Lattices of Equational Theories as Church Algebras

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A Question

- Algebras from computation
  - Lambda abstraction algebras (LAA)
  - Combinatory algebras (CA)

- Algebras from logic and other fields
  - Boolean algebras (BA)
  - Heyting algebras (HA)
  - Rings with unit

- Do these algebras share interesting algebraic properties?

- How do the LATTICES of EQUATIONAL THEORIES relate to these algebras?

Church Algebras

- Do these algebras share interesting algebraic properties?

  YES

  because they are all *Church algebras*, i.e., algebras with two constants $0, 1$ and a ternary term "if" formalizing the "if-then-else" construct of programming languages

  $$\mathrm{if}(1, x, y) = x; \quad \mathrm{if}(0, x, y) = y$$

- The lattices of equational theories are congruence lattices of Church algebras

Lattices of Equational Theories

- An *equational theory* is a set of identities closed under deduction
- Birkhoff and Malcev asked to characterize which lattices are isomorphic to *lattices of equational theories* (et-lattices), i.e., lattices isomorphic to

$$L(\Sigma) = \{T : \Sigma \subseteq T, T \text{ equational theory}\}$$

for a given equational theory Σ.

- Every et-lattice is an algebraic lattice with compact unit, but the converse is false.
- (Lampe 86) Every et-lattice satisfies the Zipper condition:

$$\vee_{i\in I} a_i = 1 \text{ and } x \wedge a_i = y \ (i \in I) \Rightarrow x = y.$$

ET-Lattices as Congruence lattices of Church Algebras

- Let τ be an algebraic similarity type and $X = \{x_0, x_1, \dots\}$ be a countable set of variables.

  $$L(\tau) \cong \mathrm{Con}(Te(X), \tau \cup \mathrm{End})$$

  Let Σ be an equational theory. Then

  $$L(\Sigma) \cong \mathrm{Con}(Te(X)/\Sigma, \tau \cup \mathrm{End})$$

- (Newrly 93): $L(\tau) \cong \mathrm{Con}(Te(X), +, 0, pred)$

  $$s + t \equiv t[s/x_0]; \quad 0 \equiv x_0; \quad pred(x_i) = x_{i-1}; \quad pred(x_0) = x_0$$

- We modify Newrly's algebra without changing its congruence lattice. The Church algebra $\mathbf{Ch}_\tau$ of type τ is defined as follows:

  $$L(\tau) \cong \mathsf{Con}(Te(X), \mathsf{if}, 0, 1, pred)$$

  where

  $$\mathsf{if}(t, s, u) \equiv t[u/x_0, s/x_1]; \quad 0 \equiv x_0; \quad 1 \equiv x_1$$

  If Σ is an equational theory of type τ then $\mathbf{Ch}_\Sigma$ is the Σ-Church algebra:

  $$L(\Sigma) \cong \mathsf{Con}(Te(X)/\Sigma, \mathsf{if}, 0, 1, pred)$$

Stone's Theorem: from Boole to Church

Theorem 1 *(Stone Representation Theorem) Every Boolean algebra is isomorphic to a Boolean product of indecomposable Boolean algebras.*

Generalization:

- Pierce: Every ring with unit is isomorphic to a Boolean product of indecomposable rings.
- Vaggione, Comer: Generalization to other classes of algebras.
- Manzonetto-S.: Combinatory algebras (CA) and λ-abstraction algebras (LAA) satisfy an analogous theorem...

Central Elements

- Every element a in a BA $\mathbf{A}$ (resp. every idempotent element in a commutative ring with unit) induce a pair of complementary factor congruences $\theta(1,a)$, $\theta(a,0)$:

$$\mathbf{A} \cong \mathbf{A}/\theta(1,a) \times \mathbf{A}/\theta(a,0)$$

Proposition 1 *Let $\mathbf{A}$ be a Church algebra of type τ. The following conditions are equivalent for $e \in A$:*

(i) e *is central, i.e.,* $\mathbf{A} \cong \mathbf{A}/\theta(1,e) \times \mathbf{A}/\theta(e,0)$

(ii) $\mathsf{if}(e,x,x) = x$

$\mathsf{if}(e,\mathsf{if}(e,x,y),z) = \mathsf{if}(e,x,z) = \mathsf{if}(e,x,\mathsf{if}(e,y,z))$

$\mathsf{if}(e,g(\overline{x}),g(\overline{y})) = g(\mathsf{if}(e,x_1,y_1),\dots,\mathsf{if}(e,x_n,y_n))$, *for all* $g \in \tau$

$e = \mathsf{if}(e,1,0)$.

The Stone Representation Theorem for Church Algebras

Theorem 2 *Let* $\mathbf{A}$ *be a Church algebra. The algebra Ce*$\mathbf{A}$ *of central elements of* $\mathbf{A}$*, with operations defined by*

$$x \wedge y = \mathsf{if}(x, y, 0),\ \ x \vee y = \mathsf{if}(x, 1, y),\ \ x^{-} = \mathsf{if}(x, 0, 1),$$

is a BA *isomorphic to the* BA *of factor congruences of* $\mathbf{A}$*.*

If I is an ideal of the Boolean algebra Ce$\mathbf{A}$, then $\phi_I = \cup_{e \in I} \theta(e, 0)$ is a congruence. In the next theorem $\mathcal{S}$ is the Boolean space of maximal ideals of Ce$\mathbf{A}$.

Theorem 3 *Let* $\mathbf{A}$ *be a Church algebra. Then the map*

$$f : A \to \Pi_{I \in \mathcal{S}} (A/\phi_I),$$

defined by $f(x) = ([x]_{\phi_I} : I \in \mathcal{S})$*, gives a weak Boolean product representation of* $\mathbf{A}$ *by directly indecomposable algebras* $\mathbf{A}/\phi_I$*.*

Church applied to et-lattices

Theorem 4 *Let Σ be an equational theory. Then the following conditions are equivalent, for every $e \in \mathbf{Ch}_\Sigma$ and term $t(x_1, x_0) \in e$:*

- (i) *e is a central element.*
- (ii) *Σ contains the identities*
  $t(x,x) = x; \quad t(x, t(y,z)) = t(x,z) = t(t(x,y),z);$
  $t(g(\overline{x}), g(\overline{y})) = g(t(x_1,y_1), \dots, t(x_n,y_n))$*, for $g \in \tau$.*
- (iii) *$\Sigma = \Sigma_0 \cap \Sigma_1$, where Σ_i is the theory axiomatized (over Σ) by $t(x_1, x_0) = x_i$.*
- (iv) *For every $\mathbf{A} \in \mathcal{V}(\Sigma)$, the function $t^{\mathbf{A}} : A \times A \to A$ is a decomposition operator, so that*

$$\mathbf{A} = \mathbf{A}_0 \times \mathbf{A}_1, \quad \text{with } \mathbf{A}_0 \in \mathcal{V}(\Sigma_0) \text{ and } \mathbf{A}_1 \in \mathcal{V}(\Sigma_1)$$

$\mathcal{V}(\Sigma)$ is decomposable as a product of $\mathcal{V}(\Sigma_0)$ and $\mathcal{V}(\Sigma_1)$

Church applied to et-lattices

Proposition 2 *Let* Σ *be an equational theory. For every algebra* $\mathbf{A} \in \mathcal{V}(\Sigma)$*, the function* $h : Ce(\mathbf{Ch}_\Sigma) \to Con\mathbf{A}$*, defined by*

$$h(e) = \{(x, y) : t^{\mathbf{A}}(x, y) = x,\ t(x, y) \in e\},$$

is a lattice homomorphism such that $(h(e), h(e^-))$ *is a pair of complementary factor congruences. The range of* h *constitutes a Boolean sublattice of Con*$\mathbf{A}$*.*

We say that a variety $\mathcal{V}$ is *decomposable as a weak Boolean product of directly indecomposable subvarieties* if there exists a family $\langle \mathcal{V}_i : i \in X \rangle$ of indecomposable subvarieties $\mathcal{V}_i$ of $\mathcal{V}$ such that every algebra $\mathbf{A} \in \mathcal{V}$ is isomorphic to a weak Boolean product $\Pi_{i \in X} \mathbf{B}_i$ of algebras $\mathbf{B}_i \in \mathcal{V}_i$.

Theorem 5 (Stone Meta-Representation Theorem) *Every variety $\mathcal{V}$ of τ-algebras is decomposable as a weak Boolean product of directly indecomposable subvarieties.*

By using the weak Boolean product representation

$$f : \mathbf{Ch}_{\mathbf{\Sigma}} \to \Pi_{I \in \mathcal{S}}(\mathbf{Ch}_{\mathbf{\Sigma}}/\phi_I)$$

with directly indecomposable algebras $\mathbf{Ch}_{\mathbf{\Sigma}}/\phi_I \cong \mathbf{Ch}_{\mathbf{\Sigma}_I}$ for a suitable equational theory $\mathbf{\Sigma}_I$.

From easy λ-terms to easy sets in Church algebras

Let $\mathbf{A}$ be a Church algebra. $a \subseteq_{fin} A$ is *an easy set* if

$$\forall(b \subseteq a)\ \theta(1,b) \vee \theta(a-b,0) \text{ consistent}$$

Theorem 6 *Let $\mathbf{A}$ be a Church algebra and a be an easy set. Then there exists a congruence ϕ such that the principal filter $\phi\!\uparrow$ is isomorphic to the free Boolean lattice with a generators.*

Corollary 1 *Let Σ be an equational theory whose Church algebra has an easy set of cardinality n. Then the variety $\mathcal{V}(\Sigma)$ contains a subvariety $\mathcal{W}$ whose lattice of subvarieties is a Boolean lattice with 2^{2^n} elements.*

Conclusion and open problems

- Is Con$\mathbf{A}$ an et-lattice for every Church algebra $\mathbf{A}$?

- Find an implication in the language of bounded lattices which holds in the congruence lattices of all Church algebras but it does not hold in the congruence lattices of all groupoids with left unit and left zero.

- (Lampe) Is every finite lattice satisfying the Zipper condition an et-lattice?

- Bounded lattices have a quaternary term operation q satisfying

$$q(1,0,x,y) = x; \quad q(0,1,x,y) = y$$

When are bounded lattices Church algebras? And residuated lattices?

- Let $\mathbf{Ch}_\Sigma$ be the Σ-Church algebra. The following term operations on $\mathbf{Ch}_\Sigma$ (not just on $\mathrm{Ce}(\mathbf{Ch}_\Sigma)$)

$$x \wedge y = \mathsf{if}(x,y,0),\ x \vee y = \mathsf{if}(x,1,y),\ x^- = \mathsf{if}(x,0,1),$$

are two monoids satisfying:

$$(x^-)^- = x; \quad x \wedge y = (x^- \vee y^-)^-; \quad x \vee y = (x^- \wedge y^-)^-$$

$$x \to y \equiv (x \wedge y^-)^-$$

Is the variety of Church algebras satisfying the above identities useful for et-lattice?

- (Nurakunov 08) A lattice L is an et-lattice if, and only if, $L \cong \mathrm{Con}\mathbf{A}$ for some et-monoid $\mathbf{A}$ (a monoid with two more unary operations satisfying some non-first-order conditions).