Lambda Abstraction Algebras: Coordinatizing Models of Lambda Calculus*

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Abstract. Lambda abstraction algebras are designed to algebraize the untyped lambda calculus in the same way cylindric and polyadic algebras algebraize the first-order logic; they are intended as an alternative to combinatory algebras in this regard. Like combinatory algebras they can be defined by true identities and thus form a variety in the sense of universal algebra. One feature of lambda abstraction algebras that sets them apart from combinatory algebras is the way variables in the lambda calculus are abstracted; this provides each lambda abstraction algebra with an implicit coordinate system. Another peculiar feature is the algebraic reformulation of (β)-conversion as the definition of abstract substitution. Functional lambda abstraction algebras arise as the “coordinatizations” of environment models or lambda models, the natural combinatory models of the lambda calculus. As in the case of cylindric and polyadic algebras, questions of the functional representation of various subclasses of lambda abstraction algebras are an important part of the theory. The main result of the paper is a stronger version of the fundamental representation theorem for locally finite lambda abstraction algebras, the algebraic analogue of the completeness theorem of lambda calculus. This result is used to study the connection between the combinatory models of the lambda calculus and lambda abstraction algebras. Two significant results of this kind are the existence of a strong categorical equivalence between lambda algebras and locally finite lambda abstraction algebras, and between lambda models and rich, locally finite lambda abstraction algebras.

Keywords: lambda calculus, cylindric algebras, polyadic algebras, abstract substitution, combinatory algebras, lambda algebras, lambda models, representation theorems

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1. Introduction

Lambda abstraction algebras\(^1\) constitute a purely algebraic theory of the untyped lambda calculus in the same spirit that Boolean algebras constitute an algebraic theory of classical propositional logic and, more to the point, cylindric and polyadic (Boolean) algebras of the first-order predicate logic. Although the untyped lambda calculus is formalized as a theory of equations, it is not an equational theory in the normal sense because the equations, unlike the associative and commutative laws of group theory for example, are not always preserved when arbitrary terms are substituted for variables. Consequently the general methods that have been developed in universal algebra and category theory, for defining the semantics of an arbitrary algebraic theory for instance, are not directly applicable. There have been several previous attempts to reformulate the lambda calculus as a purely algebraic theory. The earliest, and best known, are the combinatory algebras of Curry.\(^2\) Combinatory algebras have a simple, pure equational characterization. Curry also specified (by a considerably less natural set of axioms) a pure equational subclass of combinatory algebras, the \(\lambda\)-algebras (see [3, 5.2.5]), that he viewed as algebraic models of the lambda calculus. It was later discovered that the combinatory algebras most closely connected to the intuitive functional models of the lambda calculus have an intrinsic characterization (up to isomorphism) as a special class of \(\lambda\)-algebras called lambda models ([3, 5.2.7]). They were first axiomatized by Meyer [23] and independently by Scott [43]; the first-order axiomatization while elegant is not equational. Lambda models are the same (up to isomorphism) as environment models. These are high-order structures consisting of sets of functions in which \(\lambda\)-terms have their natural interpretation. The notion of an environment model originated with Hindley and Longo [20] and was developed by Meyer [23], who describes them as “the natural, most general formulation of what might be meant by mathematical models of the untyped lambda calculus”.

Combinatory algebras (CA’s) and lambda abstraction algebras (LAA’s) are both defined by universally quantified equations and thus form varieties in the universal algebraic sense. There are important differences however that result in theories of very different character. Functional application is taken as a fundamental operation in both CA’s and LAA’s. Lambda (i.e., functional) abstraction is also fundamental in LAA’s but in CA’s is defined in terms of the combinators \(k\) and \(s\). A more important difference is connected with the role variables play in the lambda calculus as place holders. In a LAA this is also abstracted. It takes the form of a system of fundamental elements (nullary operations) of the algebra which can be viewed as specifying an implicit coordinate system, and a LAA may be thought of as the “coordinatization” of a CA with respect to this system. This is a crucial feature of LAA’s that is borrowed from cylindric and polyadic algebras and has no direct analogue in CA’s. One important consequence of the abstraction of variables is the abstraction of term-for-variable substitution in LAA’s by inverting \((\beta)\)-conversion in a natural way. In contrast, in CA’s substitution is simulated by application of special combinators.

The natural LAA’s, the ones the axioms were intended to characterize, are algebras of functions. Functional lambda abstraction algebras (FLA’s, for short) arise as the “coordinatizations” of environment models or lambda models and consist of multiple-argument functions. It is natural to conjecture that FLA’s constitute the entire class of LAA’s in the sense that every LAA is isomorphic to a FLA. The conjecture has been recently verified by Salibra and Goldblatt in [40]. Partial results in this direction were obtained by the present authors in a series of papers [29, 30, 31, 32]. The algebraic analogue of the completeness theorem of the lambda calculus ([23]), namely that every lambda theory consists of precisely

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\(^1\)Essentially the same notion was discovered independently by Diskin [9] under the influence of ideas from algebraic logic.

\(^2\)More recently, several purely algebraic theories of the lambda calculus within the context of category theory have been developed; these are discussed at the end of the paper in Sec. 6.
the equations valid in some environment model, is the functional representation theorem for locally finite LAA’s [29]; a LAA is locally finite if each element depends only on a finite number of coordinates.

The functional representation of LAA’s is also connected with the so-called “finitization problem” of algebraic logic. There are various representations of first-order predicate logic as a variety of algebras of sets of sequences of elements, over some domain, in which terms have their natural interpretation with respect to the model-theoretic meaning of the logical expressions they are intended to represent. Some examples are the varieties of representable cylindric algebras and representable polyadic algebras with equality, as algebras of sets of \( \omega \) sequences, and representable relation algebras, as algebras of binary relations. These varieties all fail to be finitely axiomatizable. The problem is whether there exists a choice of operations, that are logical in the sense of Tarski, by means of which first-order logic can be represented by a finitely axiomatizable variety of algebras of sets of sequences. For a long time negative results dominated this field. Now there seems to be a convergence of positive results (see [35, 36, 37, 25, 26, 27, 40]).

In this paper we study the connections between the combinatory models of the lambda calculus and lambda abstraction algebras. Two significant results of this kind are the existence of a strong categorical equivalence between \( \lambda \)-algebras and locally finite LAA’s (Thm. 3.2), and between lambda models and rich, locally finite LAA’s (Cor. 4.1 and Thm. 6.4), where a LAA is rich if it satisfies the algebraic analogue of the term rule of the lambda calculus. We show that a suitable combinatory reduct of a rich, locally finite LAA is a lambda model and, conversely, that every lambda model can be obtained this way (Thm. 4.2). It turns out that the locally finite LAA’s are in essentially the same relation to \( \lambda \)-algebras as rich, locally finite LAA’s are to lambda models. We also prove that a \( \lambda \)-algebra \( C \) is a lambda model if and only if the free extension of \( C \) in the variety of combinatory algebras is equal to the free extension of \( C \) in the variety generated by \( C \).

An even stronger connection can be established between lambda models and rich, locally finite LAA’s than the one described above; it ties the functional character of the two kinds of algebras more closely together by establishing a categorical equivalence between the functional categories they abstract. A functional LAA is functionally rich if the environment model it coordinatizes is isomorphic to a suitable reduct of it. A surjective homomorphism between two functional LAA’s is a domain homomorphism if it is induced in a natural way by a surjective mapping between the environment models that the two LAA’s coordinatize. We show that every surjective homomorphism between functionally rich locally finite FLA’s is a domain homomorphism (Thm. 6.3). We use this to establish an equivalence between the category of environment models and domain morphisms, and the category of functionally rich locally finite FLA’s and surjective homomorphisms.

The main tool of our investigations is the strong representation theorem for locally finite LAA’s, which asserts that every locally finite LAA can be isomorphically embedded in a functionally rich LAA of functions (Thm. 5.2). We also obtain a representation theorem for simple LAA’s (Thm. 6.5). It corresponds roughly to the main representation theorems for polyadic Boolean algebras and cylindric algebras, which can be viewed as the natural algebraic analogues of completeness theorem for first-order logic.

LAA’s prove useful in the lambda calculus as a framework for applying the methods of universal algebra. We expect that their use will enrich the theory in the same way Boolean algebras, Heyting algebras, and modal algebras have enriched propositional and modal logic. A LAA can be associated with each lambda theory \( T \); its elements are sets of \( \lambda \)-terms that are pairwise equivalent under \( T \). We call this the term LAA of \( T \). The fact that each \( \lambda \)-term contains only a finite number of variables is reflected in the fact that the term LAA of every theory is locally finite. This connection with locally finite LAA’s provides new insight into lambda theories. For example, as corollary of the strong functional representation theorem for locally finite LAA’s (Thm. 5.2), we can show that every lambda theory can be
conservatively extended to a lambda theory closed under the term rule (Thm. 5.3). In this connection see also Diskin [9, 10].

Every lambda theory is uniquely determined by its restriction to closed \( \lambda \)-terms (Cor. 3.3); this is a consequence of the categorical equivalence between \( \lambda \)-algebras and locally finite \( \text{LAA} \)'s mentioned previously. A \( \lambda \)-algebra is (isomorphic to) the term \( \lambda \)-algebra of the closed theory of exactly one lambda theory (Cor. 3.4 and Thm. 3.2), while a \( \lambda \)-model is the term \( \lambda \)-algebra of the closed theory of exactly one lambda theory closed under the term rule of the lambda calculus (Cor. 4.1 and Cor. 4.4).

1.1. Outline of paper

In the second section of this paper we review the basic definitions of the lambda calculus and summarize all definitions and results from [32] that will be needed in the subsequent part of the paper; in particular, we recall the formal definition of lambda abstraction algebras and the theory of abstract substitution.

The relationship between \( \text{LAA} \)'s, combinatory algebras and \( \lambda \)-algebras is investigated in detail in Sec. 3. We prove that suitable subreducts of \( \text{LAA} \)'s, those whose universes consist of the elements invariant under substitution, coincide with \( \lambda \)-algebras (see Curry [8], Meyer [23]) and thus form a variety. As a consequence of this result, the category of \( \lambda \)-algebras and the category of locally finite \( \text{LAA} \)'s are shown to be equivalent in a rather strong sense.

In Sec. 4 lambda models (Meyer [23]) are characterized as suitable reducts of rich, locally finite \( \text{LAA} \)'s. As a consequence of this result, the category of lambda models and the category of rich, locally finite \( \text{LAA} \)'s are shown to be equivalent.

The main representation results are presented in Sec. 5. We prove that every rich, locally finite \( \text{LAA} \) is isomorphic to a functionally rich \( \text{LAA} \) of functions. This result serves as a basis for the main result of the paper, the strong representation theorem, according to which every locally finite \( \text{LAA} \) is isomorphically embeddable in a functionally rich, locally finite \( \text{LAA} \) of functions. As a consequence of this result, we also obtain a new proof of the equivalence of environmental and lambda models. We conclude the section with a theorem characterizing dimension-complemented \( \text{LAA} \)'s in terms of environment models.

In Sec. 6 we use the representation results in Sec. 5 to obtain the categorical equivalence of environmental models and functionally rich, locally finite \( \text{LAA} \)'s of functions. A representation theorem for simple, locally finite \( \text{LAA} \)'s is also proved in this section. This result corresponds roughly to the stronger version of the representation theorem for polyadic Boolean algebras ([16, (17.3)]) and to the representation theorem for locally finite cylindric algebras ([18, Part II, Thm. 3.2.8]). These latter two representation results are the natural algebraic analogues of the completeness theorem for first-order logic.

At the end of the paper, in Sec. 7, we discuss some of the results in [32, 40] which have consequences for the functional representation of \( \text{LAA} \)'s.

2. Basic Notions and Notations

To keep this article self-contained, we summarize definitions and results from [32] that we will need in the subsequent part of the paper. With regard to the lambda calculus we generally follow the notation and terminology of Barendregt (see [3]).

2.1. Lambda abstraction algebras

Let \( I \) be a nonempty set. The similarity type of lambda abstraction algebras of dimension \( I \) is 
\[ \langle \cdot, \langle \lambda x : x \in I \rangle, \langle x : x \in I \rangle \rangle, \]
where \( \cdot \) is a binary operation symbol formalizing application, \( \lambda x \) is a unary operation symbol for every \( x \in I \), and \( x \) is a constant symbol (i.e., nullary
operation symbol) for every \( x \in I \). Note that \( \lambda x \) is to be viewed as an indivisible complex symbol; alternatively, \( \{ \lambda x : x \in I \} \) can be viewed as a system of unary operations indexed by elements of \( I \). Similarly, \( \{ x : x \in I \} \) can be viewed as system of constants indexed by \( I \).

The elements of \( I \) are to be thought of as the variables of lambda calculus although in their algebraic transformation they no longer play the role of variables in the usual sense. We will refer to them as \( \lambda \)-variables. The actual variables of the lambda abstraction theory will be referred to as context variables and denoted by the greek letters \( \xi, \upsilon, \) and \( \mu \), possibly with subscripts. The terms of the language of lambda abstraction theory are called \( \lambda \)-contexts. They are constructed in the usual way: every \( \lambda \)-variable \( x \) and context variable \( \xi \) is a \( \lambda \)-context; if \( t \) and \( s \) are \( \lambda \)-contexts, then so are \( t \cdot s \) and \( \lambda x(t) \). Because of the similarity between \( \lambda \)-contexts and terms of the lambda calculus we use the standard notational conventions of the latter. The application operation symbol \( "\cdot" \) is normally omitted, and the application of \( t \) and \( s \) is written as juxtaposition \( t \cdot s \). When parentheses are omitted, association to the left is assumed. The left parenthesis delimiting the scope of a lambda abstraction is replaced with a period and the right parenthesis is omitted. For example, \( \lambda x(t) \) is written \( \lambda x.t \). Successive \( \lambda \)-abstractions \( \lambda x \lambda y \lambda z \cdots \) are written \( \lambda x y z \cdots \).

A word of caution for those readers familiar with the lambda calculus. When dealing with models of the lambda calculus one often allows terms that contain constant symbols representing the elements of the models. These constants should not be confused with context variables; they play a much different role. Our notion of a \( \lambda \)-context coincides with the notion of context defined in Barendregt [3, Def. 14.4.1]; our context variables correspond to Barendregt’s notion of a “hole”. For example, the \( \lambda \)-context \( \{ \lambda x.\xi \mu \} y \) corresponds to Barendregt’s context \( (\lambda x[\xi])_y \).

An occurrence of a \( \lambda \)-variable \( x \) in a \( \lambda \)-context is bound if it falls within the scope of the operation symbol \( \lambda x \); otherwise it is free. The free \( \lambda \)-variables of a \( \lambda \)-context are the \( \lambda \)-variables that have at least one free occurrence. A \( \lambda \)-context without any context variables is said to be a \( \lambda \)-term. A \( \lambda \)-context without free \( \lambda \)-variables is said to be closed. Note that \( \lambda \)-terms of lambda abstraction theory coincide with ordinary terms of the lambda calculus.

We now give the formal definition of lambda abstraction algebras. Readers unfamiliar with the notation of the lambda calculus may want to go directly to the reformulation of the axioms in terms of the substitution operations that is given later.

**Definition 2.1.** By a lambda abstraction algebra of dimension \( I \) we mean an algebraic structure of the form

\[
A := \langle A, \cdot^A, \{ \lambda x^A : x \in I \}, \{ x^A : x \in I \} \rangle
\]

satisfying the following quasi-identities for all \( x, y, z \in I \) (subject to the indicated conditions) and all \( \xi, \mu, \nu \in A \).

\[
\begin{align*}
(\beta_1) \quad (\lambda x.x)\xi &= \xi; \\
(\beta_2) \quad (\lambda x.y)\xi &= y, \quad x \neq y; \\
(\beta_3) \quad (\lambda x.\xi)x &= \xi; \\
(\beta_4) \quad (\lambda x.\xi)\mu &= \lambda x.\xi; \\
(\beta_5) \quad (\lambda x.\xi)\nu &= (\lambda x.\xi)\nu((\lambda x.\mu)\nu); \\
(\beta_6) \quad (\lambda y.\mu)z &= \mu \rightarrow (\lambda xy.x)\xi = \lambda y.(\lambda x.\xi)\mu, \quad x \neq y, z \neq y; \\
(\alpha) \quad (\lambda y.\xi)z &= \xi \rightarrow \lambda x.\xi = \lambda y.(\lambda x.\xi)\mu, \quad z \neq y.
\end{align*}
\]

\( I \) is called the dimension set of \( A \). \( \cdot^A \) is called application and \( \lambda x^A \) is called lambda abstraction with respect to \( x \).

The class of lambda abstraction algebras of dimension \( I \) is denoted by \( \text{LAA}_I \) and the class of all lambda abstraction algebras of any dimension by \( \text{LAA} \). We also use \( \text{LAA}_I \) as shorthand for the phrase "lambda abstraction algebra of dimension \( I \)". An \( \text{LAA}_I \) is infinite dimensional if \( I \) is infinite.
In the sequel $A$ will be an arbitrary LAA$_I$, unless otherwise noted. The dimension set $I$ is arbitrary, in particular it can be finite unless otherwise specified. We assume however that it contains at least three variables, since many of the results we obtain in this section require this.

We will omit the superscript $A$ on $\cdot^A$, $\lambda x^A$, and $x^A$ whenever we are sure we can do so without confusion. This will also apply to defined notions introduced below, such as $\Delta^A$.

In the presence of the other axioms, $(\beta_0)$ and $(\alpha)$ are equivalent to identities

$$(\beta_0') \quad (\lambda y. \xi)((\lambda y. \mu) z) = \lambda y. (\lambda x. \xi)((\lambda y. \mu) z), \quad x \neq y, z \neq y.$$

$$(\alpha') \quad \lambda x. (\lambda y. \xi) z = \lambda y. (\lambda x. (\lambda y. \xi) z) y, \quad z \neq y.$$

Thus LAA$_I$ is a variety for every dimension set $I$.

We note here one very useful immediate consequence of the axioms: in any LAA $A$ the functions $\lambda x$ are always one-one, i.e., for all $x \in I$,

$$\lambda x. a = \lambda x. b \text{ iff } a = b, \quad \text{for all } a, b \in A.$$  

For if $\lambda x. a = \lambda x. b$, then by $(\beta_3)$, $a = (\lambda x. a) x = (\lambda x. b) x = b$.

A LAA with only one element is said to be trivial. It is interesting that there do not exist nontrivial finite models. In fact, any nontrivial LAA$_I$ of positive dimension is infinite, since the one-one map $\lambda x$ is not onto. To see this, assume by way of contradiction that $x$ is in the range of $\lambda x$; then $x = \lambda x. b$ for some element $b \in A$. Since $A$ is nontrivial, there exists an element $a \in A$ such that $a \neq x$. Then a contradiction results from the chain of equalities: $a = (\lambda x. x)a = (\lambda x. x)b = \lambda x. b = x$.

### 2.2. Substitution and dimension

When transformed into the equational language of lambda abstraction theory, $(\beta)$-conversion becomes the definition of abstract substitution. It takes the following form: For any set $A$, let $A^*$ be the set of all finite strings of elements of $A$.

**Definition 2.2.** Let $A$ be a LAA$_I$.

1. $S^\xi(a) = (\lambda x.a)b$ for all $x \in I$ and $a, b \in A$.
2. $S^\xi(a) = S^\xi_1(\cdots(S^\xi_{b_1}(a))\cdots)$ for all $x = x_1 \cdots x_n \in I^*, b = b_1 \cdots b_n \in A^*$, and $a \in A$.

$S$ is called the (abstract) substitution operator, and $S^\xi(a)$ may be thought of as “$a$ with $b$ substituted for free $x$”.

**Definition 2.3.** Let $A$ be a LAA$_I$. Let $a \in A$ and $x \in I$. $a$ is said to be algebraically dependent on $x$ (over $A$) if $S^\xi(a) \neq a$ for some $z \in I$; otherwise $a$ is algebraically independent of $x$ (over $A$). The set of all $x \in I$ such that $a$ is algebraically dependent on $x$ over $A$ is called the dimension set of $a$ and is denoted by $\Delta^A a$; thus

$$\Delta^A a = \{ x \in I : S^\xi(a) \neq a \text{ for some } z \in I \}.$$  

$a$ is finite (infinite) dimensional if $\Delta a$ is finite (infinite). An element $a$ is called zero-dimensional if $\Delta a = \emptyset$. We denote the set of zero-dimensional elements by $Zd A$.

It is convenient to treat algebraic dependency as a symmetric relation and speak of “$x$ being algebraically dependent on (independent of) $a$”. The following are two useful alternative characterizations of algebraic dependency and consequently of dimension set ([32, Lem. 1.6]): $x \notin \Delta a$ iff $S^\xi(a) = a$ for some $z \in I \setminus \{x\}$ iff $S^\xi(a) = a$ for all $b \in A$.

It is obvious that the axioms for lambda abstraction algebras can be reformulated in the following way:

$$(\beta_1) \quad S^\xi(x) = \xi;$$
\[(\beta_1) \ S_{\xi}(y) = y, \ y \neq x; \]
\[(\beta_3) \ S_{\xi}(\xi) = \xi; \]
\[(\beta_4) \ S_\mu(\lambda x, \xi) = \lambda x, \xi; \]
\[(\beta_5) \ S_\mu^\nu(\xi) = S_\mu^\nu(\xi) S_\mu^\nu(\mu); \]
\[(\beta_6) \ y \notin \Delta \mu \Rightarrow S_\mu^\nu(\lambda y, \xi) = \lambda y, S_\mu^\nu(\xi), \ x \neq y; \]
\[(\alpha) \ y \notin \Delta \xi \Rightarrow \lambda x, \xi = \lambda y, S_\mu^\nu(\xi). \]

Note that the two occurrences of \(x\) in \((\beta_4)\) have very different meanings, something that is hidden by our streamlined notation. This becomes apparent when we interpret \((\beta_4)\) in an actual \(\text{LAA}_I\) and explicitly relativize all the operations:

\[
(S^A)_x a(a) = ((\lambda x)^A, a)x^A = a \quad \text{for all } a \in A \text{ and } x \in I.
\]

We will use notation like \("(S^A)_x a\"\) very rarely because it is so cumbersome. We leave it to context to determine the particular algebra in which \(S\) is being applied.

In the following three propositions we give some basic properties of substitution and dimension set that will be used repeatedly in the sequel. The proofs of Props. 2.1, 2.2 and 2.3 can be found in [32].

**Proposition 2.1.** Let \(A \in \text{LAA}_I\), \(a, b \in A\), and \(x \in I\).

(i) \(\Delta(a, b) \subseteq \Delta(a) \cup \Delta(b). \)

(ii) \(\Delta(\lambda x, a) = \Delta a \setminus \{x\}. \)

(iii) \(\Delta(S_x^A(a)) \subseteq (\Delta a \setminus \{x\}) \cup \Delta b. \)

(iv) \(\Delta x \subseteq \{x\}, \) with equality holding if \(A\) is nontrivial.

**Proposition 2.2.** For all \(x, y, z \in I\) and \(a, b, c \in A\) we have:

(i) \(x \notin \Delta c \Rightarrow S_x^y S_y^c(a) = S_x^y S_y^c(a); \)

(ii) \(y \notin \Delta b \Rightarrow S_y^b S_y^c(a) = S_y^b S_y^c(a); \)

(iii) \(y \notin \Delta a \Rightarrow S_y^a(a) = a; \)

(iv) \(x \notin \Delta c, y \notin \Delta b \Rightarrow S_x^y S_y^b(a) = S_x^y S_y^b(a), \ x \neq y; \)

(v) \(z \notin \Delta a \cup \Delta b \Rightarrow S_z^c(a) = S_z^c(a). \)

For any set \(A\), let \(A^*\) denote the set of all finite strings of elements of \(A\) without repetitions.

**Proposition 2.3.** Let \(A\) be a \(\text{LAA}_I\), \(x = x_1 \cdots x_n \in I^*, \) and \(b = b_1 \cdots b_n \in A^*. \) If \(b_i\) is independent of \(x_1, \ldots, x_{i-1}\) for \(i = 2, \ldots, n, \) in particular, if each \(b_i\) is independent of all the \(x_j, \) then

\[
S_x^b(a) = (\lambda x_1 \cdots x_n, a) b_1 \cdots b_n \quad \text{for all } a \in A.
\]

### 2.3. Lambda calculus and locally finite LAA’s

The set \(\Lambda_I(C)\) of ordinary terms of lambda calculus over a set \(I\) of \(\lambda\)-variables and a set \(C\) of constants is constructed as usual [3]: every \(\lambda\)-variable \(x \in I\) and every constant \(c \in C\) is a \(\lambda\)-term; if \(t\) and \(s\) are \(\lambda\)-terms, then so are \((st)\) and \(\lambda x.t\) for each \(\lambda\)-variable \(x.\) An occurrence of a \(\lambda\)-variable \(x\) in a \(\lambda\)-term is *bound* if it lies within the scope of a lambda abstraction \(\lambda x;\) otherwise it is *free*. A \(\lambda\)-term is *closed* if all occurrences of variables are bound. A \(\lambda\)-term \(s\) is *free for \(x\) in \(t\) if no free occurrence of \(x\) in \(t\) lies within the scope of a lambda abstraction with respect to a \(\lambda\)-variable that occurs free in \(s.\) \([s/x]\) is the result of substituting \(s\) for all free occurrences of \(x\) in \(t\) subject to the usual provisos about renaming bound \(\lambda\)-variables in \(t\) to avoid capture of free \(\lambda\)-variables in \(s.\)

The axioms of the \(\lambda\)-calculus are as follows: \(t\) and \(s\) are arbitrary \(\lambda\)-terms and \(x, y\) \(\lambda\)-variables.
\( \alpha \) \( \lambda x.t = \lambda y.t[y/x] \), for any \( \lambda \)-variable \( y \) that does not occur free in \( t \);

\( \beta \) \( (\lambda x.t)s = t[s/x] \), for every \( s \) free for \( x \) in \( t \);

\( t = s \) implies \( s = t \);

\( t = s \), \( s = r \) imply \( t = r \);

\( t = s \), \( u = r \) imply \( tu = sr \);

\( t = s \) implies \( \lambda x.t = \lambda x.s \).

\( \beta \)-conversion expresses the way of calculating a function \( (\lambda x.t) \) on an argument \( s \), while \( \alpha \)-conversion says that bound \( \lambda \)-variables can be replaced in a term under the obvious condition. A lambda theory \( T \) over \( \Lambda I(C) \) is any set of equations that is closed under \( \alpha \)- and \( \beta \)-conversion and the five equality rules. We will write \( T \vdash t = s \) for \( t = s \in T \). \( \lambda/\beta \) denotes the minimal lambda theory, while we often write \( t =_\beta s \) for \( \lambda/\beta \vdash t = s \).

There is a strong connection between the lambda theories and the subclass of LAA’s whose elements are finite dimensional.

**Definition 2.4.** A lambda abstraction algebra \( A \) is locally finite if it is of infinite dimension (i.e., \( I \) is infinite) and every \( a \in A \) is of finite dimension (i.e., \( |\Delta a| < \omega \)).

The class of locally finite LAA’s (LAA’s) is denoted by LFA (LFA), which is also used as shorthand for the phrase “locally finite lambda abstraction algebra (of dimension \( I \)”).

Let \( T \) be a lambda theory in the language \( \Lambda I(C) \) and let

\[ \Lambda I(C) := \langle \Lambda I(C), \cdot, \lambda x \Lambda I(C): x \in I, \langle x^\Lambda I(C): x \in I \rangle \rangle \]

be the absolutely free algebra of \( \lambda \)-terms. Then \( T \) is a congruence over the algebra \( \Lambda I(C) \). We call \( \Lambda I(C)/T \) the term algebra of lambda theory \( T \).

The following proposition is the algebraic analogous of Prop. 1 and Prop. 3 in [21, Chapter VII].

**Proposition 2.4.** An algebra \( A \) is (isomorphic to) the term algebra of a lambda theory if and only if it is a LFA.

**Proof:**

\( (\Rightarrow) \) It can be shown in a straightforward way that a lambda theory \( T \) is a congruence over \( \Lambda I(C) \) making \( \Lambda I(C)/T \) a LAA. The elements of \( \Lambda I(C)/T \) are sets of \( \lambda \)-terms that are pairwise equivalent under \( T \). The local finiteness is a direct consequence of the trivial fact that every \( \lambda \)-term is a finite string of symbols and hence contains only finitely many \( \lambda \)-variables.

\( (\Leftarrow) \) Let \( A \) be a LFA and \( Zd A \) be the set of zero-dimensional elements of \( A \). Consider the unique homomorphism \( h \) from the absolutely free algebra \( \Lambda I(Zd A) \) into \( A \) that is the identity on \( Zd A \). The map \( h \) is onto. If \( a \in A \) and \( \Delta A a = \{x_1, \ldots, x_n\} \), then we have \( a = bx_1^A \ldots x_n^A \), where \( b = \lambda x_1 \ldots x_n \cdot a \) is a zero-dimensional element of \( A \). So, \( a = h(bx_1 \ldots x_n) \) and \( h \) is onto. The LFA \( A \) is isomorphic to the quotient algebra \( \Lambda I(Zd A)/\theta \) where the congruence \( \theta \) is the relation-kernel of \( h \) (i.e., \( t \theta u \iff h(t) = h(u) \)). To obtain the conclusion of the proposition it suffices to show that \( \theta \) is a lambda theory, i.e., that it is closed under \( \alpha \) and \( \beta \)-conversion. But this is easily verified.

Note that the set of zero-dimensional elements of a term LAA is the set of all those elements which are equivalence classes of closed \( \lambda \)-terms, i.e., terms without free \( \lambda \)-variables.

Only recently has a general consensus developed as to what the models of the lambda calculus should be. (A brief but illuminating history of the process can be found in [23].) The notion of an environment model (the name is due to Meyer [23]) originated with Hindley and Longo [20]. The notion of a syntactical model defined in Barendregt [3] is closely related. Environment models (see [23]) turn out to be special kinds of functional domains. We give the formal definition of functional domain.
Definition 2.5. Let $V = \langle V, \cdot^V, \lambda^V \rangle$ be a structure where $V$ is a nonempty set, $\cdot^V$ is a binary operation on $V$, and $\lambda^V : V^V \xrightarrow{\rho^V} V$ is a partial function assigning elements of $V$ to certain functions from $V$ into itself. $V$ is called a functional domain if for each $f$ in the domain of $\lambda^V$, 
\[
f(v) = (\lambda^V(f)) \cdot^V v, \quad \text{for all } v \in V.
\]
This definition of functional domain differs slightly from the one in Meyer [23] where it is assumed that each function of the form $\langle u : V : v \in V \rangle$ is in the domain of $\lambda^V$.

Let $I$ be the set of $\lambda$-variables. An element $p$ of $V^I$, i.e., an assignment of elements of $V$ to the set of $\lambda$-variables, is called an environment. $p_x$ is the value $p$ assigns to $x$ for each $x \in I$. For any $v \in V$ and $x \in I$, $p[v/x] \in V^I$ is the new environment such that, for all $y \in I$,
\[
p[v/x]_y := \begin{cases} 
v, & \text{if } y = x \\
p_y, & \text{otherwise}.
\end{cases}
\]

A lambda polynomial over a functional domain $V$ is a $\lambda$-term over a set of constants that includes a constant symbol $\overline{v}$ for each $v \in V$.

Let $V$ be a functional domain. Define a partial mapping $\llbracket t \rrbracket^V : V^I \xrightarrow{\rho^V} V$ by recursion on the structure of lambda polynomials over $V$: for all $p \in V^I$,
\[
\begin{align*}
\llbracket v \rrbracket^V(p) &= v, \quad \text{for all } v \in V,    \\
\llbracket x \rrbracket^V(p) &= p_x, \quad \text{for all } x \in I,    \\
\llbracket t s \rrbracket^V(p) &= \llbracket t \rrbracket^V(p) \cdot^V \llbracket s \rrbracket^V(p),    \\
\llbracket \lambda x.t \rrbracket^V(p) &= \lambda^V \langle \llbracket t \rrbracket^V(p[v/x]) : v \in V \rangle.
\end{align*}
\]

Definition 2.6. (Meyer [23]) $V$ is an environment model if $\llbracket t \rrbracket^V$ is defined for all $p \in V^I$ and all lambda polynomials $t$ over $V$.

The completeness theorem for the lambda calculus says that every lambda theory consists of precisely the equations valid in some environment model (see [23]). With the aid of the precise connection between lambda theories and LFA’s established in Prop. 2.4, the completeness theorem for the lambda calculus can be also obtained as a corollary of the functional representation theorem for LFA’s (Thm. 2.1 and Thm. 5.2 below).

2.4. Functional Lambda Abstraction Algebras

The most natural LAA’s, the algebras that the axioms are intended to characterize, are algebras of functions. Not surprisingly, they are closely related to the environment models of lambda calculus. Indeed, they are obtained by coordinatizing environment models by the $\lambda$-variables in a natural way.

Definition 2.7. Let $V = \langle V, \cdot^V, \lambda^V \rangle$ be a functional domain and let $I$ be a nonempty set. Let $V_I = \{ f : f : V^I \xrightarrow{\rho^V} V \}$, i.e., the set of all partial functions from $V^I$ to $V$. By the $I$-coordinatization of $V$ we mean the algebra
\[
V_I = \langle V_I, \cdot^V_I, \langle \lambda x^V_I : x \in I \rangle, \langle x^V_I : x \in I \rangle \rangle,
\]
where for all $a, b : V^I \xrightarrow{\rho^V} V$, $x \in I$, and $p \in V^I$:

- $(a^V I b)(p) = a(p) \cdot^V b(p)$, provided $a(p)$ and $b(p)$ are both defined; otherwise $(a^V I b)(p)$ is undefined.
- $(\lambda x^V_I a)(p) = \lambda^V \langle \langle a(p[v/x]) : v \in V \rangle \rangle$, provided $\langle a(p[v/x]) : v \in V \rangle$ is in the domain of $\lambda^V$ (note this implies $a(p[v/x])$ is defined for all $v \in V$); otherwise $(\lambda x^V_I a)(p)$ is undefined.
- $x^V_I(p) = p_x$. 

Definition 2.8. Let $V$ and $I$ be as in preceding definition. A subalgebra $A$ of total functions of $V_I$, i.e., a subalgebra such that $(\lambda x^V.a)(p)$ is defined for all $a \in A$ and $p \in V^I$, is called a functional lambda abstraction algebra. $I$ is the dimension set of $A$ and $V$ is its value domain.

In the sequel a subalgebra of $V_I$ of total functions will be called a total subalgebra of $V_I$.

The proof of the following proposition can be found in [32].

Proposition 2.5. Every functional lambda abstraction algebra is a lambda abstraction algebra.

The class of all functional lambda abstraction algebras of dimension $I$ is denoted by $FLA_I$, and the class of functional lambda abstraction algebras of arbitrary dimension is denoted by $FLA$. As in the case of $LAA$ and $LFA$ we also use $FLA$ ($FLA_I$) as shorthand for the phrase “functional lambda abstraction algebra (of dimension $I$).”

The following result connecting Meyer’s environment models and $FLA$’s was proved in [32] (see the remarks following Def. 5.2 and Def. 5.7 in that paper).

Proposition 2.6. A functional domain $V$ is an environment model if and only if there exists at least one $FLA_I$ with value domain $V$ (i.e., the coordinatization of $V_I$ has at least one total subalgebra).

In our view the most natural models of the lambda calculus are functional $LAA$’s. They correspond via coordinatization exactly to environment models. This highlights the main difference between our approach to models of the lambda calculus and the traditional one: the latter focuses attention on functional domains while we focus on their coordinatization.

Let $A = \langle A, \cdot, A, \lambda x^A, x^A \rangle, \forall x \in I$ be an arbitrary $LAA_I$. The functional domain $V = \langle V, \cdot, V, \lambda^V \rangle$ associated with $A$ is defined as follows: $V = A$ and $\cdot \cdot \cdot = A$. The domain of $\lambda^V : V^I \rightarrow V$ is

$$\text{dom}(\lambda^V) = \{ \langle S^V_x(a) : v \in V \rangle : a \in A \text{ and } x \in I \},$$

and for each function in this set we define

$$\lambda^V(\langle S^V_x(a) : v \in V \rangle) := \lambda x^A.a.$$

It can be shown that $\langle S^V_x(a) : v \in V \rangle = \langle S^V_y(b) : v \in V \rangle$ implies $\lambda x^A.a = \lambda y^A.b$. Thus $\lambda^V$ is well defined. It is easily checked that $V$ is a functional domain (see [32]).

The following representation theorem is the main result of [29]. It was independently proved by Diskin and Beylin [10]. It is the algebraic analogue of the completeness theorem for lambda calculus.

Theorem 2.1. (Functional Representation of $LFA$’s). Every locally finite lambda abstraction algebra $A$ is isomorphic to a functional lambda abstraction algebra. More precisely, $A$ is isomorphic to a total subalgebra of the $I$-coordinatization of its associated functional domain.

Let $V$ be the functional domain associated with $A$. The mapping $\Psi$ from $A$ into a $V_I$ is defined as follows: For each $a \in A$, let $x = x_1 \cdots x_n \in I^*$ such that $\Delta a \subseteq \{ x_1, \ldots, x_n \}$ (recall that $*$ denotes the set of finite strings without repetitions). Choose $z = z_1 \cdots z_n \in I^*$ such that the $z_i$ are distinct from the $x_j$ and independent of both $a$ and $p \circ x$. Define

$$\Psi(a)(p) = S^z_x \cdot x \cdot S^z_x(a), \text{ for every } a \in A \text{ and } p \in V^I.$$

In [29] it is shown that $\Psi$ is well defined and an isomorphism from $A$ onto a total subalgebra of $V_I$.

The function that takes each $LFA_I$ into its canonical functional representation is the object map part of a natural equivalence between the category of $LFA_I$’s and its full subcategory of locally finite $FLA_I$’s.

In [40] Salibra and Goldblatt have shown the following general representation theorem.
Theorem 2.2. (Functional Representation of LAA’s). For any infinite set $I$, every LAA$_I$ is isomorphic to a functional LAA$_I$, in symbols,

$$\mathbb{IFLA}_I = \text{LAA}_I.$$ 

Thus $\mathbb{IFLA}_I$ is a finitely axiomatized variety.

3. Combinatory algebras and $\lambda$-algebras

In a combinatorial algebra lambda abstraction can be simulated by combinators, so it is possible to interpret $\lambda$-terms in a combinatorial algebra. However, not all the equations provable in the lambda calculus are true in every combinatorial algebras. $\lambda$-algebras constitute exactly the class of combinatory algebras in which all the equations provable in the lambda calculus are true.

In this section, after reviewing briefly the concept of combinatorial algebra and $\lambda$-algebra, we show that the zero-dimensional subreduct of a LFA$_I$ is a $\lambda$-algebra and vice versa. This leads to a natural equivalence between the category of $\lambda$-algebras and the category of LFA’s. We also show that the free extension of a $\lambda$-algebra $C$ (by a set $I$) in the variety of combinatorial algebras can be turned into a LFA$_I$ whose zero-dimensional subreduct is $C$. Some applications of this result to the lambda calculus are also obtained.

3.1. Combinatory algebras

We review briefly the concept of combinatory algebra by giving a proof of the combinatory completeness lemma. We then show that the “combinatory reducts” of arbitrary LFA’s are combinatorial algebras.

We begin with the definition of a basic notion in combinatory logic and lambda calculus.

Definition 3.1. (Curry [8]) Let $C = \langle C, \cdot^C, k^C, s^C \rangle$ be an algebra where $\cdot^C$ is a binary operation and $k^C, s^C$ are constants. $C$ is a combinatory algebra if it satisfies the following identities: (as usual the symbol $\cdot$ and the superscript $^C$ are omitted, and association, when in doubt, is to the left)

$$kxy = x; \quad sxyz = xz(yz).$$

The equational theory axiomatized by these two identities is denoted by CL.

$k$ and $s$ are called combinators. In the equational language of combinatory algebras the derived combinator $i$ and $1$ are defined as follows:

$$i := ssk \quad \text{and} \quad 1 := s(ki),$$

and note that every combinatory algebra satisfies the identities

$$ix = x \quad \text{and} \quad 1xy = xy.$$

Let $C$ and $D$ be combinatory algebras. A mapping $h : C \to D$ is an applicative homomorphism if $h(c \cdot^C c') = h(c) \cdot^D h(c')$ for all $c, c' \in C$. An applicative homomorphism $h$ such that $h(k^C) = k^D$ and $h(s^C) = s^D$ is called a combinatory homomorphism.

Suitable reducts of arbitrary LAA’s turn out to be combinatory algebras. Let $A$ be a LAA$_I$. By the combinatorial reduct of $A$ we mean the algebra

$$\text{Cr} A = \langle A, \cdot^A, k^A, s^A \rangle$$

where

$$k^A = (\lambda xy.x)^A \quad \text{and} \quad s^A = (\lambda xyz.xz(yz))^A.$$
The $\lambda$-variables $x$, $y$, and $z$ are assumed to be distinct. Note that by a repeated application of axiom (a) it is possible to prove that $k^A$ and $s^A$ are uniquely defined (i.e., independent of the choice of $x, y, z$) if $I$ has enough $\lambda$-variables ([32, Prop. 4.5]). In the sequel we will assume this is always the case unless otherwise specified.

A subalgebra of the combinatory reduct of a $\text{LAA}_I A$ (i.e., a subset of $A$ containing $k^A$ and $s^A$ and closed under $\cdot^A$) is called a combinatory subreduct of $A$. The zero-dimensional subreduct of $A$ is the combinatory subreduct

$$\text{Zd } A = \langle \text{Zd } A, \cdot^A, k^A, s^A \rangle,$$

where $\text{Zd } A = \{ a \in A : \Delta^A a = 0 \}$, the set of zero-dimensional elements of $A$.

The following result is a consequence of Thm. 29 in [40].

**Proposition 3.1.** The combinatory reduct $\text{Cr } A$ of a $\text{LAA}_I A$ is a combinatory algebra.

In this paper we will often use this result under the hypothesis of locally finiteness. For the sake of completeness, we provide here the easy proof of this restricted result. Let $A$ be a $\text{LFA}_I$. For any $a, b, c \in A$, let $x, y, z \in I$ be distinct $\lambda$-variables that are independent of all three finite-dimensional elements. Then

$$s^{abc} = (\lambda xyz.xz(yz))^{abc} = S^x_a S^y_b S^z_c (xz(yz)) = ac(bc);$$

the second equality uses Prop. 2.3 and the last ($\beta_5$). A similar argument works for $k$. So $A$ is a combinatory algebra.

Environment models can also be given a combinatory structure in a natural way. Let $V = \langle V, \cdot^V, \lambda^V \rangle$ be an environment model and let $A$ be a $\text{FLA}_I$ with value domain $V$, i.e., a total subalgebra of $V_I$. (Recall that by Prop. 2.6 at least one such $A$ must exist.) Set

$$k^A := (\lambda x y x)^A \text{ and } s^A := (\lambda x y z. x z(y z))^A,$$

where $x, y, z$ are any three distinct $\lambda$-variables.

**Lemma 3.1.** If $V$ is an environment model, then $k^A$ and $s^A$ are total, constant functions from $V^I$ to $V$, i.e., $k^A(p) = k^A(q)$ and $s^A(p) = s^A(q)$ for all $p, q \in V^I$.

**Proof:**

Let $p \in V^I$ be arbitrary.

$$(\lambda x y x)^A(p) = \lambda^V \langle \lambda^V (\lambda^V x^A(p v/x) : v \in V) : v \in V \rangle$$
$$= \lambda^V \langle \lambda^V (\lambda^V (u v/x) : u \in V) : v \in V \rangle$$
$$= \lambda^V \langle \lambda^V (v : u \in V) : v \in V \rangle.$$

Thus the value of $k^A(p)$ is independent of the choice of $p$. A similar argument works for $s$.

We denote the constant values of $k^A$ and $s^A$ in $V$ respectively by $k^V$ and $s^V$. They do not depend on the particular total subalgebra $A$ of $V_I$ we choose.

**Proposition 3.2.** Every environment model $V = \langle V, \cdot^V, k^V, s^V \rangle$ is a combinatory algebra.

**Proof:**

Let $A$ be any total subalgebra of $V_I$. By the defining condition of a functional domain and by Lem. 3.1, we have for all $v, u \in V$

$$k^V v u = (\lambda^V \langle \lambda^V (v : u \in V) : v \in V \rangle) v u$$
$$= (\lambda^V (v : u \in V)) u$$
$$= v.$$

A similar calculation gives

$$s^V v u w = v u (w u) \text{ for all } v, u, w \in V.$$
3.2. Combinatory completeness

In the equational logic of combinatory algebras it is traditional to let \( \lambda \)-variable’s play the role of real variables. We follow this convention in the next definition. By a **combinatory term** we mean a term of the equational logic of combinatory algebras in the usual sense. Thus \( k, s, \) and \( x \), for every \( \lambda \)-variable \( x \), are combinatory terms. If \( s \) and \( t \) are combinatory terms, so is \( st \). A combinatory term is **ground** if it contains no variables. Note that context variables do not occur in combinatory terms. Let \( C \) be a combinatory algebra. Let \( \overline{c} \) be a new symbol for each \( c \in C \). Extend the language of combinatory algebras by adjoining \( \overline{c} \) as a new constant symbol for each \( c \in C \). A term \( t \) in this extended language is called a **combinatory polynomial over** \( C \). The set all such polynomials is denoted by \( P_t(C) \). If \( t = t(x_1, \ldots, x_n) \), where \( x_1, \ldots, x_n \) includes all the \( \lambda \)-variables occurring in \( t \), and \( v_1, \ldots, v_n \in C \), then \( t^C(v_1, \ldots, v_n) \) will denote the value of \( t \) in \( C \) when \( x_i \) is interpreted as \( v_i \) and each new constant \( \overline{c} \) as \( c \).

The following result is well-known (Meyer [23], Barendregt [3, Thm. 5.1.10], Curry-Feys [8]); we reproduce its simple proof for completeness.

**Proposition 3.3.** (Combinatory Completeness Lemma). Let \( C \) be a combinatory algebra and let \( t(x_1, \ldots, x_n) \) be a combinatory polynomial over \( C \) whose variables all occur in the list \( x_1, \ldots, x_n \). Then there exists an element \( c \) in \( C \) such that, for all \( v_1, \ldots, v_n \in C \),

\[
    t^C(v_1, \ldots, v_n) = cv_1 \cdots v_n.
\]

This combinatory completeness lemma depends on the following lemma that shows that some aspects of lambda abstraction can be simulated in combinatory algebras.

Let \( C \) be a combinatory algebra. For each \( \lambda \)-variable \( x \) define a transformation \( \lambda^* x \) of the set \( P_t(C) \) of combinatory polynomials over \( C \) as follows: \( \lambda^* x(x) = i \). Let \( t \) be a combinatory term different from \( x \). If \( x \) does not occur in \( t \), define \( \lambda^* x(t) = kt \); in particular, \( \lambda^* x(\overline{c}) = k\overline{c} \) for every \( v \in C \). Otherwise, \( t \) must be of the form \( sr \) where \( s \) and \( r \) are combinatory terms, at least one of which contains \( x \); in this case define \( \lambda^* x(t) = s(\lambda^* x(r))(\lambda^* x(s)) \). For any finite sequence \( x_1, \ldots, x_n \) of variables define

\[
    \lambda^* x_1 \cdots x_n(t) = \lambda^* x_1(\lambda^* x_2(\cdots(\lambda^* x_n(t)\cdots))).
\]

**Lemma 3.2.** Let \( C \) be a combinatory algebra, \( t \) a combinatory polynomial over \( C \), and \( x \) a variable.

(i) \( x \) does not occur in \( \lambda^* x(t) \). More precisely, the variables that occur in \( \lambda^* x(t) \) are exactly the variables different from \( x \) that occur in \( t \).

(ii) Let \( y_1, \ldots, y_n \) be any list of variables that includes all variables occurring in \( t \) except \( x \), and write \( t = t(x, y_1, \ldots, y_n) \) and \( \lambda^* x(t) = (\lambda^* x(t))(y_1, \ldots, y_n) \). Then for all \( v, u_1, \ldots, u_n \in C \),

\[
    t^C(v, u_1, \ldots, u_n) = (\lambda^* x(t))^C(u_1, \ldots, u_n)v,
\]

i.e., the combinatory algebra \( C \) satisfies the equation \( (\lambda^* x(t))x = t \), in symbols,

\[
    C \models (\lambda^* x(t))x = t.
\]

**Proof:**

The proofs of both (i) and (ii) are straightforward inductions on the structure of \( t \); compare Barendregt [3, Prop. 5.1.9].

The combinatory completeness lemma now follows easily. \( \lambda^* x_1 \cdots x_n(t) \) is ground by Lem. 3.2(i) (i.e., it contains no variables). Hence it defines a unique element

\[
    c = (\lambda^* x_1 \cdots x_n(t))^C
\]
of $C$. We get $t^C(v_1, \ldots, v_n) = cv_1 \cdots v_n$ by repeated applications of Lem. 3.2(ii).

$\lambda^* x$ is an operation on combinatory terms; it does not define directly an operator on combinatory algebras. It can be used to define translations, $CL : \lambda I(C) \rightarrow P I(C)$ and $\lambda : P I(C) \rightarrow \lambda I(C)$ from lambda terms to combinatory polynomials and vice versa [3, Def. 7.3.1].

(Recall that $x, y, z$, possibly with subscripts, denote arbitrary distinct $\lambda$-variables in $L$.)

$$x_\lambda = x, \quad \overline{\alpha}_\lambda = \overline{\alpha}, \quad k_\lambda = \lambda y x, \quad s_\lambda = \lambda y z. x z(yz), \quad \overline{(tu)}_\lambda = t_\lambda u_\lambda;$$

$$x_{CL} = x, \quad \overline{\alpha}_{CL} = \overline{\alpha}, \quad \overline{(tu)}_{CL} = t_{CL} u_{CL}, \quad \overline{(\lambda x.t)}_{CL} = \lambda^* x(t_{CL}).$$

Combinatory logic is weaker than lambda calculus in the sense that the translation of an identity of the lambda calculus needs not be an identity of combinatory logic. For example, $s_\lambda x$, $s_t x$ and $k_\lambda x$ are not normal forms in the lambda calculus, while $s x$, $s y x$ and $k x$ are normal forms in combinatory logic. The best we can obtain by these translations is summarized in the following proposition [3, Lem. 7.3.3; Thm. 7.3.10(i)]:

**Proposition 3.4.** Let $CA$ be the variety of combinatory algebras.

(i) $CA \models t = u \implies \lambda \beta \models t_\lambda = u_\lambda$, for all combinatory terms $t, u$, while the converse is not true. For example, $\lambda \beta \models s_\lambda k_\lambda = k_\lambda i_\lambda$, while $CA \not\models s k = k i$.

(ii) $\lambda \beta \models t = u \iff CA \models t_{CL} = u_{CL}$, for all $\lambda$-terms $t, u$, while the converse is not true. Indeed, the set $\{ t = u : CA \models \overline{t} = \overline{u} \}$ does not constitute a lambda theory.

(iii) $\lambda \beta \models t_{CL, \lambda} = t$, for every $\lambda$-term $t$.

(iv) It is not the case that $CA \models t_{\lambda, CL} = t$ for every combinatory term $t$. For example, $CA \not\models k = k_{\lambda, CL}$.

The way to strengthen combinatory logic to make it equivalent to lambda calculus was discovered by Curry and will be outlined in the next subsection.

We conclude this subsection by showing that $\lambda x^*$ simulates lambda abstraction when the combinatory algebra is a reduct of a LAA. This is shown in the Prop. 3.5 below.

As a matter of notation, if $A$ is a LAA and $t \in P I(A)$ is a combinatory polynomial, $t^A$ will denote the element $(t_\lambda)^A$ of $A$, i.e., each $\lambda$-variable $x$ in $t$ is interpreted as a constant $\overline{x}$ as $a$, and the combinators $k, s$ respectively as $(\lambda x y x)^A$ and $(\lambda x y z. x z(yz))^A$.

Note that, if $t = t(x_1 \cdots x_n)$ is a combinatory polynomial, then $t^A$ and $t^{C \! A}$ have different interpretations because variables are interpreted differently in lambda abstraction algebras than they are in a combinatory algebra; in the former they represent constants while in the latter they serve as place holders. Thus $t^A$ denotes a fixed element of $A$ while $t^{C \! A}$ is a function from $A^n$ into $A$.

**Lemma 3.3.** Let $A$ be a LAA and $t = t(x_1, \ldots, x_n)$ be a combinatory term. Then, for every $a = a_1 \cdots a_n \in A^*$, we have $t^{C \! A}(a_1, \ldots, a_n) = S^*_{\alpha}(t^A)$.

**Proof:**

The proof is by induction over the complexity of $t$.

Here and in the sequel we make the following convention. We will retain the superscript $A$ in definitions and the statements of theorems, propositions, and lemmas. But we will often omit it in proofs and interconnecting text if no confusion is likely.

**Lemma 3.4.** Let $A$ be a LAA.

(i) $\overline{t}^A = (\lambda x. x)^A$.

(ii) $1^A = (\lambda x y. x)^A$. 

**Proof:**
At several places in this proof we make use of the fact that $k$ and $i$ are zero-dimensional.

\[ i = s k k \quad \text{(i)} \]
\[ = \left( \lambda x y z. x z (y z) \right) k k \]
\[ = S^g_k S^g_k (\lambda z. x z (y z)) \text{, by Prop. 2.3.} \]
\[ = \lambda z. k z (k z) \text{, by } (\beta_1), (\beta_4), \text{ and } (\beta_5) \]
\[ = \lambda z. (\lambda x. z)(\lambda x. z) \text{, by } (\beta_3) \text{ and } k = \lambda z. x. z \]
\[ = \lambda z. z \text{, by } (\beta_2) \]
\[ = \lambda x. x \text{, by axiom } (\alpha). \]

\[ 1 = s (k i) \quad \text{(ii)} \]
\[ = \left( \lambda x y z. x z (y z) \right)(k i) \]
\[ = S^e_{ki} (\lambda y z. x z (y z)) \]
\[ = \lambda y z. k i z (y z) \]
\[ = \lambda y z. y z \text{, by } (i), (\beta_6) \text{ and } (\beta_1) \]
\[ = \lambda x y. x y \text{, by axiom } (\alpha). \]

**Proposition 3.5.** Let $A$ be a $\text{LA}_1$, $Zd A$ be the subset of zero-dimensional elements of $A$, and $t \in P_1(Zd A)$ be a combinatorial polynomial. Then for any $x \in I$, $\lambda x^A. t^A = (\lambda^* x(t))^A$.

**Proof:**
By induction on the structure of $t$. If $t = x$,

\[ (\lambda^* x(t))^A = i^A = (\lambda x. x)^A = \lambda x^A.t^A. \]

If $x$ does not occur in $t$,

\[ (\lambda^* x(t))^A = (k t)^A = (\lambda y x. y)^A t^A \]
\[ = S^g_{t A} (\lambda x. y)^A \]
\[ = \lambda x^A.t^A, \text{ since } x \notin \Delta^A t^A. \]

Let $t = s r$ with at least one of $s$ and $r$ containing $x$. Let $y, z \neq x$ be two distinct $\lambda$-variables not occurring in $t$. From Prop. 2.1 it follows that the dimension set of $t^A$ is finite, so that $t^A$ is a finite-dimensional element of $A$ (even if $A$ itself is not locally finite). From this remark and from Prop. 2.1(ii), it follows that $y, z$ are independent of $\lambda x^A.s^A$ and $\lambda x^A.r^A$. 
\[(\lambda^* x(s) r)^A = s^A(\lambda^x(s))^A(\lambda^* x(r))^A
\]
\[= s^A(\lambda^x)^A, r^A\]
\[= (\lambda y z x, y x(z x))^A(\lambda x^A)^A, r^A)\]
\[= S_{\lambda^x}^A, r^A(\lambda x, y x(z x))^A\]
\[= \lambda x^A, (\lambda x^A, s^A)^A, r^A = \lambda x^A, s^A, r^A = \lambda^x, r^A\]

**Proposition 3.6.** Let \(A\) be a \(\text{LAA}_I\) and let \(t, s\) be combinatory terms. Then, \(\text{Cr} A \vdash t = s\) if and only if \(t^A = s^A\).

**Proof:**
Let \(t = t(x_1, \ldots, x_n)\) and \(s = s(x_1, \ldots, x_n)\). Then \(t^A = t^A(x_1^A, \ldots, x_n^A)\) and similarly for \(s\). So, the ‘only if part’ follows. The ‘if’ part is a consequence of the above lemma.

### 3.3. Lambda algebras

Those combinatory algebras for which the combinatory polynomial transformation \(\lambda^x\) simulates lambda abstraction form a variety. They are called \(\lambda\)-algebras; the concept is essentially due to Curry. We shall see that they coincide with the zero-dimensional subreducts of arbi-

**Definition 3.2.** A combinatory algebra \(C\) is a \(\lambda\)-algebra if it satisfies the following condition for all combinatory terms \(t, u\):

\[\lambda^x t = u \iff C \vdash t = u.\]

The class of all \(\lambda\)-algebras is denoted by \(\text{LA}\).

The hypothesis that \(t, u\) range over the set of combinatory terms and not over the set \(P_i(C)\) of combinatory polynomials, as in the standard definition in Barendregt’s book, is not restrictive (see [41] for a simple proof of this fact).

**Lemma 3.5.** Every \(\lambda\)-algebra satisfies the following two identities \(k = \lambda^x y.x\) and \(s = \lambda^x y z.x(z(yz))\).

**Proof:**
By Prop. 3.4(iii) we have \(\lambda^x t = k_{\lambda, CL, \lambda} \iff C \vdash k = k_{\lambda, CL} = \lambda^x y.x\).

A similar argument works for \(s\).

**Proposition 3.7.** The class of \(\lambda\)-algebras, \(\text{LA}\), forms a variety that is axiomatized by the defining identities of combinatory algebras together with all identities \(t = u\) between ground combinatory terms (no variables are involved) such that \(\lambda^x t = u\).
Proof:
Let \( C \) be a combinatorial algebra satisfying all the identities \( t' = u' \) between ground combinatorial terms for which \( \lambda \beta \vdash t'_\lambda = u'_\lambda \). Assume \( \lambda \beta \vdash t_\lambda = u_\lambda \) for combinatorial terms \( t = t(x_1,\ldots,x_n) \) and \( u = u(x_1,\ldots,x_n) \). We have to show that \( C \models t = u \). Since

\[
\lambda \beta \vdash t_\lambda = u_\lambda \iff \lambda \beta \vdash \lambda x_1 \cdots x_n.t_\lambda = \lambda x_1 \cdots x_n.u_\lambda
\]

\[
\iff \lambda \beta \vdash (\lambda x_1 \cdots x_n.t_\lambda)_{CL,\lambda} = \lambda (x_1 \cdots x_n.u_\lambda)_{CL,\lambda}
\]

\[
\iff \lambda \beta \vdash (\lambda^* x_1 \cdots x_n(t_{\lambda,CL}))_{\lambda} = \lambda^* x_1 \cdots x_n(u_{\lambda,CL})_{\lambda},
\]

by applying the hypothesis we obtain

\[
C \models \lambda^* x_1 \cdots x_n(t_{\lambda,CL}) = \lambda^* x_1 \cdots x_n(u_{\lambda,CL}).
\]

Then Lem. 3.2(ii) implies

\[
C \models t_{\lambda,CL} = u_{\lambda,CL}.
\]

We obtain the conclusion if \( C \models t_{\lambda,CL} = t \) holds for all combinatorial terms \( t \). This can be easily proved by induction on the complexity of \( t \) provided that \( C \) satisfies the identities of \( k = k_{\lambda,CL} \) and \( s = s_{\lambda,CL} \). But these identities are consequences of the assumptions on \( C \) because the combinatorial terms involved are closed and \( \lambda \beta \vdash k_{\lambda} = k_{\lambda,CL,\lambda} \) and \( \lambda \beta \vdash s_{\lambda} = s_{\lambda,CL,\lambda} \).

Curry discovered that only a finite number of identities between ground combinatorial terms are sufficient for axiomatizing \( \lambda \)-algebras over combinatorial algebras (see [3, Chap. 7]).

The axioms of a \( \lambda \)-algebra are designed expressly to prove the next lemma. We require a definition.

Let \( C \) be a combinatorial algebra. Recall that \( P_I(C) \) is the set of combinatorial polynomials over \( C \). Recall also that the members of \( P_I(C) \) are constructed from \( \lambda \)-variables in \( I \) and constant symbols \( k, s, \) and \( \overline{c} \) for all elements \( c \) of \( C \).

Let \( D_C \) be the \textit{equational diagram} of \( C \), i.e., the set of all equations of the form \( \overline{c} \overline{d} = \overline{e} \) for \( c,d,e \in C \) such that \( c.d = e \); we also include the two equations \( k = \overline{c} \) and \( s = \overline{d} \) where \( c = k^C \) and \( d = s^C \). Let \( \equiv_C \) be the equivalence relation of \( P_I(C) \) such that \( t \equiv_C s \) iff the equation \( t = s \) is a logical consequence of \( D_C \) together with the axioms of combinatorial logic.

Lemma 3.6. Let \( C \) be a \( \lambda \)-algebra and let \( t, s \) be combinatorial polynomials over \( C \). Then \( t \equiv_C s \) implies \( \lambda^* x(t) \equiv_C \lambda^* x(s) \) for every \( x \in I \).

Proof:
See the proof of Lem. 7.12 in Meyer [23].

For the sake of completeness, we provide a proof of the following well-known result [3, Thm. 7.3.10].

Proposition 3.8. \( \begin{array}{ll}
(1) & \lambda \beta \vdash t = u \iff LA \models t_{CL} = u_{CL}, \text{ for all } \lambda \text{-terms } t, u. \\
(2) & LA \models t_{\lambda,CL} = t, \text{ for every combinatorial term } t.
\end{array} \)

Proof:
(1) \( (\Rightarrow) \) From Prop. 3.4(iii) and \( \lambda \beta \vdash t = u \) it follows that \( \lambda \beta \vdash t_{CL,\lambda} = u_{CL,\lambda} \); then an application of the defining condition of \( \lambda \)-algebras gives the result.

(1) \( (\Leftarrow) \) Since the identities characterizing \( LA \) are equations between ground combinatorial terms, they are logical consequences of the equational diagram \( D_C \) of every \( \lambda \)-algebra \( C \). This implies that \( t \equiv_C u \) for every identity such that \( LA \models t = u \). By this remark and by Lem. 3.6 the set \( \{ t = u : t, u \in A_I, LA \models t_{CL} = u_{CL} \} \) is a lambda theory. We have to show that it is the minimal lambda theory. Since the term \( LAA A \) associated with the minimal
lambda theory $\lambda \beta$ is locally finite, from Prop. 3.6 and from the definition of the term LAA it follows that $\text{Cr } A \models t = u$ iff $t^A = u^A$ iff $\lambda \beta \models t_\lambda = u_\lambda$, for all combinatory terms $t, u$. So, $\text{Cr } A$ is a $\lambda$-algebra and \{ $t = u : \text{Cr } A \models t_{CL} = u_{CL}$ \} is the minimal lambda theory.

(ii) From Prop. 3.4(iii) it follows that $\lambda \beta \models t_{\lambda,CL,\lambda} = t_\lambda$. Then, by applying the defining condition of $\lambda$-algebras the conclusion follows.

(iii) $(\Rightarrow)$ By Def. 3.2.

(iii) $(\Rightarrow)$ If $\text{LA } \models t = u$, then by (ii) we have $\text{LA } \models t_{\lambda,CL} = u_{\lambda,CL}$. Now we can apply (i) for obtaining the conclusion.

We denote by $C[I]$ the free extension of $C$ by $I$ in the variety of combinatory algebras. If $\Sigma$ is the algebraic similarity type of combinatory algebras enriched by a constant $\bar{c}$ for each element $c \in C$, then $C[I]$ is also characterized as the free $\Sigma$-algebra over $I$ in the variety axiomatized by the equational diagram of $C$ and the axioms of combinatory logic. $C[I]$ is an expansion of $C$ defined up to isomorphism by the following universal mapping properties: (1) $C[I]$ is the universe of $C[I]$; (2) $C[I]$ is a combinatory algebra; (3) for every homomorphism $h : C \to A$ from $C$ into a combinatory algebra $A$ and every mapping $g : I \to A$ there exists a unique homomorphism $f : C[I] \to A$ extending both $h$ and $g$. Let $t$ be a combinatory polynomial over $C$. $t^{C[I]}$ denotes the unique interpretation of $t$ in $C[I]$ when each variable $x$ in $t$ is interpreted as $x^{C[I]}$, each constant $\bar{c}$ as $c$, and the combinators $k, s$ as $k^{C}, s^{C}$. It follows easily from basic principles of universal algebra that $t^{C[I]} = s^{C[I]}$ if $t \equiv_{C} s$.

We define $\lambda$-abstractions $\lambda x^{C[I]}$ on $C[I]$ for all $x \in I$ as follows: Let $a \in C[I]$. Choose any $t \in P_I(C)$ such that $t^{C[I]} = a$. Define

$$\lambda x^{C[I]}, a = (\lambda^* x(t))^{C[I]}.$$ 

Lem. 3.6 guarantees $\lambda x^{C[I]}$ is well defined. The algebra obtained by adjoining these operations will also be denoted by $C[I]$.

**Theorem 3.1.** Let $C$ be a $\lambda$-algebra and $I$ an infinite set. Then $C[I]$ is a LFA$_I$ whose zero-dimensional subreduct is $C$. Moreover, it is universal with respect to this property in the sense that, if $h : C \to \mathbb{Z}d A$ is any combinatorial-algebra homomorphism of $C$ into the zero-dimensional subreduct of a LAA$_I$ $A$, then $h$ extends uniquely to a lambda-abstraction-algebra homomorphism $h_I$ from $C[I]$ into $A$.

**Proof:**
We verify only axiom $(\beta_3)$ of LAA. The verifications of the others are similar. Let $a, b, c \in C[I]$. Choose $t, s, u \in P_I(C)$ such that $a = t^{C[I]}$, $b = s^{C[I]}$, and $c = u^{C[I]}$. (Here and in the sequel we omit the superscript $C[I]$ on $\lambda x^{C[I]}$.) We want to prove $(\lambda x.ab)c = (\lambda x.a)c((\lambda x.b)c)$; this clearly equivalent to

$$\lambda^* x(ts)u \equiv_{C} \lambda^* x(t)u(\lambda^* x(s)u).$$

The conclusion follows since every combinatory algebra satisfies the identities $\lambda^* x(t)u = t[u/x]$ for all combinatory polynomials $t, u$ (see Lem. 3.2(ii) and [3, Prop. 7.1.6]), so that $\lambda^* x(ts)u \equiv_{C} (t[s]u/x) = (t[u/x])(s[u/x]) \equiv_{C} \lambda^* x(t)u(\lambda^* x(s)u)$. So $C[I]$ is a LAA.

To see $C[I]$ is locally finite, observe that, if $x$ does not occur in $t \in P_I(C)$, then, for any variable $z$ distinct from $x$, $\lambda^* x(t)z = ktz \equiv_{C} t$. (The last equivalence holds because the axiomatization of $\equiv_{C}$ includes the axioms of a combinatory algebra). Thus, if $a = t^{C[I]}$, then $\Delta a$ can include only variables that occur in $t$. A similar argument shows that $\mathbb{Z}d (C[I]) = C$.

$C$ is the zero-dimensional subreduct of $C[I]$ if we can prove that

$$(\lambda xy, x)^{C[I]} = k^{C} \quad \text{and} \quad (\lambda xy, z(\lambda z(y)))^{C[I]} = s^{C}.$$

These equalities are clearly equivalent respectively to

$$\lambda^* x(\lambda^* y(x)) \equiv_{C} k \quad \text{and} \quad \lambda^* x(\lambda^* y(\lambda^* x(\lambda^* z(y)))) \equiv_{C} s.$$
Since the identities characterizing the variety LA of λ-algebras are equations between closed combinatorial terms, they are logical consequences of the equational diagram of every λ-algebra. This implies that \( t \equiv_C u \) holds whenever \( LA \vdash t = u \). From this observation and Lem.3.5 we get the desired conclusion.

Finally, let \( A \in \text{LAA}_I \) and \( h : C \to Zd A \) be any homomorphism. By definition of \( C[I] \), \( h \) extends uniquely to a homomorphism \( h_I : C[I] \to A \) of combinatorial algebras such that \( h_I(x^{C[I]}) = x^A \) for every \( x \in I \). It only remains to show that \( h_I \) preserves λ-abstraction. Let \( a \in C[I] \) and \( t \in P_I(C) \) such that \( t^{C[I]} = a \). Since \( \lambda^*x(t) \) is a combinatorial polynomial over \( C \) and \( h_I \) is a combinatorial homomorphism,

\[
h_I(\lambda x.a) = h_I((\lambda^*x(t))^{C[I]}) = (\lambda^*x(t))^A
\]

where \( (\lambda^*x(t))^A \) is the unique interpretation of \( \lambda^*x(t) \) in \( A \) when each constant \( \bar{e} \) is interpreted as \( h(e) \). But \( (\lambda^*x(t))^A = \lambda x^A, t^A = \lambda x^A, h_I(a) \) by Prop. 3.5.

Recall that by definition every locally finite \( \text{LAA} \) is infinite-dimensional.

**Corollary 3.1.** Let \( C \) be a combinatorial algebra. The following are equivalent.

(i) \( C \) is a λ-algebra;
(ii) \( C \subseteq Zd A \) for an infinite-dimensional \( \text{LAA}_I \) \( A \);
(iii) \( C \subseteq Zd A \) for an infinite-dimensional \( \text{LAA}_I \) \( A \);
(iv) \( C \) is a combinatorial subreduct of a locally finite \( \text{LAA}_I \);
(v) \( C \) is a combinatorial subreduct of an infinite-dimensional \( \text{LAA}_I \).

**Proof:**
The implication (i) \( \Rightarrow \) (ii) follows directly from the theorem. The implications (ii) \( \Rightarrow \) (iii), (iii) \( \Rightarrow \) (iv) and (iv) \( \Rightarrow \) (v) are trivial.

(iv) \( \Rightarrow \) (i). It is sufficient to prove that the combinatorial reduct \( \text{Cr} A \) of a \( \text{LFA}_I \) \( A \) is a λ-algebra. Let \( t, u \) be combinatorial terms such that \( \lambda \beta \vdash t_\lambda = u_\lambda \). By Prop. 3.6 we have that \( \text{Cr} A \vdash t = u \) if \( t^A = u^A \). Moreover, Prop. 2.4 implies that the set \( \{w = s : w, s \in A_I, w^A = s^A\} \) is a lambda theory. The conclusion is now immediate.

(v) \( \Rightarrow \) (i). By Thm. 29 in [40].

We conclude the section with a theorem which establishes the equivalence of the categories of the λ-algebras and of the \( \text{LFA}_I \)’s.

Let \( \mathcal{C} \) and \( \mathcal{D} \) be categories. We recall that, given two functors \( F, G : \mathcal{C} \to \mathcal{D} \), a natural transformation from \( F \) to \( G \) is a map \( \varepsilon \) from \( \mathcal{C} \)-objects to \( \mathcal{D} \)-arrows satisfying the following conditions.

(i) For each \( \mathcal{C} \)-object \( A \), \( \varepsilon_A \) is a \( \mathcal{D} \)-arrow \( \varepsilon_A : F(A) \to G(A) \);
(ii) For each \( \mathcal{C} \)-arrow \( f : A_1 \to A_2 \), \( G(f) \circ \varepsilon_{A_1} = \varepsilon_{A_2} \circ F(f) \).

If each component \( \varepsilon_A \) is an isomorphism in \( \mathcal{D} \), \( \varepsilon \) is called a natural isomorphism.

Two categories \( \mathcal{C} \) and \( \mathcal{D} \) are equivalent if there exist functors \( F : \mathcal{C} \to \mathcal{D} \) and \( G : \mathcal{D} \to \mathcal{C} \) such that \( FG \) and \( GF \) are naturally isomorphic respectively to the identity functors on \( \mathcal{D} \) and \( \mathcal{C} \).

**Theorem 3.2.** The category of λ-algebras and the category of \( \text{LFA}_I \)’s (with \( I \) infinite) are equivalent.
Proof:
Denote by \( \mathcal{L} \) the category of \( \lambda \)-algebras and by \( \mathcal{K}_I \) the category of locally finite lambda abstraction algebras of dimension \( I \). The function \( F \) that assigns to each \( \lambda \)-algebra \( C \) the \( \text{LFA}_I \) \( C[I] \) and to each homomorphism \( h : C \to D \) of \( \lambda \)-algebras the homomorphism \( h_I : C[I] \to D[I] \) (see Thm. 3.1 for the definition of \( h_I \)) is a functor from the category \( \mathcal{L} \) to the category \( \mathcal{K}_I \).

Conversely, the function \( G \) that assigns to each \( \text{LFA}_I \) \( A \) its zero-dimensional combinatorial subreduct \( ZdA \) and to each homomorphism \( h : A \to B \) of lambda abstraction algebras its restriction to \( ZdA \) is a functor from the category \( \mathcal{K}_I \) to the category \( \mathcal{L} \). Note that if \( h : A \to B \) is a homomorphism of \( \text{LAA} \)'s and \( a \in A \) such that \( h(a) = b \), then \( \Delta^B b \subseteq \Delta^A a \). So, \( b \) is zero-dimensional if \( a \) is such.

The conclusion of the theorem follows if we show that \( GF \) and \( FG \) are respectively naturally isomorphic to the identity functors on \( \mathcal{L} \) and \( \mathcal{K}_I \). This is obvious for \( GF \) since it is the identity functor on \( \mathcal{L} \). We prove the result for the other functor \( FG \). For each \( \text{LFA}_I \) \( A \), define \( \iota_A \) to be the unique homomorphism from \( FG(A) = (ZdA)[I] \) to \( A \) extending the identity map on \( ZdA \) (see Thm. 3.1 for its definition). We claim that \( \iota \) is a natural isomorphism.

We prove that each \( \iota_A \) is an isomorphism. Let \( B = (\text{ZdA})[I] \). If \( a \in A \) then \( a = b x_1^A \cdots x_n^A \), where \( b = \lambda x_1^A \cdots x_n^A \) and \( x_1 \cdots x_n \) is any enumeration without repetitions of \( \Delta^A a \). Since \( b \) is a zero-dimensional element of \( A \) and the restriction of \( \iota_A \) to \( ZdA \) is the identity map, then

\[
\iota_A(b x_1^B \cdots x_n^B) = b x_1^A \cdots x_n^A = a,
\]

so that \( \iota_A \) is onto.

Let \( a_1, a_2 \) be two elements of \( B \) such that \( \iota_A(a_1) = \iota_A(a_2) \) and let \( x_1 \cdots x_n \) be any enumeration without repetitions of \( \Delta^B a_1 \cup \Delta^B a_2 \). Then we have

\[
\begin{align*}
\iota_A(\lambda x_1 \cdots x_n^B \cdot a_1) &= \lambda x_1 \cdots x_n^A \cdot \iota_A(a_1) \\
&= \lambda x_1 \cdots x_n^A \cdot \iota_A(a_2) \\
&= \iota_A(\lambda x_1 \cdots x_n^B \cdot a_2).
\end{align*}
\]

Thus \( \lambda x_1 \cdots x_n^B \cdot a_1 = \lambda x_1 \cdots x_n^B \cdot a_2 \) since \( \iota_A \) restricted to zero-dimensional elements is one-one. In conclusion,

\[
\begin{align*}
a_1 &= (\lambda x_1 \cdots x_n^B \cdot a_1)x_1^B \cdots x_n^B \\
&= (\lambda x_1 \cdots x_n^B \cdot a_2)x_1^B \cdots x_n^B \\
&= a_2.
\end{align*}
\]

Thus \( \iota_A \) is also one-one.

Finally, we show that \( f \circ \iota_A = \iota_B \circ GF(f) \) for every homomorphism \( f : A \to B \) of lambda abstraction algebras. Let \( A' = (\text{ZdA})[I] \) and \( B' = (\text{ZdB})[I] \). If \( a \in A' \) then \( a = b x_1^{A'} \cdots x_n^{A'} \) where \( b = \lambda x_1 \cdots x_n^{A'} \) is zero-dimensional and \( x_1 \cdots x_n \) is any sequence of variables including \( \Delta^{A'} a \). Then

\[
\begin{align*}
f(\iota_A(a)) &= f(\iota_A(b x_1^{A'} \cdots x_n^{A'})) \\
&= f(b x_1^{A'} \cdots x_n^{A'}) \\
&= f(b)x_1^B \cdots x_n^B \\
&= \iota_B(f(b)x_1^B \cdots x_n^B) \\
&= \iota_B(GF(f)(a)).
\end{align*}
\]
The following result is a trivial corollary of the previous theorem.

**Corollary 3.2.** Let $A$ and $B$ be two LFA$_I$'s. Then any combinatory homomorphism $h$ from $\mathbb{Z}d A$ to $\mathbb{Z}d B$ extends uniquely to a lambda abstraction algebra homomorphism from $A$ to $B$. Moreover, if $h$ is one-one and/or onto, so is its extension. Thus, $A$ and $B$ are isomorphic if their zero-dimensional subreducts are isomorphic.

The following are some applications of the above results to the lambda calculus. Let $\Lambda I(C)$ be the set of $\lambda$-terms over a set $I$ of variables and a set $C$ of constants. Recall that a lambda theory is any set equations that is closed under $(\alpha)$- and $(\beta)$-conversion and the equality axioms. For every lambda theory $T$ on $\Lambda I(C)$, the term LAA $\Lambda I(C)/T$ of $T$ is a LFA$_I$.

Let $T$ be a lambda theory. The term $\lambda$-algebra of $T$ is the combinatory reduct of the term LAA of $T$. It is a $\lambda$-algebra by Prop. 3.1 and Cor. 3.1. The restriction of $T$ to closed $\lambda$-terms is denoted by $T^0$ and is called the closed theory of $T$. The term $\lambda$-algebra of $T^0$ is the zero-dimensional combinatory subreduct of the term LAA of $T$.

**Corollary 3.3.** Every lambda theory is uniquely determined by its closed theory.

**Proof:** Let $T_1$ and $T_2$ be lambda theories on $\Lambda I(C_1)$ and $\Lambda I(C_2)$ respectively. Let $T_1^0$ and $T_2^0$ be the respective closed theories. Then, by Cor. 3.2 any isomorphism between the term $\lambda$-algebras of $T_1^0$ and $T_2^0$ can be uniquely extended to an isomorphism between the term LAA's of $T_1$ and $T_2$.

**Corollary 3.4.** Every $\lambda$-algebra is (isomorphic to) the term $\lambda$-algebra of the closed theory of exactly one lambda theory.

### 4. Lambda models

Lambda models were introduced by Meyer [23] as an alternative first-order characterization of environment models. In fact, they form an elementary class, while the definition of environment model is higher order. We will eventually show however, in Thm. 5.4 below, that the two notions are essentially the same.

In this section we introduce rich LAA's and show that the zero-dimensional subreduct of a rich LFA$_I$ is a lambda model and vice versa. This leads to a categorical equivalence similar to the one for $\lambda$-algebras given in Thm. 3.2. We also show that the free extension of a lambda model $C$ (by a set $I$) in the variety generated by $C$ can be turned into a LFA$_I$ whose zero-dimensional subreduct is $C$. This result is the basis for characterizing lambda models as those $\lambda$-algebras $C$ whose free extensions in the variety of combinatory algebras and in the variety generated by $C$ coincide. The last part of the section is devoted to the applications of the above results to the lambda calculus.

**Definition 4.1.** (Meyer [23]; Scott [43]) A lambda model is a $\lambda$-algebra $C$ satisfying the following condition, for all $u, w \in C$:

$$uw = wv \text{ for all } v \in C, \text{ then } 1u = 1w.$$  

This condition is called the *Meyer-Scott axiom*. In the first-order language of combinatory algebras it takes the following form

$$\forall x \forall y(\forall z (xz = yz) \Rightarrow 1x = 1y).$$

The particular form of the definition of lambda model given in Def. 4.1 is taken from Barendregt [3, Def. 5.2.7].

The following result is Prop. 5.2.9 in [3].
Proposition 4.1. A $\lambda$-algebra $C$ is a lambda model if and only if it satisfies the following condition, for all combinatory polynomials $t, u \in P_l(C)$:

$$C \models t = u \quad \text{implies} \quad C \models \lambda x.t(x) = \lambda x.u(x).$$

Proposition 4.2. The combinatory reduct of every LFA$_1$ is a lambda model.

Proof:
Let $A$ be a LFA$_1$. $\text{Cr } A$ is a $\lambda$-algebra by Cor. 3.1(iv). Suppose the hypothesis of the Meyer-Scott axiom holds for $a, b \in A$, i.e.,

$$ac = bc \quad \text{for all } c \in A.$$

Let $y$ be independent of both $a$ and $b$. Then

$$1a = (\lambda xy.xy)a, \quad \text{by Lem. 3.4(ii)}$$

$$= S^x_c(\lambda y.y)$$

$$= \lambda y.ay, \quad \text{by } (\beta_0) \text{ and } (\beta_1)$$

$$= \lambda y.by, \quad \text{by assumption}$$

$$= 1b.$$

So the Meyer-Scott axiom holds.

Let $C$ be a lambda model. We denote by $C^*[I]$ the free extension of $C$ by $I$ in the variety generated by $C$. If $\Sigma$ is the algebraic similarity type of combinatory algebras enriched by a constant $c$ for each element $c \in C$, then $C^*[I]$ is also characterized as the free $\Sigma$-algebra in the variety generated by $C$. $C^*[I]$ is an expansion of $C$ by $I$ in the variety generated by $C$. It is defined up to isomorphism by the following universal mapping property. For every mapping $g : I \to C$ there exists a unique homomorphism $f : C^*[I] \to C$ extending both $g$ and the identity map on $C$. Let $t \in P_l(C)$, i.e., a combinatory polynomial over $C$. $t^C[I]$ denotes the unique interpretation of $t$ in $C^*[I]$ when each variable $x$ in $t$ is interpreted as $x^C[I]$, each constant $c$ as $c$, and the combinators $k, s$ as $k^C, s^C$. For all $t, u \in P_l(C)$ define

$$t \simeq_C u \quad \text{iff} \quad C \models t = u. \quad (1)$$

This is equivalent to saying that $t^C(v_1, \ldots, v_n) = u^C(v_1, \ldots, v_n)$ for all $v_1, \ldots, v_n \in C$, where $t = t(y_1, \ldots, y_n)$, $u = u(y_1, \ldots, y_n)$ and $y_1, \ldots, y_n$ is any sequence of variables occurring in $t$ or $u$. Then it follows easily from the basic principles of universal algebra that $t^C[I] = u^C[I]$ iff $t \simeq_C u$.

We define $\lambda$-abstractions $\lambda x^C[I]$ on $C^*[I]$ as follows: Let $a \in C^*[I]$. Choose any $t \in P_l(C)$ such that $t^C[I] = a$. Then we define

$$\lambda x^C[I].a = (\lambda x.t(x))^C[I].$$

The transformation $\lambda x^C[I]$ is well defined by Prop. 4.1.

The algebra obtained by adjoining these new operations $\lambda x^C[I]$ will also be denoted by $C^*[I]$.

Theorem 4.1. Let $C$ be a lambda model. Then $C^*[I]$ is a LFA$_1$ whose zero-dimensional subreduct is $C$.

Proof:
The proof of this result is similar to the proof of Thm. 3.1. Thus we verify only axiom $(\beta_0)$ and that $C$ is the zero-dimensional subreduct of $C^*[I]$. The verifications of the others axioms of LAA are similar. Let $a, b, c \in C^*[I]$. Choose $t, s, u \in P_l(C)$ such that $a = t^C[I]$,
\[ b = s^{C^s[t]} \text{ and } c = u^{C^s[t]} \text{.} \] (We omit the superscript \( C^s[t] \) on \( \lambda x^{C^s[t]} \).) We have to prove that \((\lambda x.ab)c = (\lambda x.a)c((\lambda x.b)c)\); this is clearly equivalent to
\[
\lambda^*x(t_s)u \approx_C \lambda^*x(t)u(\lambda^*x(s)u),
\]
that is,
\[
C \models \lambda^*x(t_s)u = \lambda^*x(t)u(\lambda^*x(s)u).\]
The conclusion follows since every combinatory algebra satisfies the identities \( \lambda^*x(t)s = t[s/x] \) for all combinatory polynomials \( t, s \) (Prop. 3.2(ii)), [3, Prop. 7.1.6]).

\( C \) is the zero-dimensional subreduct of \( C^s[I] \) if we can prove that
\[
(\lambda xy.x)^{C^s[I]} = k^C \quad \text{and} \quad (\lambda yz.z(yz))^{C^s[I]} = s^C.
\]
The first equality is clearly equivalent to \( C \models \lambda^*y(x) = k \). Since \( C \) is also a \( \lambda \)-algebra the result is consequence of Lem. 3.5.

### 4.1. A characterization of lambda models

In this subsection we provide several different characterizations of lambda models in terms of lambda abstraction algebras. In the first part of this subsection we introduce the concept of a rich lambda abstraction algebra.

A lambda theory \( T \) of the lambda calculus is **extensional** if it is closed under the following rule of extensionality [3, Def. 2.1.27].

\[
\text{tx} = \text{sx} \in T \text{ with } x \text{ not free in } ts \text{ implies } t = s \in T. \quad (\text{ext})
\]

Recall that the **closed restriction** \( T^0 \) of a lambda theory \( T \) is defined as the set of all \( t = s \in T \) such that \( t \) and \( s \) are closed \( \lambda \)-terms. The closed restriction \( T^0 \) is **extensional** if, for every pair of closed \( \lambda \)-lambda terms \( t, s \), the condition \( tu = su \in T^0 \) for every closed \( \lambda \)-term \( u \) implies \( t = s \in T^0 \). It is not true in general that the extensionality of a lambda theory implies the extensionality of its closed restriction. The **\( \omega \)-rule** of the lambda calculus [3, Def. 4.1.10],

\[
tu = su \in T \text{ for every closed } \lambda \text{-term } u \text{ implies } t = s \in T, \quad (\omega)
\]

was introduced to study this situation. The closed restriction \( T^0 \) of a lambda theory \( T \) satisfying the \( \omega \)-rule is obviously extensional. In [3, Lem. 4.1.12] it is proved that a lambda theory is closed under the \( \omega \)-rule if and only if it is extensional and the following **term rule** holds.

\[
tu = su \in T \text{ for every closed } \lambda \text{-term } u \text{ implies } tx = sx \in T \text{ for every variable } x. \quad (\text{tr})
\]

It is easy to verify that the term rule is completely equivalent to the following rule.

\[
(\lambda x.t)u = (\lambda x.s)u \in T \text{ for every closed } \lambda \text{-term } u \text{ implies } t = s \in T \text{ for every variable } x.
\]

We will show in Cor. 4.4 below that a lambda model is the term \( \lambda \)-algebra of the closed theory of exactly one lambda theory closed under the term rule.

We now introduce the algebraic version of the term rule. We say that an element \( a \) of a \( \text{LAA}_I \) is **finite** if the dimension set of \( a \) is a finite set. It is simple to prove that \( \text{Fi} \ A = \{ a \in A : |\Delta a| < \omega \} \) is a subuniverse of \( A \).

**Definition 4.2.** A \( \text{LAA}_I \) \( A \) is **rich** if, for all \( a, b \in \text{Fi} \ A \) and all \( x \in I \), we have that

\[
(\forall c \in \text{Zd} \ A : S^c_x(a) = S^c_x(b)) \Rightarrow a = b. \quad (\text{tr})
\]
Rich LAA\(_I\)'s correspond roughly to rich polyadic Boolean algebras ([16]).

**Proposition 4.3.** A LAA\(_I\) \(A\) is rich if and only if it satisfies the following condition for all \(a, b \in \text{Fi } A\) and all \(x \notin \Delta a \cup \Delta b:\)

\[
(\forall c \in \text{Zd } A : ac = bc) \Rightarrow ax = bx.
\]

**Proof:**
Let \(a, b \in \text{Fi } A\).

(\(\Rightarrow\)) If \(ac = bc\) for every zero-dimensional element \(c\), then \(S_c^a(ax) = S_c^b(bx)\) \((x \notin \Delta a \cup \Delta b)\).

Then the hypothesis that \(A\) is rich yields \(ax = bx\).

(\(\Leftarrow\)) If \((\lambda x.a)c = (\lambda x.b)c\) for every zero-dimensional element \(c\), then from the hypothesis it follows that \((\lambda x.a)x = (\lambda x.b)x\), so that \(a = b\) by (\(\beta_3\)).

Let \(\mathcal{V}\) be an arbitrary variety of algebras and \(A \in \mathcal{V}\). Then \(A\) is said to be *generic* (in \(\mathcal{V}\)) if an identity holds in \(A\) iff it holds in \(\mathcal{V}\); equivalently, \(A\) is generic iff it generates \(\mathcal{V}\) as a variety; (see Grätzer [15, p. 383]).

Recall that CL is the equational theory axiomatized by the usual axioms of combinatorial logic, and, for every combinatorial algebra \(C\), \(D_C\) is the equational diagram of \(C\).

**Theorem 4.2.** Let \(C\) be a combinatorial algebra. The following are equivalent.

(i) \(C\) is a lambda model;
(ii) \(C = \text{Zd } A\) for a rich LFA\(_I\) \(A\).
(iii) \(C = \text{Zd } A\) for a rich, infinite-dimensional LAA\(_I\) \(A\).
(iv) \(C\) is a \(\lambda\)-algebra and \(C[I] = C^*[I]\), i.e., the free extension of \(C\) by \(I\) in the variety of combinatorial algebras is equal to the free extension of \(C\) by \(I\) in the variety generated by \(C\).
(v) \(C\) is generic in the variety axiomatized by \(\text{CL} \cup D_C\).

**Proof:**
(i) \(\Rightarrow\) (ii): From Thm. 4.1 it follows that \(C\) is the zero-dimensional subreduct of the LFA\(_I\) \(C^*[I]\). We have to show that \(C^*[I]\) is rich. Let \(a, b \in C^*[I]\) be (necessarily finite) elements such that \((\lambda x.a)c = (\lambda x.b)c\) for all \(c \in C\). (We omit the superscript \(C^*[I]\) on \(\lambda x C^*[I]\).) Let \(\{y_1, \ldots, y_n\} = (\Delta a \cup \Delta b) \setminus \{x\}\). Then by a repeated application of axiom (\(\beta_3\)) we have, for all \(c \in C\):

\[
(\lambda xy_1 \cdots y_n.a)c = \lambda y_1 \cdots y_n.(\lambda x.a)c
= \lambda y_1 \cdots y_n.(\lambda x.b)c
= (\lambda xy_1 \cdots y_n.b)c.
\]

Let \(a' = \lambda xy_1 \cdots y_n.a\) and \(b' = \lambda xy_1 \cdots y_n.b\). \(a'\) and \(b'\) are both zero-dimensional and hence elements of \(C\). Thus the hypothesis that \(C\) is a lambda model implies

\[1^C a' = 1^C b'.\]

But, since \(C\) is the zero-dimensional subreduct of \(C^*[I]\), \(k^C = k^{C^*[I]}\), \(s^C = s^{C^*[I]}\) and \(1^C = 1^{C^*[I]}\). So \(1^{C^*[I]} = (\lambda xy.xy)^{C^*[I]}\) by Lem. 3.4. It follows that the equality \(1^C a' = 1^C b'\) is equivalent to \(\lambda x.a'x = \lambda x.b'x\). The injectivity of the lambda abstractions immediately gives \(a'x = b'x\). Axiom (\(\beta_1\)) and the injectivity of the lambda abstractions together with the definitions of \(a'\) and \(b'\) give the conclusion \(a = b\).

(ii) \(\Rightarrow\) (i): Let \(a, b\) be two zero-dimensional elements such that \(ac = bc\) for all \(c \in \text{Zd } A\). Recalling that \(1^A = (\lambda xy.xy)^A\) in any LAA \(A\) by Lem 3.4(ii), we have to show that \(1^A a = \lambda y.ay = \lambda y.by = 1^A b\). By the injectivity of the lambda abstractions this is equivalent to
\( ay = by \). Since \( ay \) and \( by \) are finite elements and \((\lambda y, ay) = ac = bc = (\lambda y, by)\) for all \( c \in \mathbb{Zd} A \), the hypothesis that \( A \) is rich implies the conclusion \( ay = by \).

(ii) \(\iff\) (iii): For every \( \lambda \text{LAA}_I A \), the set of its finite elements form a subalgebra of \( A \) with the same zero-dimensional subdirect as \( A \).

(i) \(\Rightarrow\) (iv): Since \( C[I] \) and \( C^*[I] \) have both \( C \) as zero-dimensional combinatorial subdirect, we have that \( C[I] \) and \( C^*[I] \) are equal by Cor. 3.2.

(iv) \(\Rightarrow\) (i): We have to prove that \( 1^C u = 1^C w \) for all \( u, w \in C \) such that \( uw = wu \) for all \( v \in C \). This last condition is equivalent to \( \bar{\bar{x}} \approx C \bar{\bar{x}} \), where \( \bar{x} \) and \( \bar{\bar{x}} \) are combinatorial polynomials and \( \approx C \) is the congruence over the algebra of combinatorial polynomials defining the free extension of \( C \) by \( I \) in the variety generated by \( C \) (see (1) above). Thus hypothesis (iv) yields \((\bar{x})^{C[I]} = (\bar{\bar{x}})^{C[I]}\). Since \( C[I] \) is a \( \lambda \text{LAA}_I \) and \( \bar{a}^{C[I]} \), \( \bar{\bar{a}}^{C[I]} \) are zero-dimensional, by applying the abstract substitution operator to the equation \((\bar{x})^{C[I]} = (\bar{\bar{x}})^{C[I]}\), we obtain that \( \bar{a}^{C[I]} = \bar{\bar{a}}^{C[I]} \) for all \( a \in C[I] \). The combinatorial reduct of \( C[I] \) is a lambda model by Prop. 4.2, so \((1^n)^{C[I]} = (1^n)^{C[I]}\). But the same relation holds in \( C \) because \( C \) is the zero-dimensional subdirect of \( C[I] \) so that \( 1^C = 1^C[I] \), and \( 1^n, 1^n \) are combinatorial polynomials without any occurrence of variables.

In view of the equivalence of (i) and (ii) it is easy to see, with the aid of Cor. 3.2, that a \( \lambda \text{LAA}_I A \) is rich iff the zero-dimensional subdirect \( \mathbb{Zd} A \) of \( A \) is a lambda model. Thus the following corollary is an immediate consequence of the equivalence of the categories of \( \lambda \) algebras and \( \lambda \text{LFA}_I \)'s (Thm. 3.2).

**Corollary 4.1.** The categories of lambda models and rich \( \lambda \text{LFA}_I \)'s are equivalent.

Thm. 6.4 below will show that the category of rich \( \lambda \text{LFA}_I \)'s and surjective homomorphisms is equivalent to a category of environment models.

The following corollary follows from Thm. 4.2(v).

**Corollary 4.2.** Let \( C \) be a lambda model. Then every \( \lambda \) algebra \( D \) which admits a homomorphic image of \( C \) as a subalgebra can be constructed from \( C \) by products, homomorphic images and subalgebras.

**Corollary 4.3.** If two \( \lambda \) algebras \( C \) and \( D \) have a common subalgebra (up to isomorphism), then there exists a lambda model \( E \) such that both \( C \) and \( D \) are in the variety generated by \( E \).

**Proof:**
Let \( A \) be a \( \lambda \) algebra which is a common subalgebra of \( C \) and \( D \). Since \( A \) is a homomorphic image of lambda model \( A[I] \), then we can apply Cor. 4.2 to obtain the conclusion.

**Corollary 4.4.** A lambda model is the term \( \lambda \) algebra of the closed theory of exactly one lambda theory closed under the term rule.

**Proof:**
Let \( C \) be a lambda model. Then by Thm. 4.2 there exists a rich \( \lambda \text{LFA} \) such that \( C = \mathbb{Zd} A \), i.e., \( C \) is the zero-dimensional combinatorial subdirect of \( A \). Since \( A \) is locally finite, by Prop. 2.4 there exists a lambda theory \( T \) such that \( A \) is isomorphic to the term \( \lambda \text{LAA} \) of \( T \). Without loss of generality we can assume \( A \) to be the term \( \lambda \text{LAA} \) of \( T \). Hence \( C \) is the term \( \lambda \) algebra of the closed restriction \( T^0 \) of \( T \). We have to show that \( T \) is (a) closed under the term rule and (b) uniquely determined.
(a). Assume that $t, u$ are lambda terms, and that $t = s \cdot u \in T$ for every $\lambda$-term $u$. This condition is equivalent to the equality $\bar{t} \cdot \bar{A} \bar{u} = \bar{s} \cdot \bar{A} \bar{u}$ holding in $A$ for every $u$, where $\bar{t}, \bar{u}$, and $\bar{u}$ are the equivalence classes of $t, s$, and $u$ with respect to the theory $T$. Since $A$ is rich, we can apply Prop. 4.3 to obtain $\bar{t} \cdot \bar{A} \bar{u} = \bar{s} \cdot \bar{A} \bar{u}$, for every variable not occurring in $t$ or $s$. We then have $t = s \cdot u \in T$, so $T$ satisfies the term rule.

(b). Let $B$ be the term $\lambda A$ of another lambda theory $W$ such that $C = Zd B$. Then, by Cor. 3.2, $B$ is also isomorphic to the term $\lambda A$ of $T$. Thus $B$ and $A$ determine the same quotient over the absolutely free algebra $\lambda(A)$. So $T$ and $W$ are the same lambda theory, up to renaming of constants.

As further corollary of Thm. 4.2 we get an alternative proof of the following results due to Barendregt, Koymans [4] and Meyer [23].

Corollary 4.5. Let $C$ be a combinatory algebra. The following are equivalent.

(i) $C$ is a $\lambda$-algebra;
(ii) $C$ is a subalgebra of a lambda model;
(iii) $C$ is a homomorphic image of a lambda model.

Proof:
Let $\mathcal{V}$ be any variety. It is obvious that $A \in \mathcal{V}$ is both a homomorphic image and a subalgebra of the free extension $A[I]$ of $A$ in $\mathcal{V}$. Let $C$ be a lambda algebra. The free extension $C[I]$ of $C$ in the variety of combinatory algebras is the combinatory reduct of a $LFA$, so that it is a lambda model by Prop. 4.2.

The following property of lambda models will be used below in the proof of Thm. 6.2.

Corollary 4.6. The identity $1 \cdot 1 = 1$ hold in every lambda model.

Proof:
By Thm. 4.2 it suffices to prove the identity holds in every $LFA_I$. By Lem. 3.4, $11 = (\lambda xy. xy)1 = \lambda y.1y = \lambda y.(\lambda z. w. z) y = \lambda y. w. y = 1\cdot 1 = 1$.

Another property of lambda models used in Thm. 6.2 is that the combinators $k$ and $s$ are completely determined by $1$ in the sense that if two lambda algebras $C$ and $D$ have the same applicative structure and $1^C = 1^D$, then $k^C = k^D$ and $s^C = s^D$. See [3, Prop. 5.6.6(i)].

5. The Strong Functional Representation Theorem

Functional lambda abstraction algebras are the natural models of the $LAA$ axioms, i.e., the models the axioms were intended to characterize. It is natural to conjecture that they are the only ones that exist. The conjecture says, in other words, that if $A$ is an arbitrary $LAA_I$, then it is possible to construct a functional domain $V$ so that $A$ is isomorphic to a functional $LAA_I$ with value domain $V$. The conjecture has been recently verified by Salibra and Goldblatt in [40]. The first partial result in this direction was obtained in [29] where the $LAA$'s of greatest interest in the lambda calculus, i.e., $LFA$'s, were shown isomorphic to functional $LAA$'s; see Thm. 2.1 above. This result is analogous to the functional representation theorem for polyadic Boolean algebras; Halmos, [16, Thm. (10.9)]. According to the latter result, every locally finite polyadic Boolean algebra is isomorphic to an algebra of functions that has a Boolean algebra as value domain. In the same paper Halmos proves a stronger representation theorem according to which every locally finite polyadic Boolean algebra is isomorphic to a subalgebra of a special kind of functional polyadic Boolean algebra that corresponds to those first-order theories that have constants witnessing every existential theorem; these special algebras are said to be functionally rich. A consequence of this result is that every simple, locally finite polyadic Boolean algebra is isomorphic to a functional polyadic Boolean algebra...
whose value domain is the 2-element Boolean algebra ([16, Thm. (17.3)]). Such functions can be interpreted as the characteristic functions over a set and the functional polyadic algebra as a natural coordinatization of a first-order model. In this way the strong representation theorem can be viewed as the natural algebraic analogue of the completeness theorem for first-order predicate logic. The standard representation theorem for locally finite cylindric algebras can also be viewed this way.

In this section we prove a strong functional representation theorem for LFA’s that corresponds roughly to the strong functional representation theorem for locally finite polyadic Boolean algebras. It depends on an analogue for FLA’s of the notion of a functionally rich functional polyadic Boolean algebra. We conclude the section with a new proof of the equivalence of lambda models and environment models based on the strong representation theorem for rich LFA’s (Thm. 5.2).

5.1. Functionally rich FLA’s

We introduce some useful notation. For all \( p, q \in V^I \) and \( J \subseteq I \), we write \( p \rightarrow_J q \) if \( p_x = q_x \) for all \( x \in J \), and \( p \rightarrow_J q \) if \( p_x = q_x \) for all \( x \in (I \setminus J) \). \( J \) and \( J \) are equivalence relations on \( V^I \): \([p]_J\) and \([p]_{J'}\) denote the corresponding equivalence classes of \( p \). An element \( a \) of a FLA \( A \) is functionally independent of \( x \in I \) if, for all \( p, q \in V^I \), \( p \{x\} \) and \( q \) imply \( a(p) = a(q) \); otherwise \( a \) is functionally dependent on \( x \). It is convenient to treat this as symmetric relation and say that \( x \) is functionally independent (dependent) of (on) \( a \).

Lemma 5.1. Let \( A \) be a FLA \( A \) with value domain \( V \). For all \( a, b \in A, x \in I \), and \( p \in V^I \),

\[
(S^p(a))(p) = a(p\{b(p)/x\}).
\]

Proof:

\[
(S^p(a))(p) = ((\lambda x \cdot V^I \cdot a \cdot b(p)) = (\lambda x \cdot V^I \cdot a \cdot b(p)) = \lambda V ((a(p\{v/x\}) : v \in V)) \cdot V \cdot b(p) = a(p\{b(p)/x\}), \quad \text{see Def. 2.5.}
\]

Proposition 5.1. Let \( A \) be a FLA \( A \) and \( a \in A \). Then \( \Delta a \) equals the set of all \( x \in I \) on which \( a \) is functionally dependent. Thus the notions of algebraic and functional dependence coincide.

Proof:

Assume \( x \notin \Delta a \). Then by Lem. 5.1 we have \( a(p) = a(p\{b(p)/x\}) \) for all \( p \in V^I \), \( b \in A \). Taking \( b = v^I \) for some \( z \in I \) with \( z \neq x \), we have \( a(p) = a(p\{v_x/x\}) \) for all \( v \in V \). Thus, taking \( p\{v/x\} \) for \( p \), we get

\[
a(p\{v/x\}) = a(p\{v/x\}\{v/x\}_z/x) = a(p\{v/x\}_z/x) = a(p\{v/x\}).
\]

Hence \( a(p) = a(p\{v_x/x\}) = a(p\{v_x\}) \) for all \( v \in V \), i.e.,

\[
p \{x\} \rightarrow q \quad \text{implies} \quad a(p) = a(q).
\]

Assume conversely that this implication holds for all \( p, q \in V^I \). By Lem. 5.1 we get

\[
(S^p(a))(p) = a(p\{b(p)/x\}) = a(p),
\]

since \( p\{b(p)/x\} \{x\} \), \( p \).
For any $p, q \in V^I$, define
\[
p \equiv_\omega q \text{ if } | \{ x \in I : p_x \neq q_x \} | < \omega.
\]
$\equiv_\omega$ is an equivalence relation on $V^I$. Let $J \subseteq I$ mean that $J$ is a finite subset of $I$. Note that $p \equiv_\omega q$ iff $p, J, q$ for some $J \subseteq I$, i.e., $\equiv_\omega$ coincides with $\bigcup_{J \subseteq I} J$. We denote the $\equiv_\omega$-equivalence class of $p$ by $[p]_\omega$.

**Corollary 5.1.** Let $\mathcal{A}$ be an FLA$_I$ with value domain $V$. Let $a \in A$. Then $\Delta a = \emptyset$ iff
\[
p \equiv_\omega q \text{ implies } a(p) = a(q), \text{ for all } p, q \in V^I.
\]

**Proof:**
Assume $\Delta a = \emptyset$ and suppose $p, J, q$ for some $J \subseteq I$. Let $J = \{x_1, \ldots, x_n\}$. Then
\[
p \{x_1\}, p\{q_{x_1}/x_1\} \{x_2\}, p\{q_{x_1}/x_1, q_{x_2}/x_2\}
\]
\[\ldots \{x_n\}, p\{q_{x_1}/x_1, q_{x_2}/x_2, \ldots, q_{x_n}/x_n\} = q.
\]
Thus $a(p) = a(q)$ by the proposition. The converse follows immediately from the proposition.

This corollary is the best we can do in the sense that in an arbitrary FLA$_I$ it is possible that $a$ has an empty dimension set but $a(p) \neq a(q)$ for some $p, q \in V^I$. But we now want to consider in some detail an important class of FLA$_I$'s $\mathcal{A}$ with the property that $\Delta a = \emptyset$ iff $a(p) = a(q)$ for all $p, q \in V^I$.

**Definition 5.1.** Let $V$ be a functional domain. An element $a$ of $V_I$ is said to be *regular* if it has the following property.

\[(4.1) \quad \text{There exists a } J \subseteq I \text{ such that } p, J, q \text{ implies } a(p) = a(q), \text{ for all } p, q \in V^I.
\]

A FLA$_I$ $\mathcal{A}$ with value domain $V$ is *regular* if each of its members is.

Every regular FLA$_I$ is locally finite, for if $J$ is as in (4.1), then $a$ is clearly functionally independent of every $x \in I \setminus J$.

For every $v \in V$, we denote by $\hat{v}$ the constant function such that $\hat{v}(p) = v$ for all $p \in V^I$.

**Proposition 5.2.** Assume $\mathcal{A}$ is a regular FLA$_I$ with value domain $V$. Let $a \in A$. Then $a \in \text{Reg } A$ iff $a = \hat{v}$ for some $v \in V$.

**Proof:**
$\hat{v}$ is obviously functionally independent of every $x \in I$. Thus $\Delta \hat{v} = \emptyset$ by Cor. 5.1. Suppose $\Delta a = \emptyset$. Then, since $\mathcal{A}$ is regular, there is a $J \subseteq I$ such that, for all $p, q \in V^I$, $p, J, q$ implies $a(p) = a(q)$. Consider arbitrary $p, q \in V^I$. Let $r \in V^I$ such that
\[
r_x = \begin{cases} p_x, & \text{if } x \in J \\ q_x, & \text{otherwise.} \end{cases}
\]
Then $p, J, r, J, q$. So $a(p) = a(r) = a(q)$; the last equality holds by Cor. 5.1 and the assumption $\Delta a = \emptyset$. Let $v$ be the common value of all $a(p)$. Then $a = \hat{v}$.

In a regular LAA's, the zero-dimensional elements can be identified with certain elements of the value domain, via the constant functions. But not every element of the value domain need be the value of a constant function. Those FLA's with this property deserve special attention.

**Definition 5.2.** A FLA$_I$ $\mathcal{A}$ with value domain $V$ is said to be *functionally rich* if it is regular and every element of $V$ is the value of a constant function in $\mathcal{A}$. 
In a functionally rich FLA$_I$, $A$, the mapping $v \mapsto \check{v}$ is clearly an applicative isomorphism between the value domain $V$ of $A$ and $\mathbb{Zd} A$. It turns out to be also a combinatory isomorphism. Recall that every environmental model $V$ can be given the structure of a combinatory algebra $\langle V, V, k^V, s^V \rangle$ (Prop. 3.2).

**Proposition 5.3.** Let $A$ be a functionally rich FLA$_I$ with value domain $V$. Then $v \mapsto \check{v}$ is a combinatory isomorphism between $\langle V, V, k^V, s^V \rangle$ and $\mathbb{Zd} A$.

**Proof:**
The mapping is injective and is surjective by Prop. 5.2. Consider any $u, v \in V$. For any $p \in V^I$ we have $$(\check{u} \cdot A \check{v})(p) = \check{u}(p) \cdot V \check{v}(p) = u \cdot V v = (u \cdot V, v)(p).$$
So $(u \cdot V, v) = (u \cdot A \check{v})$. By definition of $k^V$ and $s^V$ we have $k^A(p) = k^V$ and $s^A(p) = s^V$. So $(k^V) = k^A$ and $(s^V) = s^A$.

Clearly functionally rich FLA$_I$'s can be characterized by the property that $v \mapsto \check{v}$ is a combinatory isomorphism between the value domain and the zero-dimensional part of the algebra.

**Proposition 5.4.** Let $V$ be an environment model. Then, the set of all regular elements of $V_I$ is the universe of a functionally rich FLA$_I$ with value domain $V$.

**Proof:**
Every constant function of $V_I$ is regular. Thus, it is sufficient to verify that regular elements are closed under the operations $\cdot V_I$, $\lambda x^V_I$ and $\lambda^V_I$ for all $x \in I$. But this is obvious.

### 5.2. The strong functional representation theorem

We prove that every LFA$_I$ is isomorphic to a subalgebra of a functionally rich FLA$_I$. This result is obtained by showing that every rich LFA (Def. 4.2) is isomorphic to a functionally rich FLA and that every LFA can be embedded in a rich LFA.

Let $A = \langle A, A, \lambda x^A, x^A \rangle_{x \in I}$ be a rich LAA$_I$. We define the functional domain $V = \langle V, V, \lambda^V \rangle$ associated with the zero-dimensional elements of $A$ as follows: $V = \mathbb{Zd} A$ and $\lambda^V = A$. The domain of $\lambda^V : V^V \to V$ is

$$\text{dom}(\lambda^V) = \{ \langle S^x(a) : v \in V \rangle : \lambda x^A.a \in \mathbb{Zd} A \text{ and } x \in I \},$$

and for each function in this set we define

$$\lambda^V(\langle S^x(a) : v \in V \rangle) := \lambda x^A.a. \quad (2)$$

It is possible to prove that $\lambda^V$ is well-defined and that the structure $V$ is a functional domain under the hypothesis that $A$ is rich.

**Lemma 5.2.** Let $A$ be a rich LAA$_I$, and let $\lambda x^A.a$ and $\lambda y^A.b$ be zero-dimensional elements of $A$. Then $\langle S^v(a) : v \in \mathbb{Zd} A \rangle = \langle S^v(b) : v \in \mathbb{Zd} A \rangle$ implies $\lambda x^A.a = \lambda y^A.b$.

**Proof:**
Assume $S^v(a) = S^v(b)$ for all $v \in \mathbb{Zd} A$. If $x = y$, then the conclusion is an immediate consequence of the hypothesis that $A$ is rich. Suppose now that $x \neq y$. Then, for all $c \in \mathbb{Zd} A$, we have

$$S^v((\lambda y.b)x) = S^v(\lambda y.b)S^v(x), \quad \text{by } (\beta_3)$$

$$= S^v(\lambda y.b)c, \quad \text{by } (\beta_1)$$

$$= (\lambda y.b)c, \quad \text{by } \Delta(\lambda y.b) = \emptyset$$

$$= S^v(b),$$

$$= S^v(a), \quad \text{by assumption.}$$
From the hypothesis that $A$ is rich we obtain $a = (\lambda y.b)x$. Finally, since $\Delta(\lambda y.b) = \Delta b \setminus \{ y \}$ and $\Delta(\lambda y.b) = \emptyset$, we have $x \notin \Delta b$, and hence axiom (a) yields the conclusion: $\lambda y.b = \lambda x.(\lambda y.b)x = \lambda x.a$.

So $\lambda^V$ and hence the structure $V$ are well defined.

**Lemma 5.3.** $V$ is a functional domain.

**Proof:**
Let $f$ be in the domain $D_A$ of $\lambda^V$, i.e., $f = \langle S^\sigma_0(a) : v \in V \rangle$, where $\lambda x^A.a \in Zd A$. Then, for each $v \in V$, $(\lambda^V(f)) \cdot V v = \lambda^V(\langle S^\sigma_0(a) : v \in V \rangle) \cdot V v = (\lambda x^A.a)^A v = S^\sigma_0(a) = f(v)$.

The following is one of the main results of the paper.

**Theorem 5.1.** (Functional Representation of Rich LFA’s) Every rich LFA $A$ is isomorphic to a functionally rich FLA$_I$. More precisely, $A$ is isomorphic to a total subalgebra of the $I$-coordinatization of the functional domain associated with the zero-dimensional elements of $A$.

**Proof:**
Let $V$ be the functional domain associated with the zero dimensional elements of $A$ and let $V_I$ be its $I$-coordinatization. Define $\Psi : A \rightarrow V_I$ as follows (recall that $V_I$ is the set of all partial functions from $V^I$ to $V$):

$$\psi(a)(p) = S_{p \circ x}^x(a), \quad \text{for every } p \in V^I,$$

where $x = x_1 \ldots x_n$ is any enumeration without repetitions of any set of $\lambda$-variables including $\Delta a$, and $p \circ x = p_{x_1} \ldots p_{x_n}$.

Note that $\Psi(a)$ is well-defined and independent of the choice of $x$ including $\Delta a$ by Prop. 2.2. Note also that $\Psi(a)$ is a total function. The injectivity of $\Psi$ is an easy consequence of the hypothesis that $A$ is rich. We now verify that $\Psi$ is a homomorphism from $A$ to $V_I$ and hence an isomorphism between $A$ and a total subalgebra of $V_I$.

Let $a \in A$, $x \in I$, and $p \in V^I$. Let $y$ be any enumeration without repetitions of $\Delta^A(\lambda x^A.a)$. Note that $x \notin y$. Then

$$\psi(\lambda x^A.a)(p) = S_{p \circ y}^y(\lambda x^A.a)$$

$$= \lambda x^A.a \cdot S_{p \circ y}^y(a), \quad \text{by } (\beta_3) \text{ since } x \notin y$$

$$= \lambda^V(\langle S_{p \circ y}^y(a) : v \in V \rangle) \quad \text{by } (2)$$

$$= \lambda^V(\langle S_{p(x/v)}^x(a) : v \in V \rangle)$$

$$= \lambda^V((\Psi(a)(p\cdot v/x) : v \in V))$$

$$= (\lambda x^V.I \cdot \Psi(a))(p).$$

Let $a, b \in A$, $x \in I$, and $p \in V^I$. Let $y$ be any enumeration without repetitions of $\Delta^A(a) \cup \Delta^A(b)$. Recall that axiom $(\beta_3)$ implies that $S^x_\varepsilon(a \cdot b) = S^x_\varepsilon(a) \cdot S^x_\varepsilon(b)$ for all $a, b, c \in A$.

$$\psi(a \cdot A.b)(p) = S_{p \circ y}^y(a \cdot A.b)$$

$$= S_{p \circ y}^y(a) \cdot V S_{p \circ y}^y(b),$$

$$= \psi(a)(p) \cdot V \psi(b)(p)$$

$$= (\psi(a) \cdot V^I \psi(b))(p).$$
Finally, the interpretations of the \( \lambda \)-variables are preserved (recalled the \( \lambda \)-variables are constant symbols in the language of lambda abstraction algebras).

\[
\Psi(x^A)(p) = S_{\psi}(x^A) \\
= p_x \\
= x^{V_I}(p).
\]

Thus \( \Psi \) is an isomorphism between \( A \) and a total subalgebra \( \Psi(A) \) of \( V_I \). It remains only to show that \( \Psi(A) \) is functionally rich. It is obvious that every \( a \in V(= \mathbb{Z}d A) \) is the value of a constant function in \( \Psi(A) \); indeed from the fact \( a \) is zero-dimensional we get from (2) that

\[
\Psi(a)(p) = S_{\psi}(a) = a, \text{ for all } p \in V_I.
\]

It only remains to show that \( \Psi(A) \) is regular. Let \( a \in A \) and let \( \Delta(a) = \{ x_1, \ldots, x_n \} \). Consider any \( p, q \in V_I \) such that \( p \Delta(a) q \) and note that, if \( x = x_1 \cdots x_n \) is any enumeration of \( \Delta(a) \), then \( p \circ x = q \circ x \). Then, by (3),

\[
\Psi(a)(p) = S_{\psi}(a) = S_{\psi}(a) = \Psi(a)(q).
\]

The converse of Thm. (5.1) also holds, i.e., every functionally rich \( \text{FLA}_I A \) is a rich \( \text{LAA}_J \). Suppose \( a, b \in A \) and

\[
S_{\psi}(a) = S_{\psi}(b) \text{ for all } c \in \mathbb{Z}d A. \tag{4}
\]

By hypothesis \( c = \hat{c} \) for some \( v \in V \). Using Lem. 5.1 we get, for every \( p \in V_I \), \( S_{\psi}(a)(p) = a(p[\hat{v}/p/x]) = a(p[v/x]) \). So (4) implies \( a(p[v/x]) = b(p[v/x]) \) for all \( p \in V_I \) and \( v \in V \). Taking \( v = p_x \) we get \( a(p) = b(p) \) for all \( p_x \), i.e., \( a = b \).

We will get the strong representation theorem for \( \text{LFA}'s \) as a corollary when we prove that every \( \text{LFA} \) can be embedded into a rich \( \text{LFA} \). Following Halmos [16] the proof is based on the concept of dilation.

### 5.3. Dilations

The process of going from a lambda abstraction algebra to one of its dilations can be thought of as the adjunction of new \( \lambda \)-variables. It is not at all obvious that new \( \lambda \)-variables can always be adjoined to an arbitrary \( \text{LAA} \). \( \text{LFA}'s \) behave particularly well in this regard, and we can prove both the existence and uniqueness of dilations in this case. This result will be the basis for the proof of the strong functional representation theorem.

Let \( A \) be a \( \text{LAA}_J \) and \( I \subseteq J \). By the \( I \)-reduct of \( A \) we mean the algebra

\[
\text{Rd}_I A = \langle A, \cdot^A, \lambda x^A, x^A \rangle_{x \in I}.
\]

Clearly this is a \( \text{LAA}_I \). Define

\[
\text{Nr}_I A = \{ a \in A : \Delta a \subseteq I \}.
\]

This set forms a subuniverse of the \( I \)-reduct. By the \( I \)-neutral reduct of \( A \) we mean the algebra

\[
\text{Nr}_I A := \langle \text{Nr}_I A, \cdot^\text{Nr}_I A, \lambda x^\text{Nr}_I A, x^\text{Nr}_I A \rangle_{x \in I},
\]

whose operations are corresponding operations of \( A \) restricted to \( \text{Nr}_I A \). \( \text{Nr}_I A \) is obviously a \( \text{LAA}_I \).

Let \( I \subseteq J \). By a \( J \)-dilation of \( \text{LAA}_J A \) we mean any \( \text{LAA}_J B \) such that \( A = \text{Nr}_I B \).
Let $B$ be a $J$-dilation of a LAA$_1 A$. The elements of $B$ which depend only on the $\lambda$-variables in $J \setminus I$ are zero-dimensional elements in the $I$-reduct $Rd_I B$ of $B$. Thus the process of forming the $J$-dilation of $A$ and then taking its $I$-reduct can be thought as the process of adjoining new ‘constants’ to $A$. It is important to emphasize that the constants we are adjoining here are definitely not zero-dimensional in the $J$-dilation $B$. The construction of the dilations is a new concept for the lambda calculus. In similar constructions in the lambda calculus when constants are adjoined to the language they are always zero-dimensional.

**Lemma 5.4.** Let $B$ be a $J$-dilation of a LFA$_1 A$. If $J \setminus I$ is infinite, then $Rd_I B$ is a rich LFA$_1$.

**Proof:**
$Rd_I B$ is a LAA$_1$ by definition and is locally finite because $B$ is. To see it is rich, assume $a, b \in B$, $x \in I$, and $S^x_\eta(a) = S^x_\eta(b)$ for all $c \in Zd \, Rd_I B$. Let $y \in J \setminus (I \cup \Delta a \cup \Delta b)$; $y$ exists since $J \setminus I$ is infinite by hypothesis and $Rd_I B$ is locally finite. $y \in Zd \, Rd_I B$ by Prop. 2.1(iv). Thus $S^x_\eta(a) = S^x_\eta(b)$ and hence, by ($\beta_3$) and Prop. 2.2(v),

\[ a = S^x_\eta(a) = S^y_\eta(S^x_\eta(a)) = S^y_\eta(S^y_\eta(b)) = S^y_\eta(b) = b. \]

**Lemma 5.5.** Let $A$ be a LFA$_1$ and let $J \supseteq I$ with $J \setminus I$ infinite. Then $(Zd \, A)[J]$ is the unique $J$-dilation of $A$ up to isomorphism.

**Proof:**
Let $C = Zd \, A$ in this proof. Recall the definition of $C[I]$, the free extension of $C$ by $I$, given in Sec. 3. By Cor. 3.2 we have that $A$ is isomorphic to $C[I]$. Consider the neat $I$-reduct of $C[J]$, $a \in N_{R} C[J]$ iff $\Delta^C[J] \in I$. Thus $a \in Zd \, N_{R} C[J]$ iff $\Delta^C[J] \in I$ (by Thm. 3.1) $a \in C$. We conclude that $A$ and the neat $I$-reduct of $C[J]$ are LFA$_1$’s with the same zero-dimensional part, so that they are isomorphic by Cor. 3.2. It follows that $C[J]$ is a $J$-dilation of $A$. The uniqueness is again a consequence of Cor. 3.2.

**Theorem 5.2.** (Strong Functional Representation of Locally Finite LAA’s)
Every LFA$_1$ $A$ is isomorphic to a subalgebra of a functionally rich FLA$_1$.

**Proof:**
Let $B$ be the $J$-dilation of $A$ with $J \setminus I$ infinite. Then $A$ is a subalgebra of $Rd_I B$ (N$_{R} B$ to be precise) and $Rd_I B$ is isomorphic to a functionally rich FLA$_1$ by Thm. 5.1 and Lem. 5.4.

We conclude this section with an application of the above results to the classical lambda calculus.

A lambda theory $T_1$ is a conservative extension of a lambda theory $T_2$ if $T_2 \subseteq T_1$ and the restriction of $T_1$ to the language of $T_2$ coincides with $T_2$.

**Theorem 5.3.** Let $T$ be a lambda theory over $\Lambda_I(C)$ and let $D$ be an extension of $C$ such that $D \setminus C$ is infinite. Then $T$ can be uniquely extended in a conservative way to a lambda theory on $\Lambda_I(D)$ closed under the term rule.

**Proof:**
Let $A$ be the term LAA$_1$ of $T$ and let $J = I \cup (D \setminus C)$. Then $B = (Zd \, A)[J]$ is the unique $J$-dilation of $A$ up to isomorphism, $Rd_I B$ is a rich LFA$_1$ by Lem. 5.4, and $A$ is the neat $I$-reduct of $B$ up to isomorphism. We will show that $Rd_I B$ is the term LAA of a lambda theory $T_1$ on $\Lambda_I(D)$ satisfying the properties stated in the theorem. Consider the unique homomorphism $h$ from the absolutely free algebra $\Lambda_I(D)$ into $Rd_I B$ extending the identity map on $D$. The map $h$ is onto. Let $b \in B$, where $B$ is the universe of $B$. Since $B$ is locally

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$^3$A preliminary version of the strong functional representation theorem of locally finite LAA’s was presented in [31].
finite, we have that \( \Delta^B b \cap D = \{ y_1, \ldots, y_n \} \) is a finite subset of \( D \setminus C \). Then \( b = ay_1 \ldots y_n \), where \( a = \lambda y_1 \ldots y_n \). Let \( a = t^A \) for a lambda term \( t \) over \( \lambda T_1(C) \). Then \( b = (t y_1 \ldots y_n)^B \), so that every element of \( B \) is the interpretation of some lambda term over \( \lambda T_1(D) \). It follows that the map \( h \) is surjective. Define \( T_1 \) to be the congruence generated on \( \lambda T_1(D) \) by map \( h \). Then \( T_1 \) is a lambda theory closed under the term rule because \( \text{Rd}_B \text{d}_A \) is such a theory, and it is a conservative extension of \( T \) because the term \( \text{LAA}_T \) of \( T \) is the neat reduct of \( B \).

5.4. An equivalence theorem

In the next theorem we prove the equivalence of lambda models and environmental models; it is essentially the same as the combinatory model theorem in Meyer [23].

We begin by recalling that an environmental model can be given the structure of a combinatory algebra in a natural way; see Lem. 3.1 and Prop. 3.2.

**Theorem 5.4.** (i) Let \( C = \langle C, \cdot^C, k^C, s^C \rangle \) be a lambda model and define \( \lambda^C : C^C \to C \) by

\[
\lambda^C(u : v \in C) := 1^C u, \text{ for all } u \in C.
\]

Then \( \langle C, \cdot^C, \lambda^C \rangle \) is an environment model. Moreover, if we take \( V = \langle C, \cdot^C, \lambda^C \rangle \), then \( k^V = k^C \) and \( s^V = s^C \).

(ii) Conversely, let \( V = \langle V, \cdot^V, \lambda^V \rangle \) be an environmental model and define \( k^V \) and \( s^V \) to be the unique values that the constant functions \( (\lambda xy.x)^A \) and \( (\lambda xy.z)(yz) \) take in \( V \) for any total subalgebra \( A \) of \( V \). Then \( \langle V, \cdot^V, k^V, s^V \rangle \) is a lambda model. Moreover, if we take \( C = \langle V, \cdot^V, k^V, s^V \rangle \) and apply the construction of (i) we get \( \lambda^C = \lambda^V \).

**Proof:**

(i). \( C \) is the zero-dimensional subreduct of a rich \( \text{LFA}_A \ A \) (Thm. 4.2). Thus \( k^C = (\lambda xy.x)^A \), \( s^C = (\lambda xyz.z)(yz)^A \) and \( 1^C = (\lambda xy.xy)^A \). The representation Thm. 5.1 implies that the functional domain \( V = \langle V, \cdot^V, \lambda^V \rangle \) associated with the zero-dimensional elements of \( A \) is an environment model. We recall that \( V = \langle C, \cdot^V, \lambda^V \rangle \), the domain of \( \lambda^V \) is \( D_A \):

\[
\text{dom}(\lambda^V) = \{ \langle S^*_v(a) : v \in V \rangle : \lambda x^A.a \in Zd A \text{ and } x \in I \},
\]

and for each function in this set,

\[
\lambda^V(\langle S^*_v(a) : v \in V \rangle) := \lambda x^A.a.
\]

The equality \( V = \langle C, \cdot^C, \lambda^C \rangle \) follows if we prove \( \lambda^V = \lambda^C \). For every \( u \in C \), we have \( \langle uv : v \in C \rangle = \langle S^*_u(u) : v \in C \rangle \) and \( \lambda y^A.uy^A \in Zd A \) since \( u \in Zd A \). Moreover, \( 1^C u = (\lambda xy.xy)^A u = \lambda y^A.uy^A \). So \( \text{dom}(\lambda^C) \subseteq \text{dom}(\lambda^V) \) and \( \lambda^C \) and \( \lambda^V \) agree on \( \text{dom}(\lambda^C) \). It is easy to see that \( \text{dom}(\lambda^V) \subseteq \text{dom}(\lambda^C) \); hence, \( \lambda^V = \lambda^C \).

To complete the proof of (i) we must show that \( k^V = k^C \) and \( s^V = s^C \). We verify only the first equality; the verification of the second is similar. Recall that since \( V \) is an environment model, \( k^V \) is defined to be the constant value of the function \( k^V \), where \( V \).
is the $I$-coordinatization of $V$. Let $p \in V^I$ be arbitrary.

$$k^V = (\lambda xy. x)^{V^I}(p)$$

$$= \lambda^C(\lambda^C \langle v : u \in C \rangle : v \in C), \text{ by proof of Lem. 3.1}$$

$$= \lambda^C(\lambda^C \langle S^C_0(v) : u \in C \rangle : v \in C)$$

$$= \lambda^C(\lambda y^A, v : v \in C)$$

$$= (\lambda x y^A, x)$$

$$= (\lambda xy.x)^A$$

$$= k^C.$$

(ii). By Prop. 5.4 there exists at least a functionally rich FLA$_I$ $A$ with value domain $V$. Every functionally rich FLA$_I$ is a rich LAA$_I$ by the remark following Thm. 5.1. Thus $\text{Zd} A$ is a lambda model by Thm. 4.2. From Prop. 5.3 it follows that $\text{Zd} A$ is isomorphic to $\langle V, \langle V, k^V, s^V \rangle \rangle$, so that this last algebra is a lambda model.

We conclude the proof by showing that the domain of $\lambda^V$ is $\{ \langle uv : v \in V \rangle : u \in V \}$ and

$$\lambda^V(\langle uv : v \in V \rangle) = 1^V u.$$ 

We begin by proving every function of the form $\langle uv : v \in V \rangle$ is in the domain of $\lambda^V$. Choose any $p \in V^I$ such that $p_v = u$. Clearly $(\lambda x y^A)(p)$ is defined and

$$= (\lambda x y x)^A(p) = \lambda^V(\langle (xy)^A(p/y) \rangle : v \in V) = \lambda^V(\langle uv : v \in V \rangle).$$

Thus $\lambda^V(\langle uv : v \in V \rangle)$ is defined and equal to $1^V u$. Conversely, let $f$ be in the domain of $\lambda^V$. Then $f(v) = (\lambda^V f)v$ by the defining condition of a functional domain, and hence $f = \langle (\lambda^V f)v : v \in V \rangle$.

We show in the next section that the equivalence between environment models and lambda models actually gives rise to a categorical isomorphism.

5. A characterization of dimension-complemented LAA's

A LAA$_I$ is dimension-complemented if it contains no element whose dimension set is all of $I$. (The reader may consult [32] for an analysis of the results obtained for this class of algebras.) In this section we provide a characterization of dimension-complemented LAA's in terms of environment models.

Assume that $A$ is a LAA$_I$. We can associate with $A$ two different functional domains. The first one, $V = \langle V, \langle V, \lambda^V \rangle \rangle$, is called the functional domain associated with $A$ and was defined in Section 2 in the remarks preceding Thm. 2.1 (see also [32]) as follows: $V = A, \cdot V = \cdot A$, and $\lambda^V(\langle S^C_0(a) : v \in A \rangle) = \lambda A^A, a$ for all $a \in A$. The second one, $W = \langle W, \langle W, \lambda^W \rangle \rangle$, is called the functional domain associated with the combinatory reduct of $A$ and is defined as follows:

$W = A, \cdot W = \cdot A$, $\text{dom}(\lambda W) = \{ \langle a \cdot A, v : v \in A \rangle : a \in A \}$ and $\lambda W(\langle a \cdot A, v : v \in A \rangle) = 1^A \cdot A$. $W$ is well defined if the combinatory reduct of $A$ is a lambda model.

It follows from the functional representation theorem for locally finite LAA's and its proof (see Thm. 2.1 and the remarks following it) that the functional domain associated with each $\text{LFA}$ is an environment model. We now characterize the class of all LAA's with this property.

**Theorem 5.5.** Let $A$ be a LAA$_I$. The following conditions are equivalent.

(i) The functional domain $V$ associated with $A$ is an environment model;
(ii) \( \mathcal{A} \) is a dimension-complemented \( \text{LAA}_I \);
(iii) The combinatorial reduct of \( \mathcal{A} \) is a lambda model, and the functional domain associated with \( \mathcal{A} \) is equal to the functional domain associated with the combinatorial reduct of \( \mathcal{A} \).

Proof:
(i) \( \Rightarrow \) (ii). Assume, by way of contradiction that (i) holds and that there exists an element \( a \in A \) such that \( \Delta^\mathcal{A} a = I \). Let \( y \in I \) be a \( \lambda \)-variable and \( p \in V^I \) be any environment such that \( p_y = a \). Since \( V \) is an environment model the element \( \lambda x^{V^I} y^{V^I} \) must be well defined. Then

\[
\lambda x y(p) = \lambda y((y(p/v)) : v \in A) = \lambda y(p : v \in A) = \lambda y(a : v \in A).
\]

But the constant map \( \langle a : v \in A \rangle \) is not in the domain of \( \lambda V \). If it were, then there would exist \( b \in A \), \( z \in I \) such that \( \langle a : v \in A \rangle = \langle S^\mathcal{C} b : v \in A \rangle \). Thus \( a = S^\mathcal{C} b \) for all \( v \in A \). In particular, we can choose \( v = z^\mathcal{A} \). Then \( a = b \). This implies that \( a = S^\mathcal{C} (a) \) for all \( v \in A \), i.e., \( z \notin \Delta^\mathcal{A} a \). Contradiction.

(ii) \( \Rightarrow \) (iii). The combinatorial reduct of a dimension-complemented \( \text{LAA} \) is a lambda model. The proof of this result is essentially the same as the proof of Prop. 4.2 since, for every finite set \( a_1, \ldots, a_n \) of elements of a dimension-complemented \( \text{LAA} \), the set \( \Delta a_1 \cup \cdots \cup \Delta a_n \) is cofinite (see [32] Prop. 2.3). Thus the functional domain \( \mathcal{W} \) associated with the combinatory reduct of \( \mathcal{A} \) is an environment model (see Thm. 5.4(i)). We obtain the conclusion by showing that \( \mathcal{W} = V \). It is immediate that \( \text{dom}(\lambda V) \subseteq \text{dom}(\lambda \mathcal{W}) \). The converse is proved as follows. Let \( \langle av : v \in A \rangle \) be an element of the domain of \( \lambda \mathcal{W} \). Since \( \mathcal{A} \) is dimension-complemented there exists \( y \) independent of \( a \). If we show that the element \( ay \) is such that \( \langle av : v \in A \rangle = \langle S^\mathcal{C} (ay) : v \in A \rangle \), then the conclusion \( \text{dom}(\lambda \mathcal{W}) \subseteq \text{dom}(\lambda V) \) is obtained. In fact, we have

\[
S^\mathcal{C} (ay) = (\lambda y.ay)v = (\lambda y.a)v((\lambda y.y)v), \quad \text{by } (\beta_3)
\]

\[
= av, \quad \text{by } (\beta_1) \text{ and } y \notin \Delta^\mathcal{A} a.
\]

Finally, we have to show that \( \lambda V(f) = \lambda \mathcal{W}(f) \) for all \( f \in \text{dom}(\lambda V) = \text{dom}(\lambda \mathcal{W}) \). Assume \( f = \langle av : v \in A \rangle = \langle S^\mathcal{C} (b) : v \in A \rangle \) for some \( a, b \in A \). Then \( \lambda V(f) = \lambda y. b \) and \( \lambda \mathcal{W}(f) = 1a \). Let \( z \notin \Delta^\mathcal{A} a \). Since \( av = (\lambda z.az)v \) for all \( v \), then \( f = \langle av : v \in A \rangle = \langle S^\mathcal{C} (az) : v \in A \rangle = \langle S^\mathcal{C} (b) : v \in A \rangle \). Then from Lem. 3.9 in [32] it follows that \( \lambda V(f) = \lambda z.az = \lambda y.b \). Moreover, \( \lambda \mathcal{W}(f) = 1a = (\lambda x z.xz)a = \lambda z.az \) gives the conclusion.

(iii) \( \Rightarrow \) (i). Since the combinatory reduct of \( \mathcal{A} \) is a lambda model, it follows from Thm. 5.4(i) that the functional domain \( \mathcal{W} \) associated with the combinatory reduct of \( \mathcal{A} \) is an environment model. The conclusion is now immediate.

Theorem 5.6. If \( \mathcal{A} \) is a \( \text{LAA} \) with \( 2 \leq |I| < \omega \), then the functional domain \( V \) associated with \( \mathcal{A} \) is not an environment model.

Proof:
There exist no dimension-complemented \( \text{LAA} \)'s of finite dimension (see items (2.1) and (2.2) in the proof of Prop. 2.3 in [32]).

Thus, if the dimension of \( \mathcal{A} \) is finite, \( \mathcal{A} \) cannot be represented by using its associated functional domain.
6. Domain Morphisms and Domain Homomorphisms

Given FLA$_I$'s $A$ and $B$, there need not be a natural correlation between the LAA$_I$ homomorphisms from $A$ to $B$ and mappings between the respective value domains. But if $A$ and $B$ are locally finite and functionally rich, then the categorical equivalence of LFA$_I$'s and their zero-dimensional parts, together with the fact that there are natural isomorphisms between the zero-dimensional parts and the respective value domains, gives rise to a strong natural connection. It is this connection that we investigate in this section. It leads to a stronger categorical equivalence for $\lambda$-algebras than the one given in the previous section. In the last part of the section we investigate simple LAA$_I$'s and prove a representation result that corresponds roughly to the representation theorem for simple, locally finite polyadic Boolean algebras ([16, Thm. (17.3)]).

Let $V, W$ be sets. Every mapping $h : V \to W$ induces a mapping $h^* : V^V \to \mathcal{P}(W \times W)$ from functions on $V$ to relations on $W$ in the obvious way: $h^*(f) = h \circ f \circ h^{-1}$ for every $f \in V^V$. More precisely, $(w, w') \in h^*(f)$ iff there exists a $v \in V$ such that $h(v) = w$ and $h(f(v)) = w'$. $h^*(f)$ is a function iff $h(f(v)) = h(f(v'))$ for all $v, v' \in V$ such that $h(v) = h(v')$. In this event we use standard functional notation for $h^*(f)$ and write $h^*(f)(w')$ for the unique $w' \in W$ such that $h(f(v)) = w'$ for all $v \in V$ such that $h(v) = w$.

**Definition 6.1.** Let $V = \langle V, \cdot, \lambda^V \rangle$ and $W = \langle W, \cdot, \lambda^W \rangle$ be functional domains. A mapping $h : V \to W$ is a domain morphism if it satisfies the following conditions:

(i) $h$ is surjective, i.e., maps $V$ onto $W$.

(ii) $h(v \cdot v') = h(v) \cdot h(v')$, for all $v, v' \in V$.

(iii) $h^*$ maps $dom(\lambda^V)$ onto $dom(\lambda^W)$.

(iv) $h(\lambda^V(f)) = \lambda^W(h^*(f))$, for all $f \in dom(\lambda^V)$.

Note that implicit in condition (iii) is the assumption that $h^*(f)$ is a function for every $f \in dom(\lambda^V)$.

If $h : V \to W$ and $g : W \to U$ are domain morphisms it is easy to check that $(g \circ h)^* = g^* \circ h^*$ and that $g \circ h : V \to U$ is a domain morphism. Since the identity map is clearly a domain morphism, the functional domains and domain morphisms form a category.

Let $V$ and $W$ be sets. Recall that $V_1 = \{ f : V^l \overset{p}{\longrightarrow} V \}$. Every mapping $h : V \to W$ induces a mapping $h^*_l : V_1 \to \mathcal{P}(W^l \times W^l)$ in the obvious way: let $a \in V^l \overset{p}{\longrightarrow} V$. Then, for each $\langle q, w \rangle \in W^l \times W$, $\langle q, w \rangle \in h^*_l(a)$ iff there exists a $p \in V^l$ such that $h \circ p = q$ and $h(a(p)) = w$.

We write $a : V^l \to V$ to indicate that $a$ is a total function from $V^l$ to $V$.

**Lemma 6.1.** Let $h : V \to W$ be a domain morphism.

(i) $h(f(v)) = (h^*(f))(h(v))$, for all $f \in dom(\lambda^V)$ and $v \in V$.

(ii) If $a : V^l \to V$ (i.e., $a$ is a total function) and $h^*_l(a) : W^l \overset{p}{\longrightarrow} W$, then $h^*_l(a) : W^l \to W$. Moreover,

$$h^*_l(a)(q) = h(a(p))$$

for any $p \in V^l$ such that $q = h \circ p$.

**Proof:**

(i) follows immediately from the hypothesis that $h^*(f)$ is a function for all $f \in dom(\lambda^V)$.

(ii) is obvious.

Let $V$ and $W$ be functional domains and $h : V \to W$ a domain morphism. We say that a map $a : V^l \to V$ is compatible with $h$ if the following property holds, for all $p, p' \in V^l$:

$$h \circ p = h \circ p' \Rightarrow h(a(p)) = h(a(p')).$$
Lemma 6.2. Let \( V \) and \( W \) be functional domains and \( h : V \to W \) a domain morphism. Let \( a \in A \). If \( a : V^I \to V \) and \( a \) is compatible with \( h \), then \( h^*_t(a) : W^I \to W \).

Proof:
Assume \( a : V^I \to V \), i.e., \( a \) is a total function from \( V^I \) to \( V \). Assume also that \( a \) is compatible with \( h \). Let \( p, p' \in V^I \) such that \( h \circ p = h \circ p' \). To show \( h^*_t(a) : W^I \to W \) it suffices to show that \( h(a(p)) = h(a(p')) \). But this follows immediately from the compatibility of \( a \) with \( h \) since the assumption that \( a \) is a total function implies \( h \circ p = h \circ p' \).

Theorem 6.1. Let \( V \) and \( W \) be functional domains and let \( h : V \to W \) be a domain morphism. If \( V \) is an environment model, then so is \( W \). More precisely, for every regular \( \text{FLA}_I \) \( A \) with value domain \( V \), \( h^*_t \) is a \( \text{LAA}_I \) homomorphism from \( A \) onto a total subalgebra of \( W_I \).

Proof:
It is easy to check that the set of all regular elements of any \( \text{FLA}_I \) \( A \) form a regular subalgebra of \( A \). Thus a functional domain \( V \) is an environment model iff there is a regular \( \text{FLA}_I \) with value domain \( V \). So by Prop. 2.6 the first part of the theorem follows from the second part.

Let \( a \in A \). Since \( A \) is locally finite, there exists at least a zero-dimensional element \( c \) such that \( a = cx_1^{V_I} \cdots x_n^{V_I} \) for every enumeration \( x_1 \cdots x_n \) of \( \Delta a \). Let \( p, p' \in V^I \) such that \( h \circ p = h \circ p' \). Then we have

\[
\begin{align*}
h(a(p)) &= h\left( (cx_1^{V_I} \cdots x_n^{V_I})(p) \right) \\
&= h\left( (c(p)x_1^{V_I}(p) \cdots x_n^{V_I}(p)) \right) \\
&= h\left( (c(p)p_{x_1} \cdots p_{x_n}) \right) \\
&= h(c(p))h(p_{x_1}) \cdots h(p_{x_n}), \quad \text{by Def. 6.1(ii)} \\
&= h(c(p'))h(p'_{x_1}) \cdots h(p'_{x_n}), \quad \text{by Prop. 5.2 and by } h \circ p = h \circ p' \\
&= h(a(p')).
\end{align*}
\]

Thus all the elements of \( A \) are compatible with \( h \).

We now prove the remaining part of the theorem. By Lem. 6.2 and by the previous conclusion, \( h^*_t(A) := \{ h^*_t(a) : a \in A \} \) is a set of total functions from \( W^I \) to \( W \). Thus to show that \( h^*_t(A) \) is the universe of a total subalgebra \( h^*_t(A) \) of \( W_I \) and that \( h^*_t : A \to h^*_t(A) \) is a surjective \( \text{LAA}_I \) homomorphism, it suffices to prove that \( h^*_t \) preserves application, lambda abstraction, and lambda variables.

Let \( a, b \in A \). Let \( q \in V^I \) and choose \( p \in V^I \) such that \( h \circ p = q \) (\( q \) exists because \( h \) is surjective).

\[
\begin{align*}
h^*_t(a \cdot V^I b)(q) &= h\left((a \cdot V^I b)(p)\right) \\
&= h\left((a(p) \cdot V b(p))\right) \\
&= h(a(p)) \cdot W h(b(p)) \\
&= h^*_t(a)(q) \cdot W h^*_t(b)(q), \quad \text{by Lem. 6.1(ii)} \\
&= (h^*_t(a) \cdot W^I h^*_t(b))(q).
\end{align*}
\]

Thus

\[
h^*_t(a \cdot V^I b) = h^*_t(a) \cdot W^I h^*_t(b), \quad \text{for all } a, b \in A.
\]

To show that \( h^*_t \) preserves lambda abstraction, we first verify that

\[
h^*_t(a(p[v/x])) : v \in V = h^*_t(a(q[w/x])) : w \in W, \quad \text{if } h \circ p = q.
\]
For each \( w \in W \) choose a \( v \in V \) such that \( h(v) = w \). Then

\[
(h^*(a(p\{v/x\})) : v \in V)(w) = h(\langle a(p\{v/x\}) : v \in V \rangle(v)),
\]
by Lem. 6.1(i)

\[= h(a(p\{v/x\})) \]
\[= h^*(a)(q\{w/x\}), \]
by Lem. 6.1(ii) since \( h(v) = w \) and \( h \circ p = q \) imply \( h \circ p\{v/x\} = q\{w/x\}, \)

\[= \langle h^*(a)(q\{w/x\}) : w \in W \rangle(w). \]

This verifies (5). Recalling that \( h \circ p = q \) we have

\[
(h^*(\lambda x V^i . a))(q) = h(\langle \lambda x V^i . a \rangle(p)), \quad \text{by Lem. 6.1(ii)}
\]
\[= h(\lambda x V^i . a)(p) \]
\[= \lambda W(\langle h^*(a(p\{v/x\}) : v \in V \rangle) \]
\[= \lambda W(\langle h^*(a)(q\{w/x\}) : w \in W \rangle), \quad \text{by (5)}
\]
\[= (\lambda x W^i . h^*(a))(q). \]

So

\[h^*(\lambda x V^i . a) = \lambda x W^i . h^*(a), \quad \text{for all } a \in A.\]

Finally, \((h^*(x V^i))(q) = h(x V^i(p)) = h(p_x) = q_x = x W^i(q)\). Thus

\[h^*(x V^i) = x W^i, \text{ for all } x \in I.\]

Let \( C = \langle C, \cdot^C, k^C, s^C \rangle \) and \( D = \langle D, \cdot^D, k^D, s^D \rangle \) be combinatory algebras. Recall that a mapping \( h : C \to D \) is a combinatory homomorphism if \( h(c \cdot^C c') = h(c) \cdot^D h(c') \), for all \( c, c' \in C \), \( h(k^C) = k^D \), and \( h(s^C) = s^D \). Recall also the equivalence between environment models and lambda models established in Thm. 5.4.

**Theorem 6.2.** (i) Every domain morphism between environment models \( V \) and \( W \) is a surjective combinatory homomorphism between the lambda models \( \langle V^*, V, k^V, s^V \rangle \)
and \( \langle W^*, W, k^W, s^W \rangle \).

(ii) Conversely, every surjective combinatory homomorphism between lambda models \( C \)
and \( D \) is a domain morphism between the corresponding environment models \( \langle C^*, \cdot^C, \lambda^C \rangle \)
and \( \langle D^*, \cdot^D, \lambda^D \rangle \).

**Proof:**

(i): Let \( h : V \to W \) be a domain morphism. Since \( V \) is an environment model, \( W \) is also one by Thm. 6.1. \( h \) is an applicative homomorphism by definition. Since \( k \) and \( s \) are uniquely determined in a lambda model by \( 1 \) ([3, Prop. 5.6.6(i)]), it suffices to prove that \( h(1^V) = 1^W \).

Let \( f \in \text{dom}(\lambda^V) \) and let \( u \in V \) such that \( f = \langle uv : v \in V \rangle \). For each \( w \in W \), choose \( v \in V \) such that \( h(v) = w \). Then \( h^*(f)(w) = h(f(v)) = h(uv) = h(u)h(v) = h(u)w \) for all \( w \in W \). Thus

\[h^*(f) = \langle h(u)w : w \in W \rangle.\]

Now \( \lambda^V(f) = 1^V u \) and \( \lambda^W(h^*(f)) = 1^W h(u) \). So

\[h(1^V u) = h(\lambda^V(f)) = \lambda^W(h^*(f)) = 1^W h(u), \quad \text{for all } u \in V. \quad (6)\]
But, since \( h \) is an applicative homomorphism, \( h(1^Vu) = h(1^V)h(u) \), for all \( u \in V \). Thus, since \( h \) is surjective, \( 1^Ww = h(1^V)w \), for all \( w \in W \). Now applying the Meyer-Scott axiom and making use of Cor. 4.6 and \((6)\) we get
\[
1^W = 1^W_1^W = 1^W_h(1^V) = h(1^V 1^V) = h(1^V).
\]

(ii): Let \( h \) be a surjective combinatory homomorphism between lambda models \( C \) and \( D \). Let \( f \in \text{dom}(\lambda^V) \). Then
\[
f = \{ uv : v \in V \} \text{ for some } u \in V. \tag{7}\]

If \( h(v) = h(v') \), then \( h(f(v)) = h(uv) = h(u)h(v) = h(u)h(v') = h(f(v')) \). So \( h^*(f) \) is a function. Moreover, since \( h \) is surjective, there exists for each \( w \in W \) a \( v \in V \) such that \( h(v) = w \), and hence \( h^*(f)(w) = h(f(v)) = h(uv) = h(u)h(v) = h(u)w \). So
\[
h^*(f) = \{ h(u)w : w \in V \}, \tag{8}\]
and \( h^*(f) \in \text{dom}(\lambda^W) \). Clearly \( h^* \) maps \( \text{dom}(\lambda^V) \) onto \( \text{dom}(\lambda^W) \) since \( h \) is surjective.

It remains to show that \( \lambda^W(h^*(f)) = h(\lambda^V(f)) \). Note that from \((7)\) and \((8)\) we have \( \lambda^V(f) = 1^V_u \) and \( \lambda^W(h^*(f)) = 1^W_h(u) \). But \( h(\lambda^V(f)) = h(1^V_u) = h(1^V)h(u) = 1^W_h(u) \).

**Definition 6.2.** Let \( A \) and \( B \) be regular \( \text{FLA}_f \)'s with value domains \( V \) and \( W \), respectively. A surjective homomorphism \( h : A \to B \) is called a *domain homomorphism* if there exists a domain morphism \( g : V \to W \) such that the restriction of \( h = g_f \).

Domain homomorphisms correspond roughly to base homomorphisms in the theory of cylindric algebras. See [18, Part II, Def. 3.1.50].

**Theorem 6.3.** Every surjective homomorphism between functionally rich, locally finite \( \text{FLA}_f \)'s is a domain homomorphism.

**Proof:**

Let \( A \) and \( B \) be functionally rich, locally finite \( \text{FLA}_f \)'s with value domains \( V \) and \( W \), respectively. Let \( h : A \to B \) be a surjective homomorphism. Let \( g : V \to W \) be defined by \( g(v) = w \) if \( h(v) = w \). \( g \) is surjective combinatory homomorphism from \( V \) and \( W \) by Prop. 5.3 and hence a domain morphism by Thm. 6.2. We show that \( h = g_f \).

By Cor. 3.2 it is sufficient to show that \( h \) and \( g_f \) have the same restriction to \( \text{Zd} A \). Let \( a \in \text{Zd} A \), and note that, if \( v \) is the unique element of \( V \) such that \( a \equiv v \), then \( h(a) = g(v) \).

Consider any \( q \in W^f \) and choose \( p \in V^f \) such that \( q = g \circ p \). Then by Lem. 6.1(iii), \( g_f(a)(q) = g(a(p)) = g(v) = g(v) \cap (q) = h(a)(q) \). Thus \( h = g_f \).

Recall that the functional domains and domain morphisms form a category. The environmental models constitute a full subcategory.

**Theorem 6.4.** The following categories are equivalent:

(i) environment models and domain morphisms;
(ii) lambda models and surjective combinatory homomorphisms;
(iii) functionally rich, locally finite \( \text{FLA}_f \)'s and domain morphisms;
(iv) rich \( \text{LFA}_f \)'s and surjective homomorphism.

**Proof:**

The equivalence (in fact isomorphism) between the categories of environment models and lambda models follows immediately from Thms. 5.4 and 6.2. The equivalence of lambda models and rich \( \text{LFA}_f \)'s was established in Cor. 4.1. Finally, the equivalence of functionally rich, locally finite \( \text{FLA}_f \)'s and rich \( \text{LFA}_f \)'s is an easy consequence of the representation theorem for rich \( \text{LFA}_f \)'s (Thm. 5.1) and Thm. 6.2.
6.1. Simple lambda abstraction algebras

We recall that an algebra is simple if it is nontrivial (i.e., contains more than one element) and admits only the two trivial congruences. It is easy to see that simple LAA’s exist. For this purpose let A be any nontrivial LAA. Any two distinct lambda-variables in A must have distinct interpretations in A, i.e., \( x^A \neq y^A \) if \( x, y \in I \) and \( x \neq y \). For suppose otherwise. Then for every \( a \in A, a = S_a(x) = S_a(y) = y \), contradicting the assumption that A is nontrivial. Thus for any congruence \( \Theta \) on A that is maximal with respect to not identifying \( x^A \) and \( y^A \), the quotient \( A/\Theta \) is simple.

Since the quotient of a LFA is again locally finite, there exist simple LFA’s and hence, by the representation theorem, simple FLA’s. By the equivalence of lambda-algebras and LFA’s we see that simple lambda-algebras exist and can be characterized as the zero-dimensional subalgebras of simple LFA’s.

Proposition 6.1. Simple rich LFA’s exist.

Proof:
We recall from Barendregt [3, Def. 16.1.1] that a lambda theory is sensible if it contains the following equations between closed lambda-terms \( \{ t = s : t, s \text{ unsolvable} \} \), where a closed lambda-term \( u \) is unsolvable if there exist no lambda-terms \( t_1, \ldots, t_n \) such that \( ut_1 \cdots t_n = \lambda x. x \). It turns out that sensible lambda theories admit a unique maximal extension \( H^* \) ([3, Thm. 16.2.6]). \( H^* \) is a lambda theory satisfying the \( \omega \)-rule ([3, Def. 4.1.10]). But, every lambda theory satisfying the \( \omega \)-rule satisfies also the term rule [3, Lem. 4.1.12]. So the term LAA of \( H^* \) satisfies the conclusion of the proposition.

A more interesting problem is whether every rich LFA admits a simple, rich LFA as a homomorphic image; we feel this is highly unlikely.

We conclude the paper with one more representation result.

We now consider the property of a congruence relation on a LAA that guarantees that associated quotient algebra is rich.

Definition 6.3. Let C be a combinatory algebra. A congruence \( \Theta \) on C is functional if it satisfies the following condition for all \( a, b \in C \).

\[
(\forall c \in C : ac \Theta bc) \Rightarrow 1a \Theta 1b.
\]

Note that if C is a lambda-algebra, i.e., \( C = Zd A \) for some LAA A, then \( 1 = \lambda y x. y x \) and the above condition becomes

\[
(\forall c \in C : ac \Theta bc) \Rightarrow \lambda x.a x \Theta \lambda x.b x.
\]

If C is a lambda-algebra, then \( \Theta \) is a functional congruence iff the quotient algebra \( C/\Theta \) satisfies the Meyer-Scott axiom and hence is a lambda model. (Recall that lambda models are lambda-algebras satisfying the Meyer-Scott axiom; Thm. 4.5.) In particular the identity congruence on a lambda-algebra C is functional iff C is a lambda model.

Theorem 6.5. (The Representation Theorem for Simple, Locally Finite LAA’s). Let A be a simple LFA and let A be isomorphically embedded in a functionally rich FLA B. Then, for every functional congruence \( \Theta \) on the value domain V of B, A can be isomorphically embedded in a functionally rich FLA with value domain V/\( \Theta \).

Proof:
\( \langle V, V, k^V, s^V \rangle \) is isomorphic to Zd B under the mapping \( v \mapsto \hat{v} \) (Prop. 5.3). Moreover, \( \langle V, V, k^V, s^V \rangle \) is equivalent to the environment model \( V \) in a strong categorical sense (Thms. 5.4, 6.2). Thus, identifying Zd B and V, it makes sense to think of \( \Theta \) as a congruence on V and the natural map \( h : V \rightarrow V/\Theta \) as a domain morphism.

Since \( \Theta \) is functional, \( V/\Theta \) is a lambda model and \( h^\ast \) is a domain homomorphism from B onto a functionally rich FLA C with value domain V/\( \Theta \) (Thm. 6.1). Let \( g \) be the restriction of \( h^\ast \) to A. Then \( g \) is a homomorphism from A onto a subalgebra of C. Since A is simple, it follows that g is an isomorphism.
This appears to be the closest we can get to an true analogue of the main representation result for polyadic Boolean algebras ([16, Thm. (17.3)]). According to the latter theorem every simple, locally finite polyadic Boolean algebra is isomorphic to a functional algebra over the two-element Boolean algebra.

If the open problem mentioned above has a positive solution, one would have as a corollary of Thm. 6.5 that every simple LFA$_1$ is isomorphic to a functionally rich simple FLA$_1$. This would be a valuable result for it would give us a considerable insight into the structure of simple, LFA’s. It would also give a true analogue of ([16, Thm. (17.3)]).

7. Conclusions

The way in which lambda abstraction theory arises from the lambda calculus almost exactly parallels the way cylindric algebras are obtained from first-order logic. The axioms of first-order logic are like those of lambda calculus in that the formula variables cannot be substituted without restriction. In both cases the source of the problem is the way substitution for individuals is handled. By dealing with substitution at the level of the object language rather than the metalanguage, i.e., by abstracting it, a pure equational formalization of lambda calculus can be developed giving rise to the theory of LAA’s. This abstraction of substitution is a characteristic feature of algebraic logic. Like cylindric algebras, and in contrast to the lambda calculus, the axioms of lambda abstraction theory are pure identities (more accurately, they turn out to be equivalent to pure identities). The key idea is the use of (β) conversion as the basis of a definition of abstract substitution:

$$S^\gamma_\beta(a) := (\lambda x.a)b$$

so that, among the seven axioms of lambda abstraction theory, the first six constitute a recursive definition of the substitution operators $S^\gamma_\beta$; they express precisely the metamathematical content of (β) conversion. The last axiom is an algebraic translation of (α) conversion.

Lambda abstraction theory has been extensively developed by the authors and Goldblatt in a series of papers [29, 30, 31, 32, 13, 14]. The axiomatization of functional lambda abstraction algebras is the central issue in the algebraic approach to the model theory of lambda calculus. In [32, Thm. 7.7] the present authors proved that the smallest variety of LAA$_1$’s that includes the functional LAA$_1$’s can be characterized as the class of algebras isomorphic to a certain kind of generalized FLA$_1$’s called point-relativized functional LAA$_1$’s (RFA$_1$, for short). The RFA$_1$’s turn out to be (up to isomorphism) exactly the LAA$_1$’s that can be neatly embedded in a LAA$_1$ of infinite higher dimension ([32, Thm. 7.4]). In the same paper the authors stated the open problem if IFLA$_1$ is also a variety and hence coincides with I RFA$_1$. The conjecture was proven true by Goldblatt [13] in June 1995; he proved that any RFA$_1$ is isomorphic to an FLA$_1$. Later in joint work Salibra and Goldblatt [40] used this result to prove that the variety IFLA$_1$ of algebras isomorphic to functional LAA$_1$’s is axiomatisable by the finitely many equational schemes defining LAA$_1$’s. More precisely, in [40] the result that every LAA$_1$ is isomorphic to an FLA$_1$ is established by showing that every LAA$_1$ is isomorphic to an RFA$_1$, and hence to an FLA$_1$.

We have also shown in [32] that, for every environment model V, there exists a (unique) largest possible FLA over V. These algebras are called full functional LAA’s. A natural notion of a full point-relativized functional LAA can also be defined and there is a corresponding existence result. Lambda models can be also characterized (up to isomorphism) as the zero-dimensional parts of full point-relativized functional LAA’s.

7.1. Connections with Other Work

There have been several attempts to reformulate the lambda calculus as a purely algebraic theory within the context of category theory: Obtulowicz and Wieger [28] via the algebraic
theories of Lawvere; Adachi [2] via monads; Curien [7] via categorical combinators. There have also been several works that present an algebraic theory of the lambda calculus very close to lambda calculus in spirit. Locally finite functional LAA’s are very similar to the functional models of the lambda calculus developed in Krivine [21]. However, Krivine’s models do not have an explicit algebraic structure. An abstractly defined class of algebras, called lambda term systems, that is even closer in spirit to LAA has recently been introduced by Diskin [9].

Lambda abstraction algebras do for lambda calculus what cylindric (and polyadic) algebras do for first-order predicate calculus. The theories of cylindric and polyadic algebras are two early contributions to the algebraization of quantifier logics and have greatly influenced our work. The main references for cylindric algebras are [18] and [19]; for polyadic algebras it is [17]; see in particular [16]. We also mention here Nemeti [24]. It contains an extensive survey of the various algebraic versions of quantifier logics. LAA’s, like cylindric and polyadic algebras, can be also viewed as a contribution to the theory of abstract substitution. However, in lambda abstraction and cylindric algebras, abstract substitution is a defined operation, while in polyadic algebras it is one of the primitive notions. The importance of abstract substitution, and lambda abstraction, has been recognized for some time among computer scientists because it leads among other things to more natural term rewriting systems, which are useful in the analysis of processes of computations. See for example [1]. In the transformation algebras and substitution algebras of LeBlanc [22] and Pinter [34] substitution is primitive and abstract quantification is defined in terms of it. A pure theory of abstract substitution has been developed by Feldman [11],[12]. This work parallels ours in many respects and we acknowledge our indebtedness to it.

Finally, we mention some recent work of ours connecting a theory of substitution in combination with abstract variable-binding operators has been recently done. See [33],[38].

References


