

Boolean Algebras and Lambda Calculus

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- What is lambda calculus?

- A theory of functions
- The name of a function contains a description of the function as a program
- Untyped world: every element in lambda calculus is contemporaneously
 - * Function
 - * A possible argument for a function
 - * A possible result of the application of a function to an argument
- No Partiality: every function can be applied to any other function including itself

Lambda terms

- λ -notation:
Expression: $a + 2$ Function: $f(a) = a + 2$ $\lambda_a(a + 2)$
- Algebraic similarity type Σ :
 - Nullary operators: $a, b, c, \dots \in A$ (formal parameters)
 - Binary operator: \bullet (application)
 - Unary operators: $\lambda_a (a \in A)$ (λ -abstractions)
- A **λ -term** is a ground Σ -term (no algebraic variable x, y, z, \dots)

$\lambda_a(a)$ YES $\lambda_a(x)$ NO

- $a =$ generic function
- $M \cdot N =$ function M applied to argument N
- $\lambda_a(M) =$ function of a whose body is expression M

How to compute (informally)

- Bound and free parameters: $\lambda_a(a \cdot b)$

- α -conversion: $\lambda_a(a \cdot b) = \lambda_c(c \cdot b)$

The name of a bound parameter does not matter

- β -conversion: $\lambda_a(a) \cdot b = b$

$$\lambda_a(aa) \cdot \lambda_a(aa) = \lambda_a(aa) \cdot \lambda_a(aa) = \dots$$

The classic λ -calculus

- The λ -term algebra is the absolutely free Σ -algebra over an empty set of generators:

$$\Lambda = (\wedge, \cdot, \lambda_a, a)_{a \in A}$$

The object of study of λ -calculus is any congruence on Λ (called λ -theory) including α - and β -conversion:

- β -conversion:

$$\lambda_a(M) \cdot N = M[N/a]$$

$M[N/a]$ is a “meta-operation” defined by induction over M .

- α -conversion:

$$\lambda_a(M) = \lambda_b(M[b/a]) \quad (b \text{ not free in } M)$$

- The lattice of λ -theories \equiv The congruence lattice of $\Lambda/\lambda\beta$
($\lambda\beta$ is the least congruence on Λ including α - and β -conversion)

Is the untyped λ -calculus algebraic? YES

- CA combinatory algebras (Curry-Schönfinkel)
- LAA lambda abstraction algebras (Pigozzi-S. 1993)

Theorem 1 (S. 2000)

1. $\text{Variety}(\Lambda/\lambda\beta) = \text{LAA}$.
2. $\text{Lattice of } \lambda\text{-theories} = \text{Lattice of eq. theories of LAAs}$.
 $\lambda\text{-theory } T \Leftrightarrow \text{variety generated by the term algebra of } T$.

Are CAs and LAAs good algebras?

The properties of a variety \mathcal{V} of algebras are usually studied through the lattice identities satisfied by the congruence lattices of all algebras in \mathcal{V}

Some negative algebraic results

Theorem 2 (Lusin-S. 2004) *Every nontrivial lattice identity fails in the congruence lattice of a suitable LAA (CA).*

Conclusion: We cannot apply thirty years of Universal Algebra to LAA (CA)!

Lambda calculus was introduced around 1930 by Alonzo Church as part of a foundational formalism of mathematics and logic based on functions as primitive. After some years this formalism was shown inconsistent. Why?

Theorem 3 *Classic logic is inconsistent with combinatory logic.*

Proof: The variety of Boolean algebras is congruence permutable. Plotkin and Simpson have shown that the Malcev conditions for congruence permutability are inconsistent with combinatory logic.

Theorem 4 *The implication fragment of classic logic is inconsistent with combinatory logic.*

Proof: An implication algebra is 3-permutable. Plotkin and Selinger have shown that the Malcev conditions for congruence 3-permutability are inconsistent with combinatory logic.

We should be pessimistic!

Boolean algebras for λ -calculus

- Let \mathbf{A} be any algebra. There exists a bijective correspondence between:
 - Pairs (ρ, ρ') of **complementary factor congruences**: $\rho \cap \rho' = \Delta$; $\rho \circ \rho' = \nabla$
 - **Factorizations** $\mathbf{A} = \mathbf{A}/\rho \times \mathbf{A}/\rho'$.
 - **Decomposition operations** $f : A \times A \rightarrow A$ defined by

$$f(x, y) = u \text{ iff } x\rho u\rho'y.$$

- Let $\mathbf{t} \equiv \lambda_a(\lambda_b(a))$ and $\mathbf{f} \equiv \lambda_a(\lambda_b(b))$.

$$(\mathbf{t}x)y = x; \quad (\mathbf{f}x)y = y.$$

(The least reflexive compatible relation on the term algebra $\Lambda/\lambda\beta$ including $\mathbf{t} = \mathbf{f}$ is trivial)

- We have for a pair (ρ, ρ') of complementary factor congruences:

$$\mathbf{t}\rho e\rho'\mathbf{f} \Rightarrow (\mathbf{t}x)y \rho (ex)y \rho' (\mathbf{f}x)y \Rightarrow x\rho (ex)y \rho'y.$$

$$f(x, y) = (ex)y$$

The Boolean algebra of central elements

Definition 1 Let \mathbf{A} be an LAA (CA). We say an element $e \in A$ is central when it satisfies the following equations, for all $x, y, z, v \in A$:

- (i) $(ex)x = x$.
- (ii) $(e((ex)y))z = (ex)z = (ex)((ey)z)$.
- (iii) $(e(xy))(zv) = ((ex)z)((ey)v)$.
- (iv) $e = (et)f$.

- e is central $\Leftrightarrow \mathbf{A} = \mathbf{A}/\theta(t, e) \times \mathbf{A}/\theta(f, e)$
- \mathbf{A} is directly indecomposable iff t, f are the unique central elements.

Theorem 5 Let \mathbf{A} be an LAA (CA). Then the algebra $(C(\mathbf{A}), \wedge, ^-)$ of central elements of \mathbf{A} , defined by

$$e \wedge d = (et)d; \quad e^- = (ef)t,$$

is a Boolean algebra.

Proof: LAAs have skew factor congruences \Rightarrow Factor congruences are a Boolean sublattice of $\text{Con}(\mathbf{A})$.

The Stone representation theorem

Theorem 6 *Let \mathbf{A} be an LAA (or a CA) and I be the Boolean space of maximal ideals of the Boolean algebra of central elements. Then the map*

$$f : A \rightarrow \prod_{i \in I} (A / \cup i),$$

defined by

$$f(x) = (x / \cup i : i \in I),$$

*gives a **weak** Boolean product representation of \mathbf{A} , where the quotient algebras $\mathbf{A} / \cup i$ are directly indecomposable.*

Proof: From a theorem by Vaggione.

Central elements at work

The directly indecomposable LAAs (CAs) (there exist a lot of them!) are the building blocks of LAA (CA).

How to use central elements and directly indecomposable LAAs (CAs) to get results on lambda calculus?

- [Church \(around 1930\)](#): Lambda calculus
- [Scott \(1969\)](#): First model
- [Meyer-Scott \(around 1980\)](#): There exists a first-order axiomatization of what is a model of λ -calculus as a particular class of CAs.

$$\mathcal{D} \text{ model} \Rightarrow \text{Th}(\mathcal{D}) = \{M = N : M \text{ and } N \text{ have the same interpretation}\}$$

- [Scott Semantics and its refinements \(1969-2007\)](#) A Scott topological space \mathcal{D} and two Scott continuous maps

$$i : \mathcal{D} \rightarrow [\mathcal{D} \rightarrow \mathcal{D}]; \quad j : [\mathcal{D} \rightarrow \mathcal{D}] \rightarrow \mathcal{D}; \quad i \circ j = id_{[\mathcal{D} \rightarrow \mathcal{D}]}$$

- A semantics \mathcal{C} of lambda calculus is [incomplete](#) if there exists a consistent λ -theory T s.t.

$$T \neq \text{Th}(\mathcal{D}), \text{ for all models } \mathcal{D} \in \mathcal{C}.$$

Central elements at work

Theorem 7 *The semantics of lambda calculus given in terms of directly indecomposable models (this includes Scott Semantics and its refinements) is incomplete.*

Proof:

1. CA_{di} 's is a universal class $\Rightarrow \text{CA}_{di}$ is closed under subalgebras \Rightarrow the directly decomposable CAs are closed under expansion.
2. The lambda theory T generated by $\lambda_a(aa) \cdot \lambda_a(aa) = \mathbf{t}$ is consistent.
3. The lambda theory S generated by $\lambda_a(aa) \cdot \lambda_a(aa) = \mathbf{f}$ is consistent.
4. $\lambda_a(aa) \cdot \lambda_a(aa)$ is a nontrivial central element in the term model of $T \cap S$
5. All the models of $T \cap S$ are directly decomposable.

Central elements at work

Theorem 8 *For every r.e. lambda theory T , the lattice interval $[T) = \{S : T \subseteq S\}$ contains a continuum of “decomposable” lambda theories.*

Theorem 9 *The set of lambda theories representable in EACH of the following semantics is not closed under finite intersection, so that it does not constitute a sublattice of the lattice of lambda theories:*

- *Graph models*
- *Filter models*
- *Continuous models*
- *Stable models.*

Finite Boolean Sublattices

T coatom containing $\Omega = \lambda xy.x$

S coatom containing $\Omega = \lambda xy.y$

Ω is nontrivial central in $T \cap S$

The interval $[T \cap S] = \{T \cap S, T, S, 1\}$

