y^c$;

The open sets are upward closed w.r.t. $\leq_X$, while the closed sets are downward closed.

The *inequality graph* of the poset $(X, \leq_X)$ has the points of X as nodes, while an arc connectes two points a and b if either $a \leq_X b$ or $b \leq_X a$.

Notation: $x{\Downarrow}_X = \{y : x \in int(y{\uparrow})\}$; and $x \prec_X y$ if $y \in int(x{\uparrow})$.

3 Nets

A *net* is a map from a (upward) directed set I into a topological space X. A net will be denoted by x_I.

Example 21 *Let X be a space. The filter $\mathcal{N}_y$ of neighborhoods of a point $y \in X$ (and, more generally, every filter on X) is an upward directed set w.r.t. the partial ordering $\supseteq$.*

Let x_I be a net.

(i) $x_{\geq i} = \{x_j : j \geq i\}$ is the *i-tail*.

(ii) $J \subseteq I$ is *cofinal* if $J \cap \{k \geq i\} \neq \emptyset$ for all $i \in I$. A cofinal J is directed.

(iii) $O \subseteq X$ is *x_I-eventual* if a tail of the net is in O. The set of all such Os is a filter.

(iv) $O \subseteq X$ is *x_I-cofinal* if $O \cap x_{\geq i} \neq \emptyset$ for all $i \in I$.

Definition 3.1
- *A net x_I* converges *to $y \in X$ (denoted by $x_I \to y$), if every neighborhood of y is x_I-eventual.*
- *A net x_I* clusters *to $y \in X$ if every neighborhood of y is x_I-cofinal.*

The convergence of nets generalizes the convergence of sequences in metric spaces.

Example 22 *Consider the real line with the Euclidean topology. The net*

$$-1, 1/2, -1, 1/4, -1, 1/8, \ldots$$

has two cluster points: -1 *and* 0*. The subnet* $-1, -1, \ldots$ *converges to* -1*, while the subnet* $1/2, 1/4, \ldots$ *converges to* 0*.*

Example 23 *Let* $r_0, r_1, \ldots, r_n \ldots$ *be a net which is an enumeration of the rational numbers. Then every real number is a cluster point of this net.*

Definition 3.2 *Let* X *be a set and* $\mathcal{L}$ *be a class of pairs* (x_I, y) *consisting of a net* x_I *and an element* y *of* X*. Then the family of sets*

$$\mathcal{O}(\mathcal{L}) = \{U \subseteq X : y \in U \;\Rightarrow\; U \text{ is } x_I\text{-eventual for all } (x_I, y) \in \mathcal{L}\}$$

is a topology on X*. It is the strongest topology* τ *making the net* x_I τ*-converging to* y *for all* $(x_I, y) \in \mathcal{L}$*.*

We write $x_I \subseteq A$ if $x_i \in A$ for all $i \in I$ and $\rightarrow_{x_I}$ for the set of all points x such that $x_I \to x$. The topology on X is completely characterized by the convergence of nets.

Proposition 24 *Let* X *be a space.*

1. *The set* $\rightarrow_{x_I}$ *is a lower set (w.r.t. the specialization preorder of* X*).*
2. *If the space* X *satisfies the separation axiom* T_2 *(see below), then a net converges at most to one point.*
3. $y \in A^c$ *iff there exists a net* $x_I \subseteq A$ *such that* $x_I \to y$*.*
4. $A \subseteq X$ *is closed iff it contains all convergence (cluster) points of nets contained within* A*.*

Proof. (1) If U is a neighborhood of z, then U is also a neighborhood of y.
(3 ⇒) The filter $\mathcal{N}_y$ of neighborhoods of a point y is a directed set w.r.t. the partial ordering $\supseteq$. Define the net $n_{\mathcal{N}_y}$ as follows: n_U is a point belonging to $U \cap A$. The above net converges to y. Let V be an open neighborhood of y. Then the tail $n_{\geq V}$ is contained within V.
(4) By (3). ■

Example 25 *Consider the finite complement topology over the set of natural numbers (see Example 33). This topology is* T_1 *but not* T_2*. The net*

$$0, 1, 2, \ldots, n, \ldots$$

converges to all the points of the space.

A net y_I is a *subnet* of a net x_J if there is a function $f : I \to J$ such that (a) $y_i = x_{f_i}$ for every $i \in I$; (b) for every tail $x_{\geq j}$ there is a tail $y_{\geq i}$ such that $f(y_{\geq i}) \subseteq x_{\geq j}$. If the net x_J converges to z, then the subnet y_I also converges to z. If I is a cofinal subset of J, then the net y_I, defined by $y_i = x_i$ for all $i \in I$, is a subnet of x_J.

Proposition 26 *Let* X *be a space.*

1. *A net clusters to x iff $x \in \overline{(x_{\geq i})}^c$ for every $i \in I$.*

2. *If a net x_I clusters to y then there exists a subnet of x_I that converges to y.*

Proof. (2) The filter $\mathcal{N}_y$ of neighborhoods of y is a directed set w.r.t. the partial ordering $\supseteq$. Consider the directed poset $D = \{(U, i) \in \mathcal{N}_y \times I : x_i \in U\}$. Define the net x_D as follows: $x_{(U,i)} = x_i$. Consider the range B of the map $f : D \to I$ defined by $f(U, i) = i$. Then the subnet x_B of x_I converges to y. ■

3.1 Convergence of filters

Let X be a space. A filter $\mathcal{F}$ on X converges to $x \in X$ if every neighborhood of x is in the filter.

Proposition 27

(i) *If x_I is a net, then the family $\mathcal{F}$ of all subsets A of X such that A is x_I-eventual is a filter.*

(ii) *If $\mathcal{F}$ is a filter, then $I = \{(y, B) : y \in B \in \mathcal{F}\}$ is directed and $x_I(y, B) = y$ defines a net x_I such that $\mathcal{F}$ is the filter of all A such that A is x_I-eventual.*

A *convergence space* $\mathcal{C}$ is set of pairs $(\mathcal{F}, x)$ consisting of a filter $\mathcal{F}$ on X and an element x of X. The following two axioms must be satisfied: $(\{x\}\uparrow, x)$ for the principal ultrafilter generated by $\{x\}$; $(\mathcal{F}, x)$ and $\mathcal{F} \subseteq \mathcal{G}$ imply $(\mathcal{G}, x)$. The topology generated by a convergence space $\mathcal{C}$ is defined by (To prove that it is a topology we do not use the properties of convergence spaces):

$$\mathcal{O}(\mathcal{C}) = \{U \subseteq X : (\mathcal{F}, x) \in \mathcal{C},\ x \in U\ \Rightarrow\ U \in \mathcal{F}\}.$$

See Stadler: Basic Properties of Filter Convergence Spaces, J Chem Inf Comput Sci, 2002.

4 Separation

Separation axioms in topology stipulate the degree to which distinct points may be separated by open sets or by closed neighborhoods of open sets, while connectedness axioms in topology examine the structure of a topological space in an orthogonal way with respect to separation axioms. They deny the existence of certain subsets of a topological space with properties of separation.

4.1 T$_0$-spaces

A space X is T_0 if, for all $x, y \in X$, there is an open U such that $U \cap \{x, y\}$ is a singleton set .

Theorem 28 *The following conditions are equivalent for a space X:*

- *X is T_0.*
- *The specialization preorder of X is a partial ordering.*

Proposition 29 1. *Every countable based T_0-space X can be embedded in the space $\mathcal{P}(\mathbb{N})$ of all subsets of natural numbers equipped with the Scott topology.*

2. *Every continuous map $f : X \to Y$ between subspaces X and Y of $\mathcal{P}(\mathbb{N})$ admits a Scott continuous extension $F : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$.*

Proof. (1) If $(U_n : n \in \mathbb{N})$ is an enumeration of the basis of X then the map $x \in X \to \{n : x \in U_n\}$ is an embedding.

(2) Define $Fx = \bigcup\{\bigcap\{f(z) : z \in X \cap y\uparrow\} : y \subseteq_{\mathbb{N}} x\}$, where $y \subseteq_{\mathbb{N}} x$ means that y is a finite subset of x. ■

Example 30 *The Scott topology is T_0 but not T_1.*

4.2 T$_1$-spaces

A space X is T_1 if, for all $x, y \in X$, there exist open sets U and V such that $U \cap \{x, y\} = \{x\}$ and $V \cap \{x, y\} = \{y\}$.

Theorem 31 *The following conditions are equivalent for a space X:*

- *X is T_1.*
- *For every $x \in X$, the intersection of all neighborhood of x is the singleton set $\{x\}$.*
- *Every point x of X is closed.*
- *The specialization preorder of X agrees with the equality relation.*

If X is T_1 and x is an accumulation point of A, then $U \cap A$ is infinite for every neighborhood of x. The set of accumulation points of A is closed.

Example 32 *Let T be a recursively enumerable lambda theory. The Visser topology over Λ/T is T_1 but not T_2.*

Example 33 *Let X be a countable set. We define a subset A of X to be open if $X - A$ is finite. This topology, called the* finite complement topology*, is T_1 but not T_2.*

4.3 T$_2$

A space X is T_2 or *Hausdorff* if, for all $x, y \in X$, there exist open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Theorem 34 *The following conditions are equivalent for a space X:*

- *X is T_2.*
- *For every $x \in X$, the intersection of all closed neighborhood of x is the singleton set $\{x\}$.*
- *The diagonal $\Delta = \{(x, x) : x \in X\}$ is a closed subset of $X \times X$.*
- *Every point $x \in X$ has a closed neighborhood, which is T_2 w.r.t. the induced topology.*

Example 35 *([3, Example 75]) The irrational slope topology is T_2 but not $T_{2_{1/2}}$.*

4.4 $\mathbf{T}_{2_{1/2}}$

A space X is $T_{2_{1/2}}$ (or *completely Hausdorff*) if for all distinct $a, b \in X$ there exist open sets U and V with $a \in U$, $b \in V$ and $U^c \cap V^c = \emptyset$.

The previous axioms of separation can be relativized to pairs of elements. For example, a and b are $T_{2_{1/2}}$-separable, if there exist open sets U and V with $a \in U$, $b \in V$ and $U^c \cap V^c = \emptyset$. T_2-, T_1-, T_0-separability for pairs of elements are similarly defined.

Example 36 *([3, Example 78]) The half-disc topology is $T_{2_{1/2}}$ but not T_3. Let $P = \{(x, y) : x, y \in \mathbb{R}, y > 0\}$ be the upper half-plane and let $L = \{(x, 0) : x \in \mathbb{R}\}$ be the real axis. We generate a topology on $P \cup L$ by defining a basis of neighborhoods: a basis element for an element $p \in P$ is an open disc contained in P, while a basis element for an element $a \in L$ consists of an open half-disc centered at a together with a itself. For example, $Z = \{(x, y) : (x - a)^2 + y^2 < \epsilon, y > 0\} \cup \{(a, 0)\}$ is a basis element for $(a, 0)$. The closure of Z is the set $Z^c = \{(x, y) : (x - a)^2 + y^2 \leq \epsilon, y \geq 0\}$. It is now easy to show that the half-disc topology is $T_{2_{1/2}}$. The half-disc topology is not T_3: $(P \cup L) - Z$ is a closed set. Every neighborhood of $(P \cup L) - Z$ intersects every neighborhood of $(a, 0)$.*

4.5 $\mathbf{T}_3$

A space X is *regular* (or T_3) if it is T_1 and it holds one of the following equivalent conditions:

- For every closed set A and for every $x \notin A$, there exists disjoint open sets U and V such that $A \subseteq U$ and $x \in V$.
- The closed neighborhoods of a point constitute a fundamental system of neighborhoods.

Example 37 *Let $\mathbb{R}$ be the set of real numbers. The topology generated by the open intervals (a, b) and the sets $(a, b) \cap \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers, is T_2 but not T_3. The set $\mathbb{Q}$ is open, while the set $F = \{\pi/n : n \geq 1\}$ of irrational numbers is closed. The point 0 and the closed set F cannot be separated.*

4.6 $\mathbf{T}_4$

A space X is *normal* (or T_4) if it is T_1 and it holds the following condition: For all disjoint closed sets A and B, there are disjoint open sets U and V such that $A \subseteq U$, $B \subseteq V$.

Example 38 *The Euclidean line is T_4.*

4.7 Totally separated space

A space is *totally separated* if arbitrary distinct points are separated by a clopen set.

5 Connected spaces

The notion of a connected space is orthogonal to that of totally separated space.

5.1 Connected spaces vs totally separated spaces

A space is *connected* if there are not distinct points separated by a clopen set (in other words, if there are no nontrivial clopen sets).

Theorem 39 *The following conditions are equivalent for a space X:*

1. *X is connected.*
2. *X is not the union of two nonempty disjoint open sets.*
3. *X is not the union of two nonempty disjoint closed sets.*
4. *If $X = A \cup B$ with $A, B \neq \emptyset$, then $(A \cap B^c) \cup (A^c \cap B) \neq \emptyset$.*

Proof. ($4 \Rightarrow 1$) Obvious.

($1 \Rightarrow 4$) Assume that there exists nonempty sets A and B such that $X = A \cup B$, $(A \cap B^c) = \emptyset$ and $(A^c \cap B) = \emptyset$. We now show that A^c and B^c are disjoint. If x is adherent to both A and B, then by $X = A \cup B$ we have that, for example, $x \in A$. Then x belongs to the open set $X - B^c$, so that it is not adherent to B. In conclusion, $X = A^c \cup B^c$ and $A^c \cap B^c = \emptyset$. This contradicts the hypothesis of connectedness. ■

A subset Y of X is connected if Y is connected as a subspace of X.

Proposition 40 *The following properties hold:*

1. *The closure of a point is connected.*
2. *If A is a non-trivial clopen of a space X, then either $Y \subseteq A$ or $Y \subseteq X - A$, for every connected subset Y of X.*
3. *The union of arbitrary connected subsets, having pairwise nonempty intersection, is connected.*
4. *If Y is connected, then the closure Y^c of Y is also connected.*

Proof. (1) If there are two open sets A and B such that $x^c = (A \cap x^c) \cup (B \cap x^c)$ and $(A \cap x^c) \cap (B \cap x^c) = \emptyset$, then, without loss of generality, we may assume that $x \in (A \cap x^c)$. Then, every point $y \in (B \cap x^c)$ cannot be in the closure of x, so that $B \cap x^c = \emptyset$. Contradiction.

(2) Trivial.

(3) Let $Y = \cup_{i \in I} Y_i$. If $Y = A \cup B$ with A, B open and disjoint, then by (2) we get $Y_i \subseteq A$ for every i or $Y_i \subseteq B$ for every i. Contradiction.

(4) Let $Y^c = A \cup B$, where A and B are open in the subspace Y^c with $A \cap B = \emptyset$. Since Y is connected, then we must have that either A or B is contained within $Y^c - Y$. This contradicts the property for Y^c to be the closure of Y. ■

5.2 Components

- The *connected component* of $x \in X$ is the greatest connected subset containing x. The connected component of x contains the closure of x. The connected component of x is closed. Connected components constitute a partition of X. If there is a unique connected component then the space is connected.

- Each clopen set contains the connected components of all of its points. The *quasicomponent* of $x \in X$ is the intersection of all clopen sets containing x. Quasicomponents constitute a partition of X. If there is a unique quasicomponent then the space is connected.
- A *path* (*arc*) is a (one-to-one) continuous function from the unit interval into X. Two points are path (arc) connected if there is a path (arc) f such that $f(0) = a$ and $f(1) = b$. A space is *path (arc) connected* if two arbitrary points are path (arc) connected. A path (arc) component is a maximal subset with respect to path (arc) connectedness.

We have:

Arc component $\subset$ Path component $\subset$ Connected component $\subset$ Quasicomponent.

Example 41 *([3, Example 12.point13]) Let X be a set and $p \in X$ be a point. The particular point topology on X is constituted by all the sets containing the particular point p. This topology is path connected (if $q \neq p$ then $f(1) = q$ and $f([0,1)) = p$ is a path), but it is not arc connected (the inverse image of the open set $\{p\}$ w.r.t. a one-to-one continuous map would be one point, which is not open in the interval $[0,1]$).*

Example 42 *([3, Example 115, 118])*

5.3 Totally disconnected spaces

A space is *totally disconnected* if every connected subset consists of a single point. Totally disconnected spaces are T_1. Totally separated spaces are totally disconnected.

Example 43 *Example of totally disconnected space which is not totally separated.*

5.4 Connected spaces vs T_i

We introduce strong properties of connectedness orthogonal to the properties of T_i-separability.

Definition 5.1 *We have the following definitions:*

(vs $T_{2_{1/2}}$) A space is co-connected *if it has no $T_{2_{1/2}}$-separable elements. In other words, if, for all nonempty open sets U and V, we have that $V^c \cap U^c \neq \emptyset$.*

(vs T_2) A space is hyperconnected *if it has no T_2-separable elements. In other words, if, for all nonempty open sets U and V, we have that $V \cap U \neq \emptyset$ (equivalently, X is hyperconnected if the closure of every open set is the entire space).*

- *A space is* ultraconnected *if, for all nonempty closed sets U and V, we have that $V \cap U \neq \emptyset$ (equivalently, X is ultraconnected if the closures of distinct points always intersect).*

Then we have the following implications:

hyperconnectedness $\Rightarrow$ co-connectedness $\Rightarrow$ connectedness

and

ultraconnectedness $\Rightarrow$ co-connectedness $\Rightarrow$ connectedness.

and

$$\text{ultraconnectedness} \Rightarrow \text{path connectedness} \Rightarrow \text{connectedness}.$$

Then co-connectedness is a sort of meeting point between ultraconnectedness and hyperconnectedness.

Proposition 44
1. *Ultraconnected spaces are path connected, while they need not be arc connected.*
2. *Hyperconnected spaces need not be path connected.*

Proof. (1) Let X be an ultraconnected space. If $a, b \in X$ and $y \in \{a\}^c \cap \{b\}^c$ then the function f which maps each point of $[0, 1/2)$ to a, each point of $(1/2, 1]$ to b and $f(1/2) = y$ is continuous. [3, Example 13] is a counterexample to the arc connectedness.

(2) See [3, Example 18]. ■

Proposition 45 *Let X be a hyperconnected space.*

1. $int(A) \neq \emptyset$ *iff A is dense (i.e., $A^c = X$).*
2. $int(A) = \emptyset$ *iff* $int(A^c) = \emptyset$. *If $A \neq X$ is closed, then $int(A^c) = \emptyset$.*
3. *Every continuous function from X into a T_2-space is constant.*
4. *Dense sets constitute a topology, which is a filter on X.*
5. *There is at most one singleton which is open. If a singleton $\{a\}$ is open, then the dense sets constitute a pointed topology (the nonempty open sets are the sets containing a).*

Example 46 *Let X denote $\mathbb{N}$, the set of natural numbers, endowed with the topology generated by taking basic neighborhoods of each $n \in \mathbb{N}$ the set $\{0, 1, \ldots, n\}$. The space X is T_0 and hyperconnected. The closure of a point n is the interval $[n, \infty)$. Then the space X is also ultraconnected.*

Example 47 ([3, Example 12(point10)]) *Let X be a set and $p \in X$ be a point. The particular point topology on X is constituted by all the sets containing the particular point p. This topology is hyperconnected. But, if X contains at least two points a and b distinct from p, it is not ultraconnected since $\{a\}$ and $\{b\}$ are disjoint closed sets.*

The following example provides a wide class of topological spaces whose topology is co-connected.

Example 48 (Co-connected spaces) *Let X be a T_0 space, whose specialization order $\leq_X$ satisfies the following property: every pair of nodes of the inequality graph of $(X, \leq_X)$ is joined by a path of length less or equal to 3 (In particular, if X admits a bottom (top) element or if $\leq_X$ is an upward (downward) direct set. Complete partial orderings with the Scott topology are co-connected). We now show that the space X is co-connected. Assume, by the way of contradiction, that there exist elements $a, b \in X$ and open sets U and V such that $a \in U$, $b \in V$ and $V^c \cap U^c = \emptyset$. Then a and b are incomparable w.r.t. the specialization ordering, otherwise either $a \in V$ or $b \in U$. By hypothesis a and b are joined by a path of length less or equal to 3. We have four possibilities.*

1. $(\exists c)\ a < c > b$. *In this case* $c \in U \cap V$.
2. $(\exists c)\ a > c < b$. *Then we have* $c \in a^c \subseteq U^c$ *and* $c \in b^c \subseteq V^c$.
3. $(\exists c, d)\ a < c > d < b$. *Then* $d \in \overline{c} \subseteq U^c$ *and* $d \in b^c \subseteq V^c$.
4. $(\exists c, d)\ a > c < d > b$. *Then* $c \in \overline{d} \subseteq V^c$ *and* $c \in a^c \subseteq U^c$.

All these possibilities contradict our assumption.

We cannot improve the above result. The following counter-example provides a space, whose topology is not co-connected, and such that every pair of nodes of the inequality graph associated with the specialization order is joined by a path of length less or equal to 4. *Let* $X = \{a, b, c, d, e\}$ *be a set partially ordered as follows:* $a > b < c > d < e$. *Then* X *is a topological space w.r.t. the Alexandroff topology (see below Section* **??**). X *is not co-connected, since* $a{\uparrow}{\downarrow} = \{a, b\}$ *and* $e{\uparrow}{\downarrow} = \{d, e\}$ *are disjoint closures of the open sets* $a{\uparrow}$ *and* $e{\uparrow}$.

Example 49 *Complete partial orderings (without top element) with Scott topology provide examples of co-connected, but not hyperconnected, spaces.*

Example 50 *Let* $X = \{0, 1, 2\}$ *be partially ordered as follows:* $0 < 2$ *and* $1 < 2$. *Then* X *with the Alexandroff topology is an example of co-connected, but not ultra-connected, space.*

6 Compact spaces

A space X is *compact* if it holds one of the following equivalent conditions:

- Every open cover of X contains a finite subcover;
- Every family of closed sets, whose intersection is empty, contains a finite family, whose intersection is empty;
- Every family of closed sets with the finite intersection property has a non-empty intersection.

Theorem 51 *The following properties hold:*

(i) *Every closed subset of a compact space* X *is compact.*

(ii) *If* A *is a compact subset of a Hausdorff space and* $x \notin A$, *then there are disjoint neighborhoods of* x *and of* A. *Then every compact subset of a Hausdorff space is closed.*

(iii) *Every compact and Hausdorff space is* T_3.

(iv) *Every compact and Hausdorff space is* T_4.

(v) $A \subseteq X$ *is compact iff every net in* A *has a cluster point.*

(vi) *In a compact space every infinite set has at least an accumulation point.*

(vii) *The product of compact spaces is compact.*

Proof. (i) Let B be closed. Then every cover $U = \{U_i : i \in I\}$ of B becomes a cover of the space X if we add the open set $X - B$.

(ii) For every $a \in A$, there are an open neighborhood U_a of a and an open neighborhood V_a of x such that $U_a \cap V_a = \emptyset$. Then the family $(U_a : a \in A)$ is a cover of A. Then there exists a finite subcover $U_{a_1}, \ldots, U_{a_n}$. The open set $U_{a_1} \cup \cdots \cup U_{a_n}$ and $V_{a_1} \cap \cdots \cap V_{a_n}$ are disjoint neighborhoods of A and of x.

(iii) By (ii) and by (i) we have that closed = compact. Then the conclusion follows from the first part of (ii).

(iv) By (iii) and by a reasoning similar to that of point (ii).

(v $\Leftarrow$) If A is not compact, consider a cover $U = \{U_i : i \in I\}$ of A without finite subcovers. Consider the set $D = \{U_{i_1} \cup \cdots \cup U_{i_k} : U_{i_j} \in U\}$. D is directed w.r.t. $\subseteq$. Let $(x)_D$ be a net in A such that $x_d \notin d$ for every $d \in D$. Then $(x)_D$ has no cluster points. In fact, if $y \in A$ is cluster, then we consider $d \in D$ such that $y \in d = U_{i_1} \cup \cdots \cup U_{i_k}$. Then $y \in U_{i_j}$ for some $1 \leq j \leq k$. Then the neighborhood U_{i_j} of y does not contain the tail $[x]_{\geq d}$. Contradiction.

(v $\Rightarrow$) If A is compact and $(x)_I$ is a net in A without clusters, then, for every $y \in A$, we consider an open neighborhood U_y of y that has empty intersection with some tail. Then $U = \{U_y : y \in A\}$ is a cover di A. Then there exists a finite subcover $U' = \{U_{y_1}, \ldots, U_{y_k}\}$. Let $[x]_{\geq i_r}$ be the tail with empty intersection with U_{y_r} for $1 \leq r \leq k$. Since I is directed, then there exists an upper bound j of the set $\{i_1, \ldots, i_k\}$. Then the element $x_j \in A$ of the net is not in the finite cover U'. Contradiction.

(vi) Let A be an infinite set. If A has no accumulation points, then A^c has no accumulation points. This means that the topology induced over A^c is discrete. Then A^c cannot be compact because it is infinite. ■

6.1 Locally compact spaces

A space is *locally compact* if there exists a basis of compact sets. Alternatively, for every x and every open U containing x, there exists a compact neighborhood Q of x such that $Q \subseteq U$.

Example 52 *A set X with the discrete topology is locally compact, but it is not compact if X is infinite.*

Example 53 *The real line is locally compact but it is not compact.*

Example 54 *The rational line is not locally compact because the rational intervals are not compact. For example, we consider the interval $(0, 1)$, an enumeration r_i of rational numbers in the interval, and a sequence of real numbers ϵ_i such that $\Sigma_{i \geq 0}\, \epsilon_i < 1/4$. Then, for every rational number $r_i \in (0, 1)$ we define the interval $(r_i - \epsilon_i, r_i + \epsilon_i)$. This is a cover of $(0, 1)$ which does not admit a finite subcover.*

Proposition 55 *For Hausdorff spaces, locally compactness is equivalent to the following condition: every point has at least one compact neighborhood.*

Proof. Let Q be a compact neighborhood of x. We have to show that, for every open U containing x, there exists a compact neighborhood Q' of x such that $Q' \subseteq U$. The intersection of a compact and a closed set is compact. Then $Q \cap U^c$ is compact. For every element $y \in U^c - U$ we find two open sets V_y and W_y such that $y \in V_y$, $x \in W_y$ and $V_y \cap W_y = \emptyset$. Then we consider a cover of the compact set $Q \cap U^c$ made by the open set $U \cup (\cup_{y \in U^c - U} V_y)$. Then there exists a finite subcover made by U and $V_{y_1}, \dots, V_{y_n}$. It follows that $U^c - U \subseteq V_{y_1} \cup \cdots \cup V_{y_n}$. Finally, the set $Q \cap U^c \cap (-V_{y_1}) \cap \cdots \cap (-V_{y_n})$ is compact, is a neighborhood of x, and it is contained within U. ■

7 Topologies on posets

Let D be a poset. How many T_0 topologies τ on D satisfy the condition that the specialization order $\leq_\tau$ agrees with the original order $\leq_D$ on D?

Definition 7.1 *Let D be a poset.*

1. (Alexandroff topology α_D) *The Alexandroff topology has all upper sets as open sets. An arbitrary intersection of Alexandroff open sets is Alexandroff open. It implies that, for all a, there exists a smallest open set $a\!\uparrow$ including a. A function is Alexandroff continuous iff it is monotone.*
2. (Upper topology u_D) *The upper topology u_D is the smallest topology for which all sets of the form $x\!\downarrow$ are closed, i.e., the topology based by sets of the form $D - (x_0\!\downarrow \cup \cdots \cup x_n\!\downarrow)$.*
3. (Lower topology $\omega_D = u_{D^{op}}$)
4. (Scott topology σ_D) *See below.*

Proposition 56 *Let D be a poset and τ be a topology on D. Then τ is T_0 with specialization order $\leq_D$ iff $u_D \subseteq \tau \subseteq \alpha_D$. Then $u_D \subseteq \sigma_D \subseteq \alpha_D$.*

Proof. ($\Rightarrow$) If τ induces $\leq_D$, then all τ-open sets are upper and all τ-closed sets are lower (w.r.t. $\leq$). Then $\tau \subseteq \alpha_D$. We now show that $x\!\downarrow$ is τ-closed, so that $u_D \subseteq \tau$. If $y \not\leq x$, then by the hypothesis T_0 there exists a τ-open set O containing y but not x. O is upper and it does not contain x. Then $O \cap x\!\downarrow$ is empty.

($\Leftarrow$) We have to show that $\leq_D = \leq_\tau$. From $\tau \subseteq \alpha_D$ it follows that $x \leq_D y$ implies $x \leq_\tau y$. The set $\{x : x \leq_D y\}$ is τ-closed by $u_D \subseteq \tau$. Then by $\leq_D \subseteq \leq_\tau$ we have that $\{x : x \leq_D y\} \subseteq \{x : x \leq_\tau y\}$. But this last set is the closure of $\{y\}$ w.r.t. τ. Then the τ-closed $\{x : x \leq_D y\}$ must agree with $\{x : x \leq_\tau y\}$. τ is T_0 because $\leq_D$ is a partial ordering. ■

The one-sided non-Hausdorff topologies can be combined as follows:

Definition 7.2 *Let D be a poset.*

1. (Interval topology $I_D = \omega_D \vee u_D$) *The interval topology is the join of the upper and lower topologies. It is the smallest topology for which the intervals $[a, b]$ are closed. It is T_1, but it is not always Hausdorff (consider a countable antichain with a 0 and 1 adjoined).*
2. (Lawson topology $\lambda_D = \omega_D \vee \sigma_D$)
3. (bi-Scott topology $\sigma_D \vee \sigma_{D^{op}}$)
4. (order-topology o_D) *A net x_I* order-converges *to y if there exist other two nets n_I and k_I such that:*

- *n_I is order preserving and k_I is order reversing.*
- $n_i \leq x_i \leq k_i$
- $\vee_{i \in I} n_i = x = \wedge_{i \in I} k_i$.

If the poset is a complete lattice, then the order-convergence can be defined by

$$limsup\, x_I = \wedge_{i \in I}(\vee x_{\geq i}) = \vee_{i \in I}(\wedge x_{\geq i}) = liminf\ x_I$$

It is easy to show in general that $liminf\ x_I \leq limsup\, x_I$. The order convergence can be also defined by requiring that

$$\vee\{y :\ y\!\uparrow \text{ is } x_I\text{-eventual}\} = \wedge\{y :\ y\!\downarrow \text{ is } x_I\text{-eventual}\}.$$

The order topology is the strongest topology for which the order-convergence implies topological convergence.

7.1 Topologies on lattices

Let L be a lattice. The interval topology is Hausdorff in a lattice L.

8 Topologies on topological spaces

Definition 8.1 *Let X be a T_0-topological space with topology τ.*

1. (Co-compact topology co_τ) *The co-compact topology has the upper compact subsets of X has a basis for closed sets.*
2. (Patch topology $patch_\tau = co_\tau \vee \tau$)
3. (Generalized Lawson topology $\lambda_\tau = \omega_{\leq_\tau} \vee \tau$)

9 The Heyting algebra of open sets

Definition 9.1 *A lattice with 0 is an* Heyting algebra *(HA) if, for every a, b, there is an element $a \to b$ such that*

$$c \leq a \to b \text{ iff } c \wedge a \leq b.$$

An HA is distributive and admits a top element $x \to x$.

Proposition 57 *Let L be a complete lattice with 0. L is an HA iff L satisfies the infinite distributive law:*

$$x \wedge \bigvee Y = \bigvee\{x \wedge y : y \in Y\}.$$

Proof. ($\Rightarrow$) Let $d \geq x \wedge y$ for all $y \in Y$. Then $y \leq x \to d$, so that $\bigvee Y \leq x \to d$, that implies $x \wedge \bigvee Y \leq d$.

($\Leftarrow$) If $Y = \{c : c \wedge a \leq b\}$, $a \to b = \bigvee Y$ because $a \wedge \bigvee Y = \bigvee\{c \wedge a : c \in Y\} \leq b$. ■

A homomorphism between complete HAs (called also frames) preserves also arbitrary joins.

Proposition 58 *The set $\mathcal{O}(X)$ of opens of a space X is a complete HA with*

$$U \to V = int(-U) \cup (U \cap V).$$

An open U is co-prime *iff the relative topology on U is indiscrete (i.e., $\emptyset$ and U are the only open sets), while it is* prime *if, for all opens $V, W \in \mathcal{O}(X)$, $V \cap W \subseteq U$ implies $V \subseteq U$ or $W \subseteq U$.*

If we define $\neg U \equiv U \to \emptyset = int(-U)$, then U is complemented iff U is clopen.

If $U = int(X - \{x\})$ then U is prime. The opposite is true iff the space is sober as we will show below.

9.1 Sober spaces and the hull-kernel topology

The family $\mathcal{N}_x$ of the *open neighborhoods* of a point x of a space X is a completely prime (i.e., $\cup_{i \in I} O_i \in \mathcal{N}_x$ implies $(\exists i)\ O_i \in \mathcal{N}_x$) filter of the lattice $\mathcal{O}(X)$.

Question: Is a completely prime filter of open sets equal to the family of the open neighborhoods of a suitable point x? In general the answer is no. Consider the natural numbers with the topology generated by the sets $[n) = \{x : x \geq n\}$. Then the set $\{[n) : n \geq 0\}$ is a completely prime filter which is not the family of open neighborhoods of a point.

Proposition 59 *There exists a bijective correspondence among the following sets:*

(i) Completely prime filters.

(ii) Nonunit prime elements of $\mathcal{O}(X)$.

(iii) Nonempty irreducible closed sets (a set A is irreducible if, for all closed C, D: $A \subseteq C \cup D \Rightarrow (A \subseteq C$ or $A \subseteq D)$).

(iv) Nonempty closed sets whose induced topology is hyperconnected. (Example: every subset of X directed w.r.t. the specialization order is irreducible).

Proof. U prime open iff $X - U$ closed and irreducible iff $\{O \in \mathcal{O}(X) : O \not\subseteq U\}$ is a completely prime filter. Vice versa, if $\mathcal{F}$ is a completely prime filter, then $-\mathcal{F}$ is a prime principal ideal. Then the open $\cup - \mathcal{F}$ is a prime element of $\mathcal{O}(X)$.

(iii) $\Rightarrow$ (iv): If the induced topology is not hyperconnected, then there exist open sets U, V such that $U \cap A$ and $V \cap A$ are non empty and $U \cap V \cap A = \emptyset$. This means that $A \subseteq -U \cup -V$, where $-U$ and $-V$ are closed. Since A is irreducible, then we get either $A \subseteq -U$ or $A \subseteq -V$. This contradicts the hypothesis $U \cap A$ and $V \cap A$ non empty.

(iv) $\Leftarrow$ (iii): Let $A \subseteq C \cup D$ with C, D closed. Then $-C \cap -D \subseteq -A$, i.e., $-C \cap -D \cap A = \emptyset$. Since A is hyperconnected it follows, for example, that $-C \cap A = \emptyset$, that is, $A \subseteq C$. ■

Example 60 *The prime open set $int(X - \{x\})$ generates the principal prime ideal of the open sets not containing x. The closure x^c of x is the irreducible set $X - int(X - \{x\})$.*

A *formal point p* is a completely prime filter of open sets (or the corresponding prime open, or the corresponding irreducible closed).

We can define a topology on the set of formal points:

- A set a of completely prime filters (prime opens) is open if there exists an open $O \in \mathcal{O}(X)$ such that $a = \{\mathcal{F} : O \in \mathcal{F}\}$ ($a = \{U$ open prime : $O \not\subseteq U\}$).

Since prime elements are more general, we can abstract the above notion as follows.

Definition 9.2 *Let* L *be a complete lattice and let* $Spec(L)$ *be the set of nonunit prime elements of* L*. The* hull-kernel topology *on* $Spec(L)$ *is the topology whose closed sets are* $(\{p \in Spec(L) : a \leq p\} : a \in L) = (a\!\uparrow \cap\, Spec(L) : a \in L)$ *and whose open sets are* $(\{p \in Spec(L) : p \not\geq a\} : a \in L)$*.*

Does the hull-kernel topology on $Spec(L)$ satisfy the separation axiom T_0?

Proposition 61 *Let* L *be a complete lattice. Then the following two conditions are equivalent:*

(i) The hull-kernel topology on $Spec(L)$ *is* T_0

(ii) L *is* spatial *(or* has enough points*) (i.e., for all distinct* $x, y \in L$*, there is a prime* $p \in Spec(L)$ *such that* $p \not\geq x$ *and* $p \geq y$ *or vice versa).*

(ii) (or (i)) implies that L *is a complete HA.*

Proof. If L is spatial, then the map $f : L \to 2^{Spec(L)}$ defined by $f(a) = \{p \in Spec(L) : a \leq p\}$ is injective and preserves arbitrary joins and finite meets. But $2^{Spec(L)}$ is a complete HA, hence so is L. ■

Example 62 *The lattice* $\mathcal{O}(X)$ *of the open sets is spatial. Let* A *and* B *distinct open sets. Then there is, for example, an element* $x \in A - B$*. Consider the prime element* $U = int(X - \{x\})$*. Then* $B \subseteq U$ *and* $A \not\subseteq U$*.*

Let $\mathcal{CL}$ be the category of complete lattice and mappings preserving arbitrary joins and finite meets.

Proposition 63 ([1, Prop. V-4.7] + definition of sober space) *The functor*

$$Spec : \mathcal{CL} \to TOP^{op}$$

mapping a complete lattice L *into the topological space* $Spec(L)$ *and an arrow* $f : L \to K$ *in* $\mathcal{CL}$ *into a continuous function* $Spec(f) : Spec(K) \to Spec(L)$*, is left adjoint to the functor*

$$\mathcal{O} : TOP^{op} \to \mathcal{CL}$$

mapping a topological space X *into the complete lattice* $\mathcal{O}(X)$ *of its open sets and a continuous function* $f : X \to Y$ *into the map* $\mathcal{O} : \mathcal{O}(Y) \to \mathcal{O}(X)$ *defined by* $\mathcal{O}(U) = f^{-1}(U)$*.*

Front and back adjunctions are given by the continuous function

$$\xi_X : X \to Spec(\mathcal{O}(X)); \quad \xi_X(y) = int(X - \{y\}) \ (or\ = \mathcal{N}_y)$$

(ξ_X *is injective iff* X *is* T_0*; if* ξ_X *is bijective we say that* X *is* sober*) and by the arrow in* $\mathcal{CL}$

$$\Delta_L : L \to \mathcal{O}(Spec(L)); \quad \Delta_L(a) = Spec(L) - a\!\uparrow.$$

The map ξ_X *is a homeomorphism iff* X *is sober (in particular,* $\xi_{Spec(L)}$ *is a homeomorphism) and the map* Δ_L *is an isomorphism iff* L *is a spatial complete lattice (or equivalently, spatial complete HA) (in particular,* $\Delta_{\mathcal{O}(X)}$ *is an isomorphism).*

The *soberification* of a space is the space of its formal points.

Recall that the closure of an irreducible set is irreducible.

Proposition 64 *The following conditions are equivalent for a space X:*

(i) X is sober;

(ii) $U \in \mathcal{O}(X)$ is a nonunit prime element iff $U = int(X - \{y\})$ for some $y \in X$;

(iii) $\mathcal{F}$ is a completely prime filter of $\mathcal{O}(X)$ iff $\mathcal{F} = \mathcal{N}_y$ for some $y \in X$;

(iv) Every irreducible closed set is the closure of a unique point.

Proposition 65 *Every sober space X is T_0 and its specialization order makes X a dcpo. If $D \subseteq X$ is directed, then the net x_D, defined by $x_D(d) = d$ for every $d \in D$, converges to sup D.*

Proof. The specialization order of X is the relation $\subseteq$ between formal points: $p \leq q$ iff $O \in p \Rightarrow O \in q$, for all open sets of X iff $p \subseteq q$. Then the union of a directed family of formal points is a formal point. ■

Proposition 66 *Let X be a sober space and $\leq$ be a partial ordering on X. Then $\leq$ is the specialization order of X iff $u_{\leq} \subseteq \mathcal{O}(X) \subseteq \sigma_{\leq}$.*

Proof. Let O be open in D and $U = \{\text{formal points } p : O \in p\}$ be the corresponding open set in the space of formal points ■

Not every topology τ such that upper $\subseteq \tau \subseteq$ Scott is sober (See Chapter II Section 1.9 pag. 46 in Johnstones book: Stone Space).

Theorem 67 (Algebraic characterization of topologies [Papert 1959]) *Spatial complete HAs are, up to isomorphism, exactly the algebras of open sets of topologies.*

Proof. Let A be a spatial complete HA. Define on the set Pt(A) of prime elements of A the topology constituted by the open sets $(\{p \in \text{Pt}(A) : p \not\geq x\} : x \in A)$. Then the HA A is isomorphic to the HA of the open sets on Pt(A). ■

Let X be a space. If $A = \mathcal{O}(X)$ is the set of open sets of X, then $\mathcal{O}(X)$ is isomorphic to the HA of the open sets on Pt$(\mathcal{O}(X))$, so that the spaces X and Pt$(\mathcal{O}(X))$ have, up to isomorphism, the same set $\mathcal{O}(X)$ of open sets. However, in general, X has less 'points' than Pt$(\mathcal{O}(X))$. In other words, Pt$(\mathcal{O}(X))$ is sober, while X is not.

9.1.1 Sobriety and nets

There exists a characterization of the sober topology in terms of nets (see [4]).

A net x_I is *observative* if $x_i \in O \in \tau$ implies x_I is eventually in O (i.e., a tail is in O).

An observative net is said to *strongly converge* to a point x (denoted by $x_I \to^* x$), if it converges to x and it satisfies the following condition:

$$x_i \in O \in \tau \;\Rightarrow\; x \in O.$$

A net is observative iff it converges to each of its points.

If a net x_I is observative, then $x_I \to^* x$ iff $x_I \to x$ w.r.t. the topology (called *b-topology*) generated by the subbase constituted by the τ-open and τ-closed sets. (Remark: Is the b-topology equivalent to the following construction? The closure bB of B is defined as the set of elements p for which there do not exist open sets O_1 and O_2 with $p \in O_2 - O_1$ and $O_1 \cap B = O_2 \cap B$. Equivalently $p \in bB$ iff $p \in (B \cap p^c)^c$).

Let $\mathcal{F}_{x_I} = \{O \in \tau : \exists i \in I\ x_{\geq i} \in O\}$. The family of sets $\mathcal{F}_{x_I}$ is always a filter. When is it completely prime? The following lemma answers the above question.

Lemma 68 *Let x_I be an observative net. Then*

1. *The filter $\mathcal{F}_{x_I}$ is completely prime.*
2. *$x_I \to^* x$ iff $\mathcal{N}_x = \mathcal{F}_{x_I}$.*

It follows that an observative net cannot strongly converge to two points.

Theorem 69 *A space is sober iff every observative net strongly converges to a unique point.*

Proof. ($\Rightarrow$) By the above lemma.

($\Leftarrow$) Let p be a formal point. Then p is directed w.r.t. the partial ordering reverse to $\subseteq$. We define a net x_p indexed by p as follows. Let $O \in p$. Then x_O is defined as an element of O such that $\mathcal{N}_{x_O} \subseteq p$. This element must exist because otherwise we may find, for every element $y \in O$, an open set U such that $y \in U \notin p$. This contradicts that p is completely prime. In conclusion, the net x_p exists. Is this net observative? Let U be an open set such that $x_V \in U$. Then $U \in p$ and the tail $x_{\geq U}$ is in U. By hypothesis the observative net x_p strongly converges to a point z. Then $p = \mathcal{N}_z$ and the space is sober. ■

Lemma 70 *A continuous (it is complete by definition) HA is spatial.*

Proof. Let $x \not\leq y$. Then there is an element $a \ll x$ such that $a \not\leq y$. By the interpoletion property of continuous poset there is a sequence of elements x_n such that

$$a \ll \ldots x_n \ll x_{n-1} \ll \ldots x_1 \ll x.$$

Consider the set $F = \{z : z \geq x_n \text{ for some } n\}$. F is a Scott open filter, so that $-F$ is Scott closed. Consider a maximal chain containing y within $-F$ (there exists by Zorn's lemma). The least upper bound of this chain is a maximal element z in $-F$. z is $\wedge$-irreducible and then, by using the distributivity it is a prime element (if $z \geq c \wedge d$ then $z = (z \vee c) \wedge (z \vee d)$). It follows that $z \geq y$ and $z \not\geq x$. ■

Every algebraic complete HA is continuous and then spatial.

Theorem 71 (Algebraic characterization of locally compact topologies) *Continuous HAs are, up to isomorphism, exactly the algebras of open sets of locally compact topologies.*

A locally compact topology is core-compact. Vice versa, a core-compact sober space is locally compact.

Example 72

1. *The algebra $\mathcal{P}(\omega)$ is a complete HA with enough supercompacts (the singleton sets). A HA with enough supercompacts is spatial.*

2. (Cantor space) *The algebra of open sets of the Cantor space* 2^ω *is an algebraic complete HA without co-primes. The topology is generated by the infinite sequences* $\sigma 2^\omega$ *having a common fixed initial segment* σ. *These sets are closed under finite intersection, because* $\sigma 2^\omega \cap \rho 2^\omega$ *is either empty or, for example,* σ *is a prefix of* ρ *so that* $\sigma 2^\omega \cap \rho 2^\omega = \rho 2^\omega$. *Let* 2^*_n *the set of binary strings of length* n. *The complement of* $\sigma 2^\omega$ *is also open because it can be defined as* $\cup_{\rho \in (2^*_n - \{\sigma\})} \rho 2^\omega$. *The Cantor space is zero-dimensional and compact.*

   Another way to consider the topology is the following: elements of the space are the subsets of ω, *while the topology is generated by the open base sets (*$a \subseteq n$, $n \in \omega$, $n = \{i : 0 \leq i < n\}$*) :* $O = \{X \subseteq \omega : a \subseteq X,\ n - a \subseteq -X\}$. *Then it is obvious that* $\{-X \subseteq \omega : a \subseteq X,\ n - a \subseteq -X\}$ *is also an element of the base. The complement of* O *is the set* $\cup_{b \neq a, b \subseteq n} \{X \subseteq \omega : b \subseteq X,\ n - b \subseteq -X\}$.

3. *The algebra of open sets of the euclidean space* R *is a continuous HA which is not algebraic.*

4. (Baire space) *The algebra of open sets of the Baire space* ω^ω *is a complete spatial HA which is not continuous. The topology is generated by the sets* $\{f : \omega \to \omega : f \geq \sigma\}$ *of functions having a common fixed initial segment* σ.

5. *The algebra* $\mathcal{P}^*(\omega)$ *of all sets of natural numbers modulo finite sets is a complete HA without primes elements.*

9.2 Representation of distributive lattices

If an HA is not spatial, prime elements can be substituted by prime ideals (or prime filters) in its representation. I is a prime ideal if I is a nonempty lower set and $a, b \in I \Rightarrow a + b \in I$; $ab \in I \Rightarrow a \in I$ or $b \in I$. If x is a prime element, then $x\!\downarrow$ is a prime ideal.

Let A be a distributive lattice. The set of prime ideals of A with the topology generated by the sets $(\{I : x \notin I\} : x \in A)$ is called the *dual Stone topology* and it is sober. The dual Stone space is isomorphic to the Stone space of prime filters with the topology generated by $(\{F : x \in F\} : x \in A)$, because the complement of a prime ideal is a prime filter and vice versa.

Every distributive lattice has enough prime ideals.

Theorem 73 (Topological characterization of HAs) *Any distributive lattice* A *is isomorphic to the algebra of compact open sets of its Stone topology, which is a coherent space. The Stone topology is compact if* A *has a top element. Vice versa, if* X *is a coherent space, then* X *is homeomorphic to the Stone space of its compact open sets.*

Proof. The map f, defined by $f(x) = \{F : x \in F\}$, is an embedding of A into the complete HA of the open sets of its Stone topology. If $x \not\leq y$ then there is a prime filter containing x but not y. Consider the filter $x\!\uparrow$ and the ideal $y\!\downarrow$. By applying the Zorn Lemma there is a maximal ideal containing y but not x. In a distributive lattice an ideal, which is maximal amongst those disjoint from a given filter, is prime (see Johnstone, Stone Spaces, Thm.2.4 page 13).

The whole space is compact if A has a top element 1. If $B \subseteq A$ determines a cover $f(B)$ of the whole space and it does not admit a finite subcover, then there is a maximal ideal I containing $b_1 \vee \cdots \vee b_k \neq 1$ for every $b_1, \ldots, b_k \in B$. The complement of I is a prime filter F such that $F \notin f(b)$ for every $b \in B$.

A similar argument works for $\{F : x \in F\}$. Let $B \subseteq A$ such that $f(B)$ is a cover of $\{F : x \in F\}$ and it does not admit a finite subcover, If $x \not\leq b_1 \vee \cdots \vee b_k$ for every $b_1, \ldots, b_k \in B$, then the ideal generated by $b_1 \vee \cdots \vee b_k$ does not contain x. ■

If the Stone space is Hausdorff, then the HA A is Boolean, because the compact open sets are closed (and then clopen) in every Hausdorff space. Let F and G be two prime filters. If F and G have disjoint neighborhoods, then there are two elements x and y not contained in a same prime filter. If $xy \neq 0$, then any prime filter containing xy contradicts the assumption. It follows that $xy = 0$.

9.3 Coherent and Boolean spaces

Definition 9.3 *A space X is*

- coherent *if it is sober, the compact open sets are closed under finite intersection and constitute a base for the topology.*
- zero-dimensional *if the clopen subsets of X form a base for the topology.*

We recall that the compact open sets U are the elements of the poset $\mathcal{O}(X)$ satisfying the condition $U \ll U$ (i.e., the finite elements of the poset $\mathcal{O}(X)$). Then, in a coherent space the frame $\mathcal{O}(X)$ admits a lattice of finite elements such that every open sets is the union of the finite elements contained within it. In conclusion, $\mathcal{O}(X)$ is an algebraic complete lattice.

A zero-dimensional T_0 space is T_3 and totally separated (Let A be closed and $x \notin A$. Then $-A$ is open. This means that $-A = \cup_{i \in I} U_i$, where U_i is clopen. Then the clopen U_j such that $x \in U_j$ solves our problem.)

Theorem 74 *A* Boolean space *X is defined by one of the following equivalent conditions:*

1. *X is a compact, Hausdorff and totally disconnected.*
2. *X is a compact and totally separated.*
3. *X is a compact, T_0 and zero-dimensional.*
4. *X is Hausdorff and coherent.*

Proof. (3) $\Rightarrow$ (4) In a compact Hausdorff space the compact open subsets are the clopen sets, since every compact in a Hausdorff space is closed (Let K compact and $x \notin K$. Then for every $y \in K$ there are open sets U_y and V_y such that $y \in U_y$, $x \in V_y$ and $U_y \cap V_y = \emptyset$. Since K is compact there is a finite subcover $U_{y_1}, \ldots, U_{y_n}$ of K. Then $K \subseteq U_{y_1} \cup \cdots \cup U_{y_n}$ and $x \in V_{y_1} \cap \cdots \cap V_{y_n}$. Moreover the open $V_{y_1} \cap \cdots \cap V_{y_n}$ is disjoint from K, so that K is closed.) Moreover, any Hausdorff space is sober (the singleton sets are the unique irreducible subsets).

(4) $\Rightarrow$ (3) We apply the subbasic Alexander Lemma (see Section **??**) to show that the space is compact.

(3) $\Rightarrow$ (2) Let x, y be distinct and let U be an open set such that $x \in U$ and $y \notin U$. Since U is union of clopen sets, then the result follows.

(2) $\Rightarrow$ (1) Let $a, b \in Y$ distinct, where Y is an arbitrary connected subspace of X. Since X is totally seprated, then there is a clopen A such that $a \in A$ and $b \in -A$. This implies that Y is not connected. Contradiction. It follows that every connected subspace is a singleton set.

(1) ⇒ (2) Let $x \in X$ and let C_x be the set of all points of X which cannot be separated from x by a clopen set. C_x is closed and $x \in C_x$. We show that $C_x = \{x\}$. By the way of contradiction, we assume that C_x is not a singleton set. Since C_x is not connected then there are two disjoint open sets A and B such that $C_x = (A \cap C_x) \cup (B \cap C_x)$. For every $z \in -C_x$ let V_z be a clopen separating z and x. Then A, B and V_z $(z \in -C_x)$ are a cover of X. Since X is compact we get a finite subcover, so that $-C_x = V_{z_1} \cup \dots \cup V_{z_n}$. It follows that C_x is clopen. Assume that $x \in B \cap C_x$. Then $A \cap C_x$ is clopen, because it is open as intersection of open sets and it is closed, as the complement is union of two open sets $-C_x$ and $B \cap C_x$. This contradicts the definition of C_x. Then $C_x = \{x\}$.

(2) ⇒ (3) Let τ be the topology of X and let ρ be the topology generated by the clopen sets of X. It is obvious that $\rho \leq \tau$. The topologies τ and ρ are both Hausdorff and compact. Since closed subset = compact subset in a Hausdorff compact space, then a compact subset of τ is also compact in ρ, so that τ and ρ have the same closed sets. ■

Every Boolean space is isomorphic to the space of the ultrafilters of the Boolean algebra of its clopen sets having the sets $(\{F \in \text{ Ultrafilters } : U \in F\} : U \text{ clopen})$ as basis.

10 Appendix: other topological concepts

An F_σ-set is a set which can be written as the union of a countable collection of closed sets; a G_δ-set is a set which can be written as the intersection of a countable collection of open sets.

1. A set A is *preopen* if $A \subseteq int(A^c)$, i.e., $A \cup B$ is open, where B is a set of elements of the boundary $A^c - int(A)$ of A. Preopen sets are closed under arbitrary union. A set is *preclosed* if its complement is preopen, i.e., if $(intA)^c \subseteq A$. The preclosure of A is the intersection of all preclosed sets including A, i.e., $pclA = A \cup (intA)^c$.

2. A set A is *a regular open set* if $A = int(A^c)$. For example, the interval $(0, 1)$ is regular open in the real line, while $(0, 1/2) \cup (1/2, 1)$ is not.

3. A set A is *semi-open* if $A \subseteq (intA)^c$. A set A is semi-open if it is between an open set and its closure, i.e., there is U open such that $U \subseteq A \subseteq U^c$. In fact, from $U \subseteq intA$ it follows that $A \subseteq U^c \subseteq (intA)^c$. A set A is semi-closed if $int(A^c) \subseteq A$

4. A set A is *semi-preopen* if it is between a preopen set and its closure, i.e., there is a preopen U such that $U \subseteq A \subseteq U^c$.

5. $\mathcal{O}(X) \subseteq SO(X) \cap PO(X) \subseteq SO(X) \cup PO(X) \subseteq SPO(X)$.

Definition 10.1 *A set A is*

- *α-closed if $(int(A^c))^c \subseteq A$; α-open $A \subseteq int((intA)^c)$.*
- *β-closed (or semi-preclosed) if $int((intA)^c) \subseteq A$; β-open $A \subseteq (int(A^c))^c$.*

Definition 10.2 *A set A is*

- *generalized closed (g-closed) if $A^c \subseteq U$ whenever $A \subseteq U$ and U is open.*
- *semi-generalized closed (sg-closed) if $A^{-_s} \subseteq U$ whenever $A \subseteq U$ and U is semi-open.*

- *generalized semi-closed (gs-closed) if $A^{-_s} \subseteq U$ whenever $A \subseteq U$ and U is open.*

The collection of all α-open subsets is a topology on X, called the α-topology, which is finer than the original one.

Definition 10.3 *A space is*

- extremally disconnected *if the closure of every open set is also open.*
- strongly irresolvable *if no open subspace is the union of two disjoint dense sets.*
- sg-submaximal *if every dense set is (semi-generalized open) sg-open.*
- *(semi-$T_{1/2}$) $T_{1/2}$ if every singleton is either (semi-open) open or (semi-closed) closed.*
- nodeg *if every nowhere dense set is g-closed (generalized closed).*

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[3] L.A. Steen and J.A. Seebach, Jr. Counterexamples in topology. Springer-Verlag, 1978.

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11 Sober spaces

The family $\mathcal{N}(x)$ of the *open neighborhoods* of a point x of a space (X, τ) is a completely prime filter of the lattice $(\tau, \subseteq)$, i.e., $\mathcal{N}(x)$ satisfies the following conditions.

Filter:

- $\emptyset \notin \mathcal{N}(x)$;
- $X \in \mathcal{N}(x)$;
- $U \in \mathcal{N}(x), U \subseteq V \;\Rightarrow\; V \in \mathcal{N}(x)$;
- $U, O \in \mathcal{N}(x) \;\Rightarrow\; U \cap O \in \mathcal{N}(x)$.

Completely Prime:

$$\cup_{i \in I} O_i \in \mathcal{N}(x) \Rightarrow\; (\exists i) O_i \in \mathcal{N}(x).$$

A *formal point* p is a completely prime filter of open sets. Question: Is a formal point p equal to the family of the open neighborhoods of a suitable point x? In general the answer is no. Consider the natural numbers with the topology generated by the sets $[n) = \{x : x \geq n\}$. Then the set $\{[n) : n \geq 0\}$ is only a formal point.

We denote by $pt(X)$ the set of formal points of the space X. We can define a topology on the set $pt(X)$ of formal points: a set U of formal points is open iff there exists an open set O in X such that $U = \{p \in pt(X) : O \in p\}$.

Define a function $f : X \to pt(X)$ by

$$f(x) = \{O : O \text{ open in } X \text{ and } x \in O\}.$$

Is f continuous? Yes. The inverse image of the above set U is just the open set O. Moreover, f is injective iff the space is T_0.

Definition 11.1 *The space X is called* sober *if f is bijective, i.e., if every formal point agrees with the family of the open neighborhoods of a unique point.*

The *soberification* of a space is the space of its formal points.

The complement $-p = \{U : U \text{ open}, U \notin p\}$ of a formal point p is a prime ideal. Because a topology is closed under arbitrary unions, then the prime ideal $-p$ is principal. The generator O of the prime ideal $-p$ is a prime element in the lattice τ of open sets, i.e.,

$$U \cap V \subseteq O \Rightarrow (U \subseteq O \text{ or } V \subseteq O).$$

The principal ideal of a prime element O in the lattice τ of open sets is prime, so that the complement is a formal point.

The complement $-O$ of a prime element $O \in \tau$ is closed and *irreducible*, i.e, the following condition holds for all closed sets C and D:

$$-O \subseteq C \cup D \Rightarrow \ (-O \subseteq C \text{ or } -O \subseteq D).$$

Example 75 *The prime ideal $-\mathcal{N}(x)$ of the open sets not containing x have a greatest element O (the set O is the union of all open sets non containing x). The complement of the greatest open set not containing x is the closure of the singleton set $\{x\}$. The closure x^- of $\{x\}$ is irreducible.*

Proposition 76 *A closed subset A of X is irreducible iff the induced topology on A is hyperconnected. (In particular, every subset of X directed w.r.t. the specialization order is irreducible).*

Proof. ($\Rightarrow$) If the induced topology is not hyperconnected, then there exist open sets U, V such that $U \cap A$ and $V \cap A$ are non empty and $U \cap V \cap A = \emptyset$. This means that $A \subseteq -U \cup -V$, where $-U$ and $-V$ are closed. Since A is irreducible, then we get either $A \subseteq -U$ or $A \subseteq -V$. This contradicts the hypothesis $U \cap A$ and $V \cap A$ non empty.

($\Leftarrow$) Let $A \subseteq C \cup D$ with C, D closed. Then $-C \cap -D \subseteq -A$, i.e., $-C \cap -D \cap A = \emptyset$. Since A is hyperconnected it follows, for example, that $-C \cap A = \emptyset$, that is, $A \subseteq C$. ■

Recall that the closure of an irreducible set is irreducible.

Proposition 77 *A space is sober iff every irreducible closed set is the closure of a unique point.*

Proof. ($\Rightarrow$) If A is closed and irreducible then $-A$ is open and prime. Then the principal ideal generated by $-A$ is prime and its complement is a formal point. Since the space is sober we are ok.

($\Leftarrow$) Easy. ■

Proposition 78 *Every sober space X is T_0. The specialization order of a sober space X makes X a dcpo. Moreover, If $D \subseteq X$ is directed, then $x = sup\ D$ is a limit point of the net $(x)_D$ such that $x_d = d$ for every $d \in D$.*

Proof. The specialization order of X is the relation $\subseteq$ between formal points: $p \leq q$ iff $O \in p \Rightarrow O \in q$, for all open sets of X iff $p \subseteq q$. Then the union of a directed family of formal points is a formal point. ■

Proposition 79 *Let* $(D, \leq)$ *be a poset and* τ *be a sober topology on* D*. Then* τ *induces* $\leq$ *iff upper* $\subseteq \tau \subseteq$ *Scott.*

Proof. Let O be open in D and $U = \{\text{formal points } p : O \in p\}$ be the corresponding open set in the space of formal points ■

Not every topology τ such that upper $\subseteq \tau \subseteq$ Scott is sober (See Chapter II Section 1.9 pag. 46 in Johnstones book: Stone Space).

11.1 Sobriety and nets

There exists a characterization of the sober topology in terms of nets (P. Sunderhauf, Sobriety in terms of nets, Applied Categorical Structures vol. 8 (2000)).

A net $(x)_I$ is *observative* if $x_i \in O \in \tau$ implies $(x)_I$ is eventually in O (i.e., a tail is in O).

An observative net is said to *strongly converge* to a point x (denoted by $x_i \to^* x$), if it converges to x and it satisfies the following condition:

$$x_i \in O \in \tau \;\Rightarrow\; x \in O.$$

A net is observative iff it converges to each of its points.

If a net $(x)_I$ is observative, then $x_i \to^* x$ iff $x_i \to x$ w.r.t. the topology (called *b-topology*) generated by the subbase constituted by the τ-open and τ-closed sets. (Remark: Is the b-topology equivalent to the following construction? The closure bB of B is defined as the set of elements p for which there do not exist open sets O_1 and O_2 with $p \in O_2 - O_1$ and $O_1 \cap B = O_2 \cap B$. Equivalently $p \in bB$ iff $p \in (B \cap p^-)^-$).

Let $\mathcal{F}_{(x)_I} = \{O \in \tau : \exists i \in I\ [x]_{\geq i} \in O\}$. The family of sets $\mathcal{F}_{(x)_I}$ is always a filter. When is it completely prime? The following lemma answers the above question.

Lemma 80 *Let* $(x)_I$ *be an observative net. Then*

1. *The filter* $\mathcal{F}_{(x)_I}$ *is completely prime.*
2. $x_i \to^* x$ *iff* $\mathcal{N}(x) = \mathcal{F}_{(x)_I}$.

It follows that an observative net cannot strongly converge to two points.

Theorem 81 *A space is sober iff every observative net strongly converges to a unique point.*

Proof. ($\Rightarrow$) By the above lemma.

($\Leftarrow$) Let p be a formal point. Then p is directed w.r.t. the partial ordering reverse to $\subseteq$. We define a net $(x)_p$ indexed by p as follows. Let $O \in p$. Then x_O is defined as an element of O such that $\mathcal{N}(x_O) \subseteq p$. This element must exist because otherwise we may find, for every element $y \in O$, an open set U such that $y \in U \notin p$. This contradicts that p is completely prime. In conclusion, the net $(x)_p$ exists. Is this net observative? Let U be an open set such that $x_V \in U$. Then $U \in p$ and the tail $[x]_{\geq U}$ is in U. By hypothesis the observative net $(x)_p$ strongly converges to a point z. Then $p = \mathcal{N}(z)$ and the space is sober. ■

12 Scott Topology

Topological spaces formalize the concept of "neighborhood". A subset X of the plane is closed if it contains all the limits of the Cauchy sequences included within X.

Definition 12.1 *Let D be a poset. $X \subseteq D$ is Scott closed if $X = X{\downarrow}$ and every directed subset of X has lub in X if this lub exists.*

Intuition: X is Scott closed if it contains all the existing limits (as lub) of directed sets included in X.

The Scott topology will be denoted by σ.

$O \subseteq D$ is Scott open if $O = O{\uparrow}$ and, for every directed subset I of D, if sup I exists and sup $I \in O$, then $I \cap O \neq \emptyset$.

Proposition 82 *Let D be a poset and a be an element of D. Then the set $a{\downarrow}$ is Scott closed. In particular, $\{\bot\}$ is Scott closed if D has a bottom element.*

A topology on a set X satisfies the separation axiom T_0 if, for all $a, b \in X$, there exists an open set O such that $O \cap \{a, b\}$ is a singleton set.

Let (X, τ) be a topological space. Define

$$a \leq_\tau b \text{ iff } (\forall O \in \tau)(a \in O \to b \in O).$$

The relation $\leq_\tau$ is a preorder (i.e., a reflexive and transitive relation) called the *specialization preorder* of τ. X is a T_0 topological space iff $\leq_\tau$ is a partial ordering.

Proposition 83 *Let $(D, \leq)$ be a poset.*

(i) The Scott topology σ on D is T_0 and it induces $\leq$ as its specialization order.

(ii) If $I \subseteq D$ is directed and sup I exists, then the net $(x)_I$, defined by $x_i = i$, converges to sup I w.r.t. the Scott topology.

Proof. (i) It is sufficient to show that $\leq_\sigma = \leq$. Let $a \leq b$ and O be a Scott open set such that $a \in O$. Since O is an upper set w.r.t. $\leq$, then $b \in O$. It follows that $a \leq_\sigma b$. Assume now $a \leq_\sigma b$. The set $B = \{x : x \not\leq b\}$ is Scott open by Prop. 82. By hypothesis a cannot be an element of B because b is not in B. Then it must be $a \leq b$.

(ii) Let O be a Scott open such that sup $I \in O$. Then there exists $d \in I$ such that $d \in O$. Since O is an upper set, then the tail $[x]_{\geq d}$ is in O. ■

Example 84 (Partial Functions)*: Let f be a partial function. Then $f{\uparrow}$ is Scott open if, and only if, the (domain of the) function f is finite. It is easy to show that the upper sets $f{\uparrow}$ with f finite generate the Scott topology in the cpo of partial functions from N into N.*

(Power Set)*: Let B be a subset of a set A. Then $B{\uparrow}$ is Scott open if, and only if, the set B is finite. It is easy to show that the upper sets $B{\uparrow}$ with B finite generate the Scott topology in the powerset of A.*

(Real Line) *Every subset of reals is a directed set. Is an open interval (a, b) Scott open? No, because it is not an upper set. Then Scott open sets are the intervals (a, ∞). To understand the difference with the usual Euclidean topology, consider a net $(x)_I$ and a real a. If $x_i \geq a$ for* *all $i \in I$, then $x_i \to a$ w.r.t. Scott topology. If $x_i \leq a$ for all $i \in I$, then $x_i \to a$ w.r.t. Scott topology iff $x_i \to a$ w.r.t. Euclidean topology. Then a sequence $(x)_\omega$ Scott converges to a iff the subsequence of the elements $x_i \leq a$ converges to a w.r.t. Euclidean topology.*

12.1 Scott topology on continuous posets

Define $d{\Uparrow} = \{x : d \ll x\}$.

Lemma 85 *Let X be a continuous poset and D be a direct subset of X for which there exists sup D. If $d \ll$ sup D, then there exists $x \in D$ such that $d \ll x$.*

Proof. Define $D' = \{y \ : \ (y \ll x) \wedge (x \in D)\}$. Then, the following properties hold:

Claim 86 *D' is an ideal.*

D' is trivially a lower set.

Claim 87 *sup $D =$ sup D'.*

If m is an upper bound of D, then it is an upper bound of D' by construction of D'. Suppose that m is an upper bound of D', then for all $x \in D$, m is an upper bound of the set $x{\Downarrow}$, hence $m \geq$ sup $x{\Downarrow}$. By continuity of X sup $x{\Downarrow} = x$, then $m \geq x$, i.e., m is also an upper bound of D. D and D' have the same upper bounds. Then sup $D =$ sup D'. This conclude the proof of the claim.

Let now $d \ll$ sup D. From Claim 86 we know that D' is an ideal, then it holds $d \leq y \in D'$, hence $d \in D'$. ■

Theorem 88 *Let X be a continuous poset. Then it holds:*

1. *The family of sets $(d{\Uparrow} : d \in X)$ constitutes a basis for the Scott topology.*
2. *The Scott topology is locally compact (and then core-compact).*

Proof. (1) We first show that $d{\Uparrow}$ is a Scott open set.

Claim 89 *$d{\Uparrow}$ is a upper set.*

Let x be an element of $d{\Uparrow}$, then $d \ll x$ by definition of $d{\Uparrow}$. By applying the Proposition 7(3) $\forall y \in X$ such that $x \leq y$ it holds $y \in d{\Uparrow}$, then $d{\Uparrow}$ is an upper set.

Claim 90 *Let D be a directed subset of X. If sup D exists and sup $D \in d{\Uparrow}$, then $D \cap d{\Uparrow} \neq \emptyset$.*

By Lemma 85.

Claim 91 *Let U be a Scott open set. Then $U = \bigcup_{d \in U} d{\Uparrow}$.*

Let x be an element of U. By the continuity of the poset X it follows that $x = \sup x{\Uparrow}$. By Claim 2 there exists an element $d \in U$ such that $d \ll x$. Then $x \in d{\Uparrow} \subseteq d{\uparrow} \subseteq U$. Hence, $U = \bigcup_{d \in U} d{\Uparrow}$.

(2) For every x, the set $x{\uparrow} = \{y : y \geq x\}$ is compact. Moreover, $x{\uparrow}$ is a neighborhood of every element y such that $x \ll y$. Let $y \in D$ and U an open neighborhood of y. We have to find a compact neighborhood Q of y such that $Q \subseteq U$. Since U is open then by (1) we have that $U = \cup_{d \in U} d{\Uparrow}$. This means that there exists $d \in U$ such that $d \ll y$. But U is an upper set, then $d{\uparrow} \subseteq U$. In conclusion, $d{\uparrow}$ is compact, $d{\uparrow} \subseteq U$ and $d{\uparrow}$ is a neighborhood of y because $y \in d{\Uparrow} \subseteq d{\uparrow}$. ■

Theorem 92 *The Scott topology is sober for every continuous dcpo.*

Proof. Let p be a formal point. Then the set $U_p = \{d \in D : d{\Uparrow} \in p\}$ is directed. Let x be the least upper bound of U_p. We show that $U_p = \{O : x \in O \in \sigma\}$. If $O \in p$, then $O = \cup_{e_i \in O} e_i{\Uparrow}$. Then from the complete primality of p there exists i such that $e_i{\Uparrow} \in p$. Then $x \in O$ since $e_i \leq x$. ■

Example 93 *We provide an example (Exercise II.1.36 Compendium) of dcpo which is not sober for its Scott topology. Let* $A = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ *with the partial order*

$$(m, n) \leq (m', n') \text{ iff either } m = m' \text{ and } n \leq n' \leq \infty \text{ or } n' = \infty \text{ and } n \leq m'.$$

Then A is a dcpo, the elements (m, ∞) are maximal, the Scott topology on A is hyperconnected (i.e., A is irredducible), but A is not the closure of a unique point.