From λ-calculus to Universal Algebra and Back

Giulio Manzonetto[1] and Antonino Salibra[1,2]

[1] Laboratoire PPS, Université Paris 7
[2] Dip. Informatica, Università Ca'Foscari di Venezia

Abstract. We generalize to universal algebra concepts originating from λ-calculus and programming to prove a new result on the lattice λT of λ-theories, and a general theorem of pure universal algebra which can be seen as a meta version of the Stone Representation Theorem. In this paper we introduce the class of *Church algebras* (which includes all Boolean algebras, combinatory algebras, rings with unit and the term algebras of all λ-theories) to model the "if-then-else" instruction of programming. The interest of Church algebras is that each has a Boolean algebra of central elements, which play the role of the idempotent elements in rings. Central elements are the key tool to represent any Church algebra as a weak Boolean product of indecomposable Church algebras and to prove the meta representation theorem mentioned above. We generalize the notion of easy λ-term of lambda calculus to prove that any Church algebra with an "easy set" of cardinality n admits (at the top) a lattice interval of congruences isomorphic to the free Boolean algebra with n generators. This theorem has the following consequence for the lattice of λ-theories: for every recursively enumerable λ-theory ϕ and each n, there is a λ-theory $\phi_n \supseteq \phi$ such that $\{\psi : \psi \supseteq \phi_n\}$ "is" the Boolean lattice with 2^n elements.
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1 Introduction

Lambda theories are equational extensions of the untyped λ-calculus closed under derivation. They arise by syntactical or semantic considerations. Indeed, a λ-theory may correspond to some operational semantics of λ-calculus, as well as it may be induced by a λ-model, which is a particular *combinatory algebra* (CA, for short) [1, Sec. 5.2]. The set of λ-theories is naturally equipped with a structure of complete lattice (see [1, Ch. 4]), whose bottom element is the least λ-theory $\lambda\beta$, and whose top element is the inconsistent λ-theory. The lattice λT of λ-theories is a very rich and complex structure of cardinality $2^{\aleph_0}$ [1, 10, 11, 2].

The interest of a systematic study of the lattice λT of λ-theories grows out of several open problems on λ-calculus. For example, Selinger's order-incompleteness problem asks for the existence of a λ-theory not arising as the equational theory of a non-trivially partially ordered model of λ-calculus. This problem can be proved equivalent to that of the existence of a recursively enumerable (r.e., for short) λ-theory ϕ whose term algebra generates an n-permutable variety of algebras for some $n \geq 2$ (see [15] and the remark after [16, Thm. 3.4]). Lipparini [9] has found out interesting non-trivial lattice identitites that hold in the congruence lattices of all algebras living in an n-permutable variety. The failure of some Lipparini's lattice identities in λT would imply that Selinger's problem has a negative answer.

Techniques of universal algebra were applied in [14, 10, 3] to study the structure of λT. In this paper we validate the inverse slogan: λ-calculus can be fruitfully applied to universal algebra. By generalizing to universal algebra concepts originating from λ-calculus and programming, we create a zigzag path from λ-calculus to universal algebra and back. All the algebraic properties we have shown in [11] for CAs, hold for a wider class of algebras, that we call *Church algebras*. Church algebras include, beside CAs, all BAs (Boolean algebras) and all rings with unit, and model the "if-then-else" instruction by two constants $0, 1$ and a ternary term $q(x, y, z)$ satisfying the following identities:

$$q(1, x, y) = x; \quad q(0, x, y) = y.$$

The interest of Church algebras is that each has a BA of central elements, which can be used to represent the Church algebra as a weak Boolean product of directly indecomposable algebras (i.e., algebras which cannot be decomposed as the Cartesian product of two other non-trivial algebras).

We generalize the notion of easy λ-term from λ-calculus and use central elements to prove that: (i) any Church algebra with an "easy set" of cardinality κ admits a congruence ϕ such that (the lattice reduct of) the free BA with κ generators embeds into the lattice interval $[\phi)$ of all congruences greater than ϕ; (ii) If κ is a finite cardinal, this embedding is an isomorphism. This theorem applies directly to all BAs and rings with units. For λT it has the following consequence: for every r.e. λ-theory ϕ and each natural number n, there is a λ-theory $\phi_n \supseteq \phi$ such that the lattice interval $[\phi_n)$ is the finite Boolean lattice with 2^n elements. It is the first time that it is found an interval of λT whose cardinality is not $1, 2$ or $2^{\aleph_0}$.

Our contribution to general Universal Algebra is the following: using Church algebras we prove a meta version of the Stone Representation Theorem that applies to all varieties of algebras and not only to the classic ones. Indeed, we show that any variety of algebras can be decomposed as a weak Boolean product of directly indecomposable subvarieties. This means that, given a variety $\mathcal{V}$, there exists a family of "indecomposable" subvarieties $\mathcal{V}_i$ $(i \in I)$ of $\mathcal{V}$ for which every algebra of $\mathcal{V}$ is isomorphic to a weak Boolean product of algebras of $\mathcal{V}_i$ $(i \in I)$.

2 Preliminaries

We will use the notation of Barendregt's classic work [1] for λ-calculus and combinatory logic, and the notation of McKenzie et al. [12] for universal algebra.

A lattice L is *bounded* if it has a top element 1 and a bottom element 0. An element $x \in L$ is an *atom* (*coatom*) if it is a minimal element in $L - \{0\}$ (maximal element in $L - \{1\}$). For $x \in L$, we set $L_x = \{y \in L - \{0\} : x \wedge y = 0\}$. L is called: *lower semicomplemented* if $L_x \neq \emptyset$ for all $x \neq 1$; *pseudocomplemented* if each $L_x \cup \{0\}$ has a greatest element (called the *pseudocomplement* of x); *complemented* if, for every $x \in L$, there exists y such that $x \wedge y = 0$ and $x \vee y = 1$; *atomic* if, for every $x \in L$, there exists an atom $y \leq x$; *coatomic* if, for every $x \in L$, there exists a coatom $y \geq x$; *Boolean* if it is distributive and complemented.

An element x of a complete lattice L is *completely join-prime* if, for every $X \subseteq L$, $x \leq \bigvee X$ implies $x \leq y$ for some $y \in X$.

We write $[x)$ for $\{y : x \leq y \leq 1\}$.

An *algebraic similarity type* Σ is constituted by a non-empty set of operator symbols together with a function assigning to each operator $f \in \Sigma$ a finite *arity*.

A *Σ-algebra* $\mathbf{A}$ is determined by a non-empty set A together with an operation $f^{\mathbf{A}} : A^n \to A$ for every $f \in \Sigma$ of arity n. $\mathbf{A}$ is *trivial* if $|A| = 1$, where $|A|$ denotes the cardinality of A.

A compatible equivalence relation ϕ on a Σ-algebra $\mathbf{A}$ is called a *congruence*. We often write $x\phi y$ for $(x, y) \in \phi$. The set $\{y : x\phi y\}$ is denoted by $[x]_\phi$.

If $\phi \subseteq \psi$ are congruences on $\mathbf{A}$, then $\psi/\phi = \{([x]_\phi, [y]_\phi) : x\psi y\}$ is a congruence on the quotient $\mathbf{A}/\phi$.

If $X \subseteq A \times A$, then we write $\theta(X)$ for the least congruence including X. We write $\theta(x, y)$ for $\theta(\{(x, y)\})$. If $x \in A$ and $Y \subseteq A$, then we write $\theta(x, Y)$ for $\theta(\{(x, y) : y \in Y\})$.

We denote by $\mathrm{Con}(\mathbf{A})$ the algebraic complete lattice of all congruences of $\mathbf{A}$, and by $\nabla = \{(x, y) : x, y \in A\}$ and $\Delta = \{(x, x) : x \in A\}$ the top and the bottom element of $\mathrm{Con}(\mathbf{A})$. We recall that the join of two congruences ϕ and ψ is the least equivalence relation containing the union $\phi \cup \psi$. A congruence ϕ on $\mathbf{A}$ is called: *trivial* if it is equal to ∇ or Δ; *consistent* if $\phi \neq \nabla$; *compact* if $\phi = \theta(X)$ for some finite set $X \subseteq A \times A$. Two congruences ϕ and ψ are *compatible* if $\phi \vee \psi \neq \nabla$; otherwise, they are *incompatible*.

An algebra $\mathbf{A}$ is *directly decomposable* if there exist two non-trivial algebras $\mathbf{B}, \mathbf{C}$ such that $\mathbf{A} \cong \mathbf{B} \times \mathbf{C}$, otherwise it is called *directly indecomposable*.

An algebra $\mathbf{A}$ is a *subdirect product* of the algebras $(\mathbf{B}_i)_{i \in I}$, written $\mathbf{A} \leq \Pi_{i \in I} \mathbf{B}_i$, if there exists an embedding f of $\mathbf{A}$ into the direct product $\Pi_{i \in I} \mathbf{B}_i$ such that the projection $\pi_i \circ f : \mathbf{A} \to \mathbf{B}_i$ is onto for every $i \in I$.

A non-empty class $\mathcal{V}$ of algebras is a *variety* if it is closed under subalgebras, homomorphic images and direct products or, equivalently, if it is axiomatizable by a set of equations. A variety $\mathcal{V}'$ is a *subvariety* of the variety $\mathcal{V}$ if $\mathcal{V}' \subseteq \mathcal{V}$. We will denote by $\mathcal{V}(\mathbf{A})$ the variety generated by an algebra $\mathbf{A}$, i.e., $\mathbf{B} \in \mathcal{V}(\mathbf{A})$ if every equation satisfied by $\mathbf{A}$ is also satisfied by $\mathbf{B}$.

Let $\mathcal{V}$ be a variety. We say that $\mathbf{A}$ is *the free $\mathcal{V}$-algebra over the set X of generators* iff $\mathbf{A} \in \mathcal{V}$, $\mathbf{A}$ is generated by X and for every $g : X \to \mathbf{B} \in \mathcal{V}$, there is a unique homomorphism $f : \mathbf{A} \to \mathbf{B}$ that extends g (i.e., $f(x) = g(x)$ for every $x \in X$). A free algebra in the class of all Σ-algebras is called *absolutely free*.

Given two congruences σ and τ on $\mathbf{A}$, we can form their relative product: $\tau \circ \sigma = \{(x, z) : \exists y \in A\ x\sigma y\tau z\}$. We have $\tau \cup \sigma \subseteq \tau \circ \sigma \subseteq \tau \vee \sigma$; in general $\tau \circ \sigma$ is not a congruence.

Definition 1. *A congruence* ϕ *on an algebra* $\mathbf{A}$ *is a* factor congruence *if there exists another congruence* ψ *such that* $\phi \cap \psi = \Delta$ *and* $\phi \circ \psi = \nabla$. *In this case we call* (ϕ, ψ) *a* pair of complementary factor congruences *or* cfc-pair*, for short.*

If (ϕ, ψ) is a cfc-pair, then, for all x and y, there is just one element u satisfying $x\ \phi\ u\ \psi\ y$.

Under the hypotheses of the above definition the homomorphism $f : \mathbf{A} \to \mathbf{A}/\phi \times \mathbf{A}/\psi$ defined by $f(x) = ([x]_\phi, [x]_\psi)$ is an isomorphism. So, the existence of factor congruences is just another way of saying "this algebra is a direct product of simpler algebras".

The set of factor congruences of $\mathbf{A}$ is not, in general, a sublattice of Con($\mathbf{A}$). Δ and ∇ are the *trivial* factor congruences, corresponding to $\mathbf{A} \cong \mathbf{A} \times \mathbf{B}$, where $\mathbf{B}$ is a trivial algebra. An algebra $\mathbf{A}$ is directly indecomposable if, and only if, $\mathbf{A}$ has no non-trivial factor congruences.

It is possible to characterize cfc-pairs in terms of certain algebra homomorphisms called *decomposition operators* (see [12, Def. 4.32] for more details).

Definition 2. *A* decomposition operator *for an algebra* $\mathbf{A}$ *is an algebra homomorphism* $f : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$ *such that* $f(x, x) = x$ *and* $f(f(x, y), z) = f(x, z) = f(x, f(y, z))$.

There exists a bijection between cfc-pairs and decomposition operators, and thus, between decomposition operators and factorizations like $\mathbf{A} \cong \mathbf{B} \times \mathbf{C}$.

Proposition 1. *[12, Thm. 4.33] Given a decomposition operator* f*, the relations* ϕ, ψ *defined by* $x\ \phi\ y$ *iff* $f(x, y) = y$; *and* $x\ \psi\ y$ *iff* $f(x, y) = x$*, form a cfc-pair. Conversely, given a cfc-pair* (ϕ, ψ)*, the map* f *defined by* $f(x, y) = u$ *iff* $x\ \phi\ u\ \psi\ y$*, is a decomposition operator.*

The Boolean product construction allows us to transfer numerous fascinating properties of BAs into other varieties of algebras (see [5, Ch. IV]). We recall that a Boolean space is a compact, Hausdorff and totally disconnected topological space, and that *clopen* means "open and closed".

Definition 3. *A* weak Boolean product *of a family* $(\mathbf{A}_i)_{i \in I}$ *of algebras is a subdirect product* $\mathbf{A} \leq \Pi_{i \in I} \mathbf{A}_i$*, where* I *can be endowed with a Boolean space topology such that:* (i) *the set* $\{i \in I : x_i = y_i\}$ *is open for all* $x, y \in A$*, and* (ii) *if* $x, y \in A$ *and* N *is a clopen subset of* I*, then the element* z*, defined by* $z_i = x_i$ *for every* $i \in N$ *and* $z_i = y_i$ *for every* $i \in I - N$*, belongs to* A*. A* Boolean product *is a weak Boolean product such that the set* $\{i \in I : x_i = y_i\}$ *is clopen for all* $x, y \in A$.

A λ-*theory* is any congruence (w.r.t. the binary operator of application and the lambda abstractions) on the set of λ-terms including (α)- and (β)-conversion (see [1, Ch. 2]). We use for λ-theories the same notational convention as for congruences. The set of all λ-theories, ordered by inclusion, is naturally equipped with a structure of complete lattice, denoted by λT, with intersection as meet. The least element of λT is denoted by $\lambda\beta$, while the top element of λT is the inconsistent λ-theory ∇. The *term algebra of a* λ*-theory* ϕ, denoted by $\mathbf{\Lambda}_\phi$, has the equivalence classes of λ-terms modulo ϕ as elements, and the natural operations induced by application and λ-abstraction. The lattice λT is isomorphic to the congruence lattice $\mathrm{Con}(\mathbf{\Lambda}_{\lambda\beta})$, while its interval sublattice $[\phi)$ is isomorphic to $\mathrm{Con}(\mathbf{\Lambda}_{\phi})$.

As a matter of notation, Ω denotes the λ-term $(\lambda x.xx)(\lambda x.xx)$.

The variety CA of *combinatory algebras* [1, Sec. 5.1] consists of algebras $\mathbf{C} = (C, \cdot, \mathbf{k}, \mathbf{s})$, where $\cdot$ is a binary operation and $\mathbf{k}, \mathbf{s}$ are constants, satisfying $\mathbf{k}xy = x$ and $\mathbf{s}xyz = xz(yz)$ (as usual, the symbol "$\cdot$" is omitted and association is made on the left).

3 Church Algebras

Our key observation is that many algebraic structures, such as CAs, BAs etc., have in common the fact that all are *Church algebras*. In this section we study the algebraic properties of this class of algebras. Applications are given in Section 5 and in Section 6.

Definition 4. *An algebra* $\mathbf{A}$ *is called* a Church algebra *if there are two constants* $0, 1 \in A$ *and a ternary term* $q(e, x, y)$ *such that* $q(1, x, y) = x$ *and* $q(0, x, y) = y$. *A variety* $\mathcal{V}$ *is called a* Church variety *if every algebra in* $\mathcal{V}$ *is a Church algebra with respect to the same term* $q(e, x, y)$ *and constants* $0, 1$.

Note that the top element ∇ of the congruence lattice $\mathrm{Con}(\mathbf{A})$ of a Church algebra $\mathbf{A}$ is a compact element because obviously $\nabla = \theta(0, 1)$.

Example 1. The following are easily checked to be Church algebras:

1. Combinatory algebras: $q(e, x, y) \equiv (e \cdot x) \cdot y$; $1 \equiv \mathbf{k}$; $0 \equiv \mathbf{sk}$
2. Boolean algebras: $q(e, x, y) \equiv (e \vee y) \wedge (e^- \vee x)$
3. Heyting algebras: $q(e, x, y) \equiv (e \vee y) \wedge ((e \to 0) \vee x)$
4. Rings with unit: $q(e, x, y) \equiv (y + e - ey)(1 - e + ex)$

Let $\mathbf{A} = (A, +, \cdot, 0, 1)$ be a commutative ring with unit. Every idempotent element $e \in A$ (i.e., satisfying $e \cdot e = e$) induces a cfc-pair $(\theta(1, e), \theta(e, 0))$. In other words, the ring $\mathbf{A}$ can be decomposed as $\mathbf{A} \cong \mathbf{A}/\theta(1, e) \times \mathbf{A}/\theta(e, 0)$. $\mathbf{A}$ is directly indecomposable iff 0 and 1 are the unique idempotent elements. Vaggione [18] generalized the notion of idempotent to any universal algebra whose top congruence ∇ is compact, and called them *central elements*. Central elements were used to investigate the closure of varieties of algebras under Boolean products. Here we give a new characterization based on decomposition operators (see Def. 2). Hereafter, we set $\theta_e \equiv \theta(1, e)$ and $\overline{\theta}_e \equiv \theta(e, 0)$.

Definition 5. *We say that an element* e *of a Church algebra* $\mathbf{A}$ *is* central, *and we write* $e \in \mathrm{Ce}(\mathbf{A})$, *if* $(\theta_e, \overline{\theta}_e)$ *is a cfc-pair. A central element* e *is called* nontrivial *if* $e \neq 0, 1$.

We now show that, in a Church algebra, factor congruences can be internally represented by central elements. The following lemma is easy to check.

Lemma 1. *Let* $\mathbf{A}$ *be a Church algebra and* $e \in A$. *Then we have, for all* $x, y \in A$:

(*a*) $x\ \theta_e\ q(e,x,y)\ \overline{\theta}_e\ y$.
(*b*) $x\theta_e y$ *iff* $q(e,x,y)\ (\theta_e \cap \overline{\theta}_e)\ y$.
(*c*) $x\overline{\theta}_e y$ *iff* $q(e,x,y)\ (\theta_e \cap \overline{\theta}_e)\ x$.
(*d*) $\theta_e \circ \overline{\theta}_e = \overline{\theta}_e \circ \theta_e = \nabla$.

Proposition 2. *Let* $\mathbf{A}$ *be a Church* Σ*-algebra and* $e \in A$. *Then the following conditions are equivalent:*
(*i*) *e is central;*
(*ii*) $\theta_e \cap \overline{\theta}_e = \Delta$;
(*iii*) *For all x and y,* $q(e,x,y)$ *is the unique element such that* $x\theta_e\ q(e,x,y)\ \overline{\theta}_e y$;
(*iv*) *e satisfies the following identities:*
1. $q(e,x,x) = x$.
2. $q(e,q(e,x,y),z) = q(e,x,z) = q(e,x,q(e,y,z))$.
3. $q(e,f(\overline{x}),f(\overline{y})) = f(q(e,x_1,y_1),\dots,q(e,x_n,y_n))$, *for every* $f \in \Sigma$.
4. $e = q(e,1,0)$.

(*v*) *The function* f_e *defined by* $f_e(x,y) = q(e,x,y)$ *is a decomposition operator such that* $f_e(1,0) = e$.

Thus a Church algebra $\mathbf{A}$ is directly indecomposable iff $\mathrm{Ce}(\mathbf{A}) = \{0,1\}$ iff $\theta_e \cap \overline{\theta}_e \neq \Delta$ for all $e \neq 0,1$.

Example 2. (i) All elements of a BA are central by Prop. 2(iv) and Example 1.
(ii) An element is central in a commutative ring with unit iff it is idempotent. This characterization does not hold for non-commutative rings with unit.
(iii) Let $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$ be the usual looping term of λ-calculus. It is well-known that the λ-theories $\theta_\Omega = \theta(\Omega, \lambda xy.x)$ and $\overline{\theta}_\Omega = \theta(\Omega, \lambda xy.y)$ are consistent (see [1, Prop. 15.3.9]). Then by Lemma 5 below the term Ω is a non-trivial central element in the term algebra of $\theta_\Omega \cap \overline{\theta}_\Omega$.

We now show that the partial ordering on $\mathrm{Ce}(\mathbf{A})$, defined by:

$$e \leq d \text{ if, and only if, } \overline{\theta}_e \subseteq \overline{\theta}_d$$

is a Boolean ordering and that the meet, join and complementation operations are internally representable. 0 and 1 are respectively the bottom and top element of this ordering.

Theorem 1. *Let* $\mathbf{A}$ *be a Church algebra. The algebra* $(\mathrm{Ce}(\mathbf{A}), \wedge, \vee, ^-, 0, 1)$, *with operations defined by* $e \wedge d = q(e,d,0)$, $e \vee d = q(e,1,d)$, $e^- = q(e,0,1)$, *is a* BA, *which is isomorphic to the* BA *of factor congruences of* $\mathbf{A}$.

Next we turn to the Stone representation theorem for Church algebras. It is a corollary of Thm. 1 and of theorems by Comer [6] and by Vaggione [18].

Let $\mathbf{A}$ be a Church algebra. If I is an ideal of the Boolean algebra $\mathrm{Ce}(\mathbf{A})$, then $\phi_I = \cup_{e\in I}\overline{\theta}_e$ is a congruence. In the next theorem $\mathcal{S}$ is the Boolean space of maximal ideals of $\mathrm{Ce}(\mathbf{A})$.

Theorem 2. (The Stone Representation Theorem) *Let* $\mathbf{A}$ *be a Church algebra. Then, for all* $I \in \mathcal{S}$ *the quotient algebra* $\mathbf{A}/\phi_I$ *is directly indecomposable and the map* $f : A \to \Pi_{I\in\mathcal{S}}(A/\phi_I)$, *defined by* $f(x) = ([x]_{\phi_I} : I \in \mathcal{S})$, *gives a weak Boolean product representation of* $\mathbf{A}$.

Note that, in general, Thm. 2 does not give a (non-weak) Boolean product representation. This was shown in [11] for combinatory algebras.

4 The Main Theorem

In λ-calculus there are *easy λ-terms*, i.e., terms, like Ω, that can be consistently equated with any other closed λ-term. In this section we generalize the notion of easiness to Church algebras and show that any Church algebra with an easy set of cardinality n admits a congruence ϕ such that the lattice interval of all congruences greater than ϕ is isomorphic to the free BA with n generators.

Let $\mathbf{A}$ be a Church algebra and $a \subseteq A$. As a matter of notation, for every $b \subseteq a$, we define

$$\delta_b \equiv \theta(1, b) \vee \theta(0, a - b). \tag{1}$$

By definition $\theta(1, \emptyset) = \theta(0, \emptyset) = \Delta$.

Definition 6. *Let $\mathbf{A}$ be a Church algebra. We say that $a \subseteq A$ is an* easy set *if $\delta_b \neq \nabla$ for every $b \subseteq a$.*

Note that, if a is an easy set, then the set of all δ_b ($b \subseteq a$) consists of $2^{|a|}$ pairwise incompatible congruences.

We say that an element x is *easy* if $\{x\}$ is an easy set. Thus, x is easy iff the congruences θ_x and $\overline{\theta}_x$ are both different from ∇.

Example 3. A finite subset a of a BA is an easy set if it holds: (i) $\bigvee a \neq 1$; (ii) $\bigwedge a \neq 0$; (iii) for all $b \subset a$, $\bigvee b \not\geq \bigwedge(a - b)$. Thus, for example, $\{\{1, 2\}, \{2, 3\}\}$ is an easy set in the powerset of $\{1, 2, 3, 4\}$.

The following lemmas are used in the proof of the main theorem.

Lemma 2. *The congruences of a Church algebra permute with its factor congruences, i.e., $\phi \circ \psi = \psi \circ \phi$ for every congruence ϕ and factor congruence ψ.*

Proof. Let $\psi = \theta_e$ for a central e and let $a\ \phi\ b\ \theta_e\ c$ for some b. We get the conclusion if $a\ \theta_e\ q(e, a, c)\ \phi\ c$. First $a\ \theta_e\ q(e, a, c)$ by Lemma 1(a). From $a\ \phi\ b$ we have $q(e, a, c)\ \phi\ q(e, b, c)$. Finally, $q(e, b, c) = c$ by $b\ \theta_e\ c$ and by Prop. 2(iii).

Definition 7. *A bounded lattice L with top 1 and bottom 0 satisfies the* Zipper condition *if, for every set I and for every $x_i, y, z \in L$ ($i \in I$), we have:*

$$\bigvee_{i \in I} x_i = 1,\ x_i \wedge y = z\ (i \in I) \implies y = z.$$

Lemma 3. *If $\mathbf{A}$ is a Church algebra, then $\mathrm{Con}(\mathbf{A})$ satisfies the Zipper condition.*

Proof. By [8] the congruence lattice of every algebra having a binary term with a right unit and a right zero satisfies the Zipper condition.

Lemma 4. *Let* $\mathbf{B}$ *be a Church algebra and* $\phi \in \mathrm{Con}(\mathbf{B})$*. Then,* $\mathbf{B}/\phi$ *is also a Church algebra and the map* $c_\phi : \mathrm{Ce}(\mathbf{B}) \to \mathrm{Ce}(\mathbf{B}/\phi)$*, defined by* $c_\phi(x) = [x]_\phi$ *is a homomorphism of* BA*s.*

Lemma 5. *Let* $\mathbf{A}$ *be a Church algebra,* $e \in A$ *and* $\phi \in \mathrm{Con}(\mathbf{A})$*. Then,*
(i) $[e]_\phi$ *is central in* $\mathbf{A}/\phi$ *iff* $\phi \supseteq \theta_e \cap \overline{\theta}_e$*.*
(ii) $[e]_{\theta_e \cap \overline{\theta}_e}$ *is a non-trivial central element in* $\mathbf{A}/\theta_e \cap \overline{\theta}_e$ *iff* $\theta_e \neq \nabla, \Delta$*.*

Theorem 3. *Let* $\mathbf{A}$ *be a Church algebra,* $a \subseteq A$ *be an easy set and* $\mathbf{B}(a)$ *be the free Boolean algebra over the set* a *of generators. Then there exists a congruence* ϕ_a *satisfying the following conditions:*

1. *The lattice reduct of* $\mathbf{B}(a)$ *can be embedded into the interval sublattice* $[\phi_a)$ *of* $\mathrm{Con}(\mathbf{A})$*;*
2. *If* a *has finite cardinality* n*, then the above embedding is an isomorphism and, hence,* $[\phi_a)$ *has cardinality* 2^{2^n}*.*

Proof. Let $\eta = \cap_{b \subseteq a} \delta_b$ (see (1) above for the definition of δ_b). We define ϕ_a as any maximal element of the set of all congruences ϕ which contain η and are compatible with each δ_b (i.e., $\phi \vee \delta_b \neq \nabla$). Note that ϕ_a exists by Zorn Lemma.

Claim 1. $[x]_{\phi_a}$ is central in $\mathbf{A}/\phi_a$ for every $x \in a$.

Proof. If we prove that $[x]_\eta$ is central in $\mathbf{A}/\eta$, then by $\eta \subseteq \phi_a$ and by Lemma 4 we get the conclusion of the claim. Since $x \in a$ is equivalent either to 1 or to 0 in each congruence δ_b, then $[x]_{\delta_b}$ is a trivial central element in $\mathbf{A}/\delta_b$, so that $[x]_\eta$ is central in $\mathbf{A}/\eta$ by Lemma 5(i) and by $\eta = \cap_{b \subseteq a} \delta_b$.

Let now $f_a : \mathbf{B}(a) \to \mathrm{Ce}(\mathbf{A}/\phi_a)$ be the unique Boolean homomorphism satisfying $f_a(x) = [x]_{\phi_a}$ $(x \in a)$.

Claim 2. f_a is an embedding.

Proof. Let $b \subseteq a$. Recall that $\phi_a \vee \delta_b \neq \nabla$. By Lemma 4 there exists a Boolean homomorphism (denoted by h_b in this proof) from $\mathrm{Ce}(\mathbf{A}/\phi_a)$ into $\mathrm{Ce}(\mathbf{A}/\phi_a \vee \delta_b)$. Since $(x, 1) \in \phi_a \vee \delta_b$ for every $x \in b$ and $(y, 0) \in \phi_a \vee \delta_b$ for every $y \in a - b$, then the kernel of $h_b \circ f_a$ is an ultrafilter of $\mathbf{B}(a)$. By the arbitrariness of $b \subseteq a$, every ultrafilter of $\mathbf{B}(a)$ can be the kernel of a suitable $h_b \circ f_a$. This is possible only if f_a is an embedding.

This concludes the proof of (1) of Thm. 3.

Hereafter, we assume that a *is finite and we let* $n = |a|$. Then $\mathbf{B}(a)$ is finite, atomic, has n generators, 2^n atoms, 2^n coatoms, and $|B(a)| = 2^{2^n}$. Recall that $\mathrm{Con}(\mathbf{A}/\phi_a)$ is isomorphic to $[\phi_a)$.

Let At_a be the set of atoms of $\mathrm{Con}(\mathbf{A}/\phi_a)$.

Claim 3. $\bigvee\{\beta \in At_a : \beta \text{ is a factor congruence}\} = \nabla$.

Proof. Let $v \in \mathrm{Ce}(\mathbf{A}/\phi_a)$ such that $v = f_a(u)$ for some atom $u \in \mathbf{B}(a)$. Consider $\tau \in \mathrm{Con}(\mathbf{A})$ such that $\tau/\phi_a = \theta(v, 0) \in \mathrm{Con}(\mathbf{A}/\phi_a)$. We claim that $\tau/\phi_a \in At_a$. By the way of contradiction, let $\sigma \in \mathrm{Con}(\mathbf{A})$ such that $\phi_a \subset \sigma \subset \tau$. By Lemma 4 we have a sequence of Boolean homomorphisms:

$$\mathbf{B}(a) \xrightarrow{f_a} \mathrm{Ce}(\mathbf{A}/\phi_a) \xrightarrow{c_\sigma} \mathrm{Ce}(\mathbf{A}/\sigma) \xrightarrow{c_{\tau/\sigma}} \mathrm{Ce}(\mathbf{A}/\tau)$$

and a Boolean homomorphism $c_\tau : \mathrm{Ce}(\mathbf{A}/\phi_a) \xrightarrow{c_\tau} \mathrm{Ce}(\mathbf{A}/\tau)$ such that $c_\tau = c_{\tau/\sigma} \circ c_\sigma$. Since u is an atom of $\mathbf{B}(a)$, then the set $\{0, u\}$ is the Boolean ideal associated with the kernel of $c_\tau \circ f_a$. If $c_\sigma(v) = 0$, then σ/ϕ_a contains the pair $(v, 0)$, i.e., $\sigma = \tau$. It follows that $c_\sigma(v) \neq 0$ and the map $c_\sigma \circ f_a : \mathbf{B}(a) \to \mathrm{Ce}(\mathbf{A}/\sigma)$ is an embedding. Since $\mathbf{B}(a)$ is free over a, for every $b \subseteq a$ there exists an atom $w \in \mathbf{B}(a)$ such that $w = (\bigwedge b) \wedge (\bigwedge\{x^- : x \in a - b\})$. Let $w' = c_\sigma(f_a(w)) \in \mathbf{A}/\sigma$ the corresponding nontrivial central element. By definition of w, the non-triviality of the factor congruence $\theta(w', 1) \in \mathrm{Con}(\mathbf{A}/\sigma)$ is equivalent to $\sigma \vee \delta_b \neq \nabla$. The arbitrariness of b and the strict inclusion $\phi_a \subset \sigma$ contradict the maximality of ϕ_a. In conclusion $\tau/\phi_a \in At_a$. Finally, the claim follows because the join of all atoms of $\mathbf{B}(a)$ is the top element.

Let Co_a be the set of coatoms of $\mathrm{Con}(\mathbf{A}/\phi_a)$. We say that the coatoms form a finite irredundant decomposition of Δ if Co_a is finite, $\cap Co_a = \Delta$, and $\cap(Co_a - \{\psi\}) \neq \Delta$ for every $\psi \in Co_a$.

Claim 4. $\mathrm{Con}(\mathbf{A}/\phi_a)$ is pseudocomplemented, complemented, atomic, and the coatoms form a finite irredundant decomposition of Δ.

Proof. $\mathrm{Con}(\mathbf{A}/\phi_a)$ satisfies the Zipper condition (by Lemma 3) and $\bigvee At_a = \nabla$ (by Claim 3). Then by [7, Prop. 2] $\mathrm{Con}(\mathbf{A}/\phi_a)$ is complemented, atomic and every coatom has a complement which is an atom. It is also pseudocomplemented by [7, Prop. 1]. Since the top element ∇ is compact, by [7, Prop. 3] we get that the coatoms form a finite irredundant decomposition of Δ.

Claim 5. Let $\xi \in \mathrm{Con}(\mathbf{A}/\phi_a)$ be a non-trivial congruence and $\gamma = \bigvee\{\delta \in At_a : \delta \subseteq \xi\}$. If $\beta \in At_a$ is a factor congruence and $\beta \not\subseteq \xi$, then $\xi \cap (\beta \vee \gamma) = \gamma$.

Proof. We always have $\gamma \subseteq \xi \cap (\beta \vee \gamma)$. We show the opposite direction. Let $(x, y) \in \xi \cap (\beta \vee \gamma)$, i.e., $x \;\xi\; y$ and $x(\beta \vee \gamma)y$. We have to show that $x \;\gamma\; y$. Since β is a factor congruence, by Lemma 2 we have $\beta \vee \gamma = \beta \circ \gamma$. Then $x \;\beta\; z \;\gamma\; y$ for some z. Since $\gamma \subseteq \xi$ and $z \;\gamma\; y$ then $z \;\xi\; y$, that together with $x \;\xi\; y$ implies $x \;\xi\; z$. Then $x(\xi \cap \beta)z$. Since β is an atom and $\beta \not\subseteq \xi$, then $\xi \cap \beta = \Delta$, so that $x = z$. This last equality and $z \;\gamma\; y$ imply $x \;\gamma\; y$. In other words, $\xi \cap (\beta \vee \gamma) = \gamma$.

Claim 6. Every $\xi \in \mathrm{Con}(\mathbf{A}/\phi_a)$ is a join of atoms.

Proof. Let At_ξ be the set of atoms included in ξ. We will show that $\xi = \bigvee At_\xi$ by applying the Zipper condition of Def. 7. Let $\gamma = \bigvee At_\xi$. By Claim 5 we have: $\bigvee\{\nu : \xi \cap \nu = \gamma\} \supseteq \bigvee\{\beta \vee \gamma : \beta \in At_a, \beta \not\subseteq \xi,\ \beta \text{ is a factor congruence}\} \supseteq \bigvee\{\beta : \beta \in At_a \text{ is a factor congruence}\}$. By Claim 3 this last element is equal to ∇, so that $\bigvee\{\nu : \xi \cap \nu = \gamma\} = \nabla$. By the Zipper condition this entails $\xi = \gamma$.

Claim 7. $\mathrm{Con}(\mathbf{A}/\phi_a)$, and hence $[\phi_a)$, is isomorphic to the power set of At_a.

Proof. $\mathrm{Con}(\mathbf{A}/\phi_a)$ is atomic and pseudocomplemented (by Claim 4), so that each atom is completely join-prime. By this and by Claim 6 every element is univocally represented as a join of atoms. The conclusion follows because every join of atoms exists by completeness.

Claim 8. $\mathrm{Con}(\mathbf{A}/\phi_a)$, and hence $[\phi_a)$, has 2^n coatoms and 2^n atoms.

Proof. Since $\phi_a \vee \delta_b \neq \nabla$ for every $b \subseteq a$, $[\phi_a)$ has at least 2^n coatoms. For every $b \subseteq a$, let c_b be a coatom including $\phi_a \vee \delta_b$. Assume now that there is a coatom ξ distinct from each c_b for every $b \subseteq a$. Consider the intersection $\cap(Co_a - \{\xi\})$, where Co_a denotes the set of coatoms of $[\phi_a)$. By Claim 4 we have that $\cap(Co_a - \{\xi\}) \neq \phi_a$. This contradicts the maximality of ϕ_a among the congruences which contains $\cap_{b\subseteq a} \delta_b$ and are compatible with δ_b.

This concludes the proof of the main theorem.

The next proposition, which follows from [7, Prop. 4], says that the main theorem cannot be improved.

Proposition 3. *Let* $\mathbf{A}$ *be a Church algebra. Then there exists no congruence* ϕ *such that the interval sublattice* $[\phi)$ *is isomorphic to an infinite Boolean lattice.*

5 The Lattice of λ-theories

The term algebra of a λ-theory ϕ is a Church algebra. This easy remark has the interesting consequence that the lattice λT of all λ-theories admits (at the top) Boolean lattice intervals of cardinality 2^k for every k.

Lemma 6. *For every r.e.* λ*-theory* ϕ*, the term algebra of* ϕ *admits an infinite easy set.*

Proof. The set consisting of all λ-terms $\Omega\hat{n}$, where $\hat{n}$ is the n-th Church numeral, is an easy set in the term algebra of $\lambda\beta$. This follows from the easiness of Ω and a compactness argument, and appears as [1, Ex. 15.4.3]. More generally, the term algebra of each r.e. λ-theory has an easy element [1, Prop. 17.1.9], and hence it has an infinite easy set, by the same compactness argument.

Theorem 4. *For every r.e.* λ*-theory* ϕ *and each natural number* k*, there is a* λ*-theory* $\phi_k \supseteq \phi$ *such that the lattice interval* $[\phi_k)$ *is isomorphic to the finite Boolean lattice with* 2^k *elements.*

Proof. By Lemma 6 and by Thm. 3 there exists a congruence ψ_k such that $\psi_k \supseteq \phi$ and $[\psi_k)$ is isomorphic to the free Boolean algebra with 2^{2^k} elements. The congruence ϕ_k of the theorem can be defined by using ψ_k and the following facts: (a) Every filter of a finite Boolean algebra is a Boolean lattice; (b) The free Boolean algebra with 2^{2^k} elements has filters of arbitrary cardinality 2^i $(i \leq 2^k)$.

Note that the λ-theory ϕ_k of Thm. 4 is not r.e. because otherwise the lattice interval $[\phi_k)$ would have a continuum of elements by [1, Cor. 17.1.11].

6 Lattices of Equational Theories

We say that L is a *lattice of equational theories* iff L is isomorphic to the lattice $L(T)$ of all equational theories containing some equational theory T (or dually, the lattices of all subvarieties of some variety of algebras). Such lattice is algebraic and coatomic, possessing a compact top element; but no stronger property was known before Lampe's discovery [8] that any lattice of equational theories obeys the Zipper condition (see Def. 7).

In this section we show the existence of Boolean lattice intervals in the lattices of equational theories as well as a meta version of the Stone representation theorem that holds for all varieties of algebras.

It is well known that any lattice of equational theories is isomorphic to a congruence lattice (see [5, 12]). Indeed, the lattice $L(T)$ of all equational theories containing T is isomorphic to the congruence lattice of the algebra $(\mathbf{F}_T, f)_{f\in \mathrm{End}(\mathbf{F}_T)}$, where $\mathbf{F}_T$ is the free algebra over a countable set of generators in the variety axiomatized by T, and $\mathrm{End}(\mathbf{F}_T)$ is the set of its endomorphisms.

We expand the algebra $(\mathbf{F}_T, f)_{f\in \mathrm{End}(\mathbf{F}_T)}$ (without changing the congruence lattice) by the operation q defined as follows (x_1, x_0 are two fixed variables) $q(t, s_1, s_0) = t[s_1/x_1, s_0/x_0]$, where $t[s_1/x_1, s_0/x_0]$ is the term obtained by substituting term s_i for variable x_i $(i = 0, 1)$ within t. The algebra $(\mathbf{F}_T, f, q)_{f\in \mathrm{End}(\mathbf{F}_T)}$ was defined by Lampe in the proof of McKenzie Lemma in [8].
If we define $1 \equiv x_1$ and $0 \equiv x_0$, from the identities $q(x_1, s_1, s_0) = s_1$ and $q(x_0, s_1, s_0) = s_0$ we get that $(\mathbf{F}_T, f, q)_{f\in \mathrm{End}(\mathbf{F}_T)}$ is a Church algebra. It will be denoted by $\mathbf{C}_T$ and called hereafter the *Church algebra of* T.

In the following lemma we characterize the central elements of $\mathbf{C}_T$.

Lemma 7. *Let T be an equational theory and $\mathcal{V}$ be the variety of Σ-algebras axiomatized by T. Then the following conditions are equivalent, for every $e \in \mathbf{C}_T$ and term $t(x_1, x_0) \in e$:*

- *(i) e is a central element.*
- *(ii) T contains the identities $t(x, x) = x$; $\quad t(x, t(y, z)) = t(x, z) = t(t(x, y), z)$ and $t(f(\overline{x}), f(\overline{y})) = f(t(x_1, y_1), \dots, t(x_n, y_n))$, for $f \in \Sigma$.*
- *(iii) For every $\mathbf{A} \in \mathcal{V}$, the function $t^{\mathbf{A}} : A \times A \to A$ is a decomposition operator.*
- *(iv) $T = T_1 \cap T_0$, where T_i is the theory axiomatized (over T) by $t(x_1, x_0) = x_i$ $(i = 0, 1)$.*

If e and t satisfy the above conditions and e is nontrivial as central element, then by Lemma 7(iii)-(iv) every algebra $\mathbf{A} \in \mathcal{V}$ can be decomposed as $\mathbf{A} \cong \mathbf{A}/\phi \times \mathbf{A}/\overline{\phi}$, where $(\phi, \overline{\phi})$ is the cfc-pair associated with the decomposition operator $t^{\mathbf{A}}$; moreover, the algebras $\mathbf{A}/\phi$ and $\mathbf{A}/\overline{\phi}$ satisfy respectively the equational theories T_1 and T_0. In this case, we say that $\mathcal{V}$ is *decomposable* as a product of the two subvarieties axiomatized respectively by T_1 and T_0 (see [17]); otherwise, we say that $\mathcal{V}$ is *indecomposable*.

Proposition 4. *Let T be an equational theory. Assume there exist n binary terms $t_0, \dots, t_{n-1}$ such that, for every function $k : n \to \{1, 2\}$, the theory axiomatized (over T) by $t_i(x_1, x_0) = x_{k(i)}$ $(i = 0, \dots, n-1)$ is consistent. Then* *there exists a theory $T' \supseteq T$ such that $L(T')$ is isomorphic to the free Boolean lattice with 2^{2^n} elements.*

The set of all factor congruences of an algebra does not constitute in general a sublattice of the congruence lattice. We now show that in every algebra there is a subset of factor congruences which always constitutes a Boolean sublattice of the congruence lattice.

We say that a variety $\mathcal{V}$ is *decomposable as a weak Boolean product of directly indecomposable subvarieties* if there exists a family $\langle \mathcal{V}_i : i \in X \rangle$ of indecomposable subvarieties $\mathcal{V}_i$ of $\mathcal{V}$ such that every algebra $\mathbf{A} \in \mathcal{V}$ is isomorphic to a weak Boolean product $\Pi_{i \in X} \mathbf{B}_i$ of algebras $\mathbf{B}_i \in \mathcal{V}_i$.

Theorem 5. (Meta-Representation Theorem) *Every variety $\mathcal{V}$ of algebras is decomposable as a weak Boolean product of directly indecomposable subvarieties.*

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