Algebra and Topology in Lambda Calculus

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Lambda calculus

- Lambda calculus was introduced around 1930 by A. Church as part of an investigation in the formal foundations of mathematics and logic.

- Why is lambda calculus important?

  - Full computational formalism

  - Functional programming languages are based on λ-calculus

- What is lambda calculus?
  - A theory of functions
  - The name of a function contains a description of the function as a program
  - Untyped world: every element in lambda calculus is contemporaneously
    - Function
    - A possible argument for a function
    - A possible result of the application of a function to an argument
  - No Partiality: every function can be applied to any other function including itself

Lambda terms

- λ-notation:
  Expression: $a + 2$ Function: $f(a) = a + 2$ $\lambda_a(a + 2)$

- Algebraic similarity type Σ:
  - Nullary operators: $a, b, c, \cdots \in A$ (formal parameters)
  - Binary operator: $\bullet$ (application)
  - Unary operators: λ_a $(a \in A)$ (λ-abstractions)

- A λ-term is a ground Σ-term (no algebraic variable $x, y, z, \ldots$)

  $\lambda_a(a)$ YES $\lambda_a(a) \cdot x$ NO

  - a = generic function
  - $M \cdot N$ = function M applied to argument N
  - $\lambda_a(M)$ = function of a whose body is expression M

How to compute (informally)

- Bound and free parameters: $\lambda_a(a \cdot b)$
- α-conversion: $\lambda_a(a \cdot b) = \lambda_c(c \cdot b)$
  The name of a bound parameter does not matter
- β-conversion: $\lambda_a(a) \cdot b = b$

$$\lambda_a(a \cdot a) \cdot \lambda_a(a \cdot a) = \lambda_a(a \cdot a) \cdot \lambda_a(a \cdot a) = \ldots$$

- $\alpha\beta$-conversion:

$$(\lambda_a(\lambda_b(a)) \cdot b) \cdot a = \lambda_b(b) \cdot a = a \quad \text{NO!}$$

$$(\lambda_a(\lambda_b(a)) \cdot b) \cdot a = (\lambda_a(\lambda_c(a)) \cdot b) \cdot a = \lambda_c(b) \cdot a = b$$

- Non-extensionality:

$$M \neq \lambda_a(M \cdot a) \quad (a \text{ not free in } M)$$

$$\lambda_a(M \cdot a) \cdot N = M \cdot N$$

- Every λ-term has a fixpoint: $(\forall M)(\exists P)\ M \cdot P = P$

The classic λ-calculus

- The λ-term algebra is the absolutely free Σ-algebra over an empty set of generators:

$$\mathbf{\Lambda} = (\Lambda, \cdot, \lambda_a, a)_{a \in A}$$

The object of study of λ-calculus is any congruence on $\mathbf{\Lambda}$ (called λ-theory) including α- and β-conversion:

  - β-conversion:

$$\lambda_a(M) \cdot N = M[N/a]$$

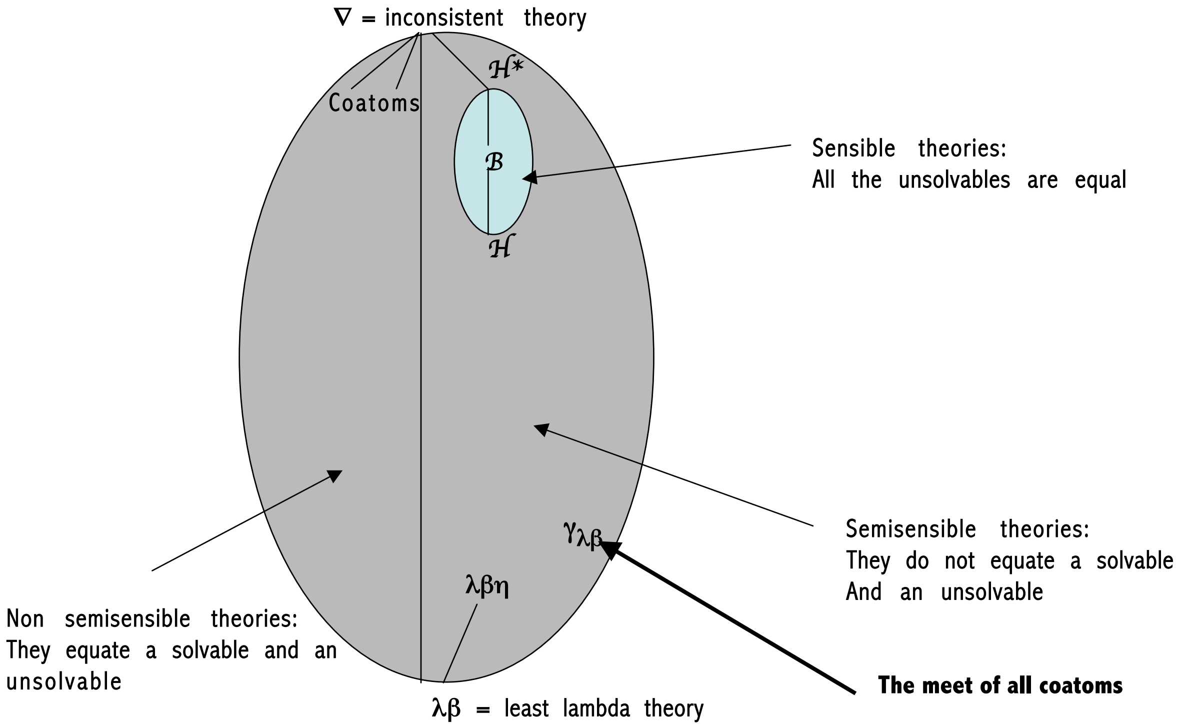
  - α-conversion:

$$\lambda_a(M) = \lambda_b(M[b/a]) \quad (b \text{ not free in } M)$$

- The lattice of λ-theories $\equiv$ The congruence lattice of $\mathbf{\Lambda}/\lambda\beta$
($\lambda\beta$ is the least congruence on $\mathbf{\Lambda}$ including α- and β-conversion)

The Lattice of λ-Theories

A continuum of λ−theories and coatoms.



The algebraic lambda calculus

- The classic λ-calculus is not interesting from the algebraic viewpoint

- Variety generated by the term algebra $\mathbf{\Lambda}/\lambda\beta$

- Pigozzi-S. (1993) introduced the variety of lambda abstraction algebras (LAAs, for brevity)

The algebraic lambda calculus

Variety LAA of lambda abstraction algebras is axiomatized by:

(β_1) $\lambda_a(a) \cdot x = x$

(β_2) $\lambda_a(b) \cdot x = b \qquad (b \neq a)$

(β_3) $\lambda_a(x) \cdot a = x$

(β_4) $\lambda_a(\lambda_a(x)) \cdot y = \lambda_a(x)$

(β_5) $\lambda_a(x \cdot y) \cdot z = (\lambda_a(x) \cdot z) \cdot (\lambda_a(y) \cdot z)$

(β_6) $\lambda_b(y) \cdot c = y \;\Rightarrow\; \lambda_a(\lambda_b(x)) \cdot y = \lambda_b(\lambda_a(x) \cdot y) \qquad (c \neq b,\ a \neq b)$

(α) $\lambda_b(x) \cdot c = x \;\Rightarrow\; \lambda_a(x) = \lambda_b(\lambda_a(x) \cdot b) \qquad (a \neq b)$

$\lambda_a(x) \cdot b = x$ $(a \neq b)$ means "x does not depend on parameter b"

The algebraic lambda calculus

- Examples of LAAs:
  - The term algebra $\mathbf{\Lambda}/T$ of a λ-theory T
  - Algebras of functions obtained by coordinatizing the models of lambda calculus

- Representation Theorems (1993-1999)

  Theorem 1 (Goldblatt-S. 1999) *Every* LAA *is representable as an algebra of functions obtained by coordinatizing a suitable model of lambda calculus.*

Universal algebra at work

Why is important to study the subvarieties of LAA?

- λ-theory $T \Leftrightarrow$ Variety generated by term algebra $\mathbf{\Lambda}/T$.
- λ-calculus problem = Problem of existence of a suitable subvariety of LAA.

Theorem 2 (S. 2000) *The lattice of the equational theories of* LAAs *is isomorphic to the lattice of λ-theories.*

Corollary 1 *Every variety of* LAAs *is generated by the term algebra $\mathbf{\Lambda}/T$ of a suitable lambda theory T. In particular,*

$$\mathrm{LAA} = \mathcal{V}(\mathbf{\Lambda}/\lambda\beta)$$

Universal algebra at work

BUT!

Theorem 3 (Lusin-S. 2004) *Every congruence identity holding in* LAA *is trivial.*

Are there interesting algebraic properties consistent with lambda calculus?

Theorem 4 (Berline-S. 2006) *There exists a congruence distributive variety of* LAAs.

The algebraic counterpart of an open problem

- Church (around 1930): Lambda calculus

- Scott (1969): First model

- Meyer-Scott (around 1980): There exists a first-order axiomatization of what is a model of λ-calculus.

  $\mathcal{D}$ model $\Rightarrow$ $\mathsf{Th}(\mathcal{D}) = \{M = N : M$ and N have the same interpretation$\}$

  We can construct an LAA expansion $\lambda(\mathcal{D})$ of $\mathcal{D}$ (algebra of functions) in such a way that

  $$\mathbf{\Lambda}/\mathsf{Th}(\mathcal{D}) \hookrightarrow \lambda(\mathcal{D}).$$

- Scott Semantics (1969-2007) A Scott model $\mathcal{D}$ is a Scott topological space (no: Hausdorff; yes: poset with bottom) and two Scott continuous maps

  $$i : \mathcal{D} \to [\mathcal{D} \to \mathcal{D}]; \quad j : [\mathcal{D} \to \mathcal{D}] \to \mathcal{D}; \quad i \circ j = id_{[\mathcal{D} \to \mathcal{D}]}$$

- Does the logical part of a λ-calculus a model (BA, HA, Lattice with λ-calculus extra operations) influence the λ-theory of the model?

The algebraic counterpart of an open problem

Definition 1 *A lambda theory T is* order-incomplete *if $T \neq Eq(\mathcal{D})$, for every model $\mathcal{D}$ such that its* LAA *expansion $\lambda(\mathcal{D})$ admits a non-trivial compatible (with respect to application and λ-abstractions) partial ordering.*

Open Question (Selinger 1996): Does it exist an order-incomplete lambda theory?

What we know:

Theorem 5 (S. 2001) *There exists a lambda theory T such that $T \neq Eq(\mathcal{D})$ for every partially ordered model $\mathcal{D}$ with bottom element.*

Fact: Order-incompleteness = Existence of an n-permutable variety of LAAs

Theorem 6 (Plotkin-Selinger-Simpson 1996) *There exists no 2-permutable and 3-permutable variety of* LAAs.

Open Question: Is n-permutability ($n \geq 4$) consistent with lambda calculus?

Boolean algebras for λ-calculus

- Let $\mathbf{A}$ be an algebra. There exists a bijective correspondence between: Pairs of compl. factor congr. $\Leftrightarrow$ Decomposition op. $\Leftrightarrow$ Factorizations

- A decomposition operation is an algebra homomorphism $f : A \times A \to A$ such that $f(x,x) = x$ and $f(f(x,y),z) = f(x,z) = f(x, f(y,z))$.

- Let $\mathbf{t} \equiv \lambda_a(\lambda_b(a))$ and $\mathbf{f} \equiv \lambda_a(\lambda_b(b))$. The least reflexive compatible relation equating

$$\mathbf{t} = \mathbf{f}$$

  is trivial.

Boolean algebras for LAAs

- A is a ring
- BA of idempotent elements of A
- A idempotent <==> $a.a = a$
- a idempotent ==> $A = A_a \times A_{-a}$
- Boolean operations:
  - Meet: $a \wedge b = a.b$;
  - Complement: $-a = 1-a$
  - Bottom: 0
  - Top: 1
- $A \to \Pi_{i \in I} A_i$, embedding

  I = Boolean space of maximal ideals of the BA of idempotent elements

  A_i = A/congruence determined by the maximal ideal i

  A_i directly indecomposable

- A is an LAA
- BA of central elements of A
- $e \in A$ is central <==> the function f: AxA --> A defined by

  $$f(x,y) = (e \cdot x) \cdot y$$

  is a decomposition operation.
- $t \equiv \lambda_a(\lambda_b(a))$; $f \equiv \lambda_a(\lambda_b(b))$.
- Boolean operations on central elements:
  - Meet: $d \wedge e = (d \cdot t) \cdot e$
  - Complement: $-d = (d \cdot f) \cdot t$
  - Bottom and Top: $0 = t$; $1 = f$
- e central ==> $A = A/\theta(1,e) \times A/\theta(1,-e)$
- $A \to \Pi_{i \in I} A_i$, embedding

  I = Boolean space of maximal ideals of the BA of central elements

  A_i = A/congruence determined by the maximal ideal i

  A_i directly indecomposable

- The Representation Theorem: Every LAA is isomorphic to a weak Boolean product of directly indecomposable LAAs.
- Proof. By using a theorem by Vaggione.

The lattice λT of λ-theories

Conjecture: Every nontrivial lattice identity fails in λT

- (S. 2000) λT is isomorphic to the lattice of equational theories of LAA's.
- (Lampe 1986) λT satisfies the Zipper condition:

$$\vee\{b : a \wedge b = c\} = 1 \;\Rightarrow\; a = c.$$

- (S. 2001) λT is not modular.
- (Berline-S. 2006) ($\exists$ λ-theory T) the interval $[T, \nabla]$ is distributive.
- (Statman 2001) The meet of all coatoms is $\neq \lambda\beta$.
- (Visser 1980)
  - Every countable poset embeds into λT by an order-preserving map.
  - Every interval $[T, S]$ with T, S r.e. has a continuum of elements.

- (S. 2006) $(\forall n)(\exists T_n)$ such that the interval sublattice $[T_n, \nabla]$ is isomorphic to the finite Boolean lattice with 2^n elements.

- (Diercks-Erné-Reinhold 1994) There exists no λ-theory T such that the interval sublattice $[T, \nabla]$ is isomorphic to an infinite Boolean lattice.