COMPITINO DI LOGICA
20 Dicembre 2010

Si svolgano 5 esercizi su 6.

1. Assumiamo i seguenti simboli di costante e predicato:

   - $F \quad : \quad Federer$
   - $N \quad : \quad Nadal$
   - $W(x, y) \quad : \quad x$ gioca con la racchetta y
   - $B(x, y) \quad : \quad x$ è di qualità migliore di y
   - $P(x) \quad : \quad x$ è una racchetta

   Quale delle formule sotto rappresenta l'enunciato "Federer gioca con una racchetta di qualità migliore di qualunque racchetta di Nadal":

   (a) $\forall x(W(F, x) \to B(x, N))$
   (b) $\forall x \exists y(B(x, y) \to W(F, y) \wedge W(N, x))$
   (c) $\exists x(P(x) \wedge W(F, x) \wedge \forall y(P(y) \wedge W(N, y) \to B(x, y)))$
   (d) Nessuna
   (e) $\exists x \forall y(W(F, x) \to W(N, y) \wedge B(x, y))$

   Soluzione: L'enunciato (c).

2. Formalizzare i seguenti enunciati:

   - "Ognuno è padre di qualcuno"
   - "P è soddisfatto da esattamente due elementi"

   Soluzione:
   $\forall x \exists y$Father(x, y);
   $\exists xy(P(x) \wedge P(y) \wedge \neg(x = y) \wedge \forall z(P(z) \to z = x \vee z = y))$.

3. Si consideri il seguente enunciato: $\phi \equiv \exists x \forall y(W(x) \to W(y) \wedge B(x, y))$. Si definisca un modello in cui l'enunciato ϕ è vero, ed un modello in cui l'enunciato ϕ è falso.

   Answer: The sentence ϕ is true in every model $\mathcal{M}$ which admits an element $a \in M$ such that $W^{\mathcal{M}}(a)$ is false. Indeed, the formula $\forall y(W(a) \to W(y) \wedge B(a, y))$ is true because the implication $W(a) \to W(y) \wedge B(a, y)$ is true for every y. The reason is that the assumption $W(a)$ is false.

   The sentence ϕ is false in every model $\mathcal{M}$ satisfying the following conditions:

- $W(a)$ is true for all $a \in M$
- $B(a, a)$ is false for all $a \in M$.

Then, however we chose $a \in M$, the formula $\forall y(W(a) \rightarrow W(y) \wedge B(a, y))$ is false, because putting $y = a$ we get that the implication $W(a) \rightarrow W(a) \wedge B(a, a)$ is false.

4. Sia $\mathcal{L}$ un linguaggio del primo ordine con un predicato binario R(x,y) ed uno unario S(x) (il simbolo di uguaglianza non è ammesso). Definire due modelli di $\mathcal{L}$ che abbiano come universo rispettivamente

   a. L'insieme $\{0, 1, 2\}$.
   b. L'insieme $\{0, 1, 2, 3\}$.

   Si scriva poi un enunciato vero nel primo modello e falso nel secondo.

   Answer: The first model $\mathcal{M}_1$ has universe $M_1 = \{0, 1, 2\}$ and the predicates are defined as follows: $R^{\mathcal{M}_1}(x, y)$ is true for all $x, y \in M_1$; $S^{\mathcal{M}_1}(x)$ is true sse $x = 1, 2$.

   The second model $\mathcal{M}_2$ has universe $M_2 = \{0, 1, 2, 3\}$ and the predicates are defined as follows: $R^{\mathcal{M}_2}(x, y)$ is true for all $x, y \in \{0, 1, 2\}$ and $R^{\mathcal{M}_2}(x, y)$ is false if either x or y is equal to 3; $S^{\mathcal{M}_2}(x)$ is true sse $x = 1, 2$.

   Then the formula $\forall xyR(x, y)$ is true in $\mathcal{M}_1$ and false in $\mathcal{M}_2$.

5. Si provi $A \vee \neg A$ in deduzione naturale.

   Answer:

$$\dfrac{A : H_1}{}\ [\vee_i]$$

$$\dfrac{A \vee \neg A \qquad \neg(A \vee \neg A) : H_2}{}\ [\neg_e]$$

$$\dfrac{\bot}{}\ [\neg_i; H_1 \text{ is discharged}]$$

$$\dfrac{\neg A}{}\ [\vee_i]$$

$$\dfrac{A \vee \neg A \qquad\qquad \neg(A \vee \neg A) : H_3}{}\ [\neg_e]$$

$$\dfrac{\bot}{A \vee \neg A}\ [RRA; H_{2,3} \text{ are discharged}]$$

6. Si provi a grandi linee che $\Gamma \vdash_{ND} \phi$ se e solo se $\Gamma \vdash_H \phi$.

   Answer: ($\Rightarrow$) The proof is by induction over the complexity of the proof tree of $\Gamma \vdash_{ND} A$. Base of the induction: the tree has complexity 1. Then it is constituted by the formula A

(i.e., A belongs to Γ). The same tree is also a proof in system H. Assume that we have a proof tree T for $\Gamma \vdash_{ND} A$.

If the last rule applied is, for example, $\wedge_i$, then $A = B \wedge C$ for some formulas B and C. Then there exist two proof subtrees T_1 and T_2 proving respectively $\Gamma \vdash_{ND} B$ and $\Gamma \vdash_{ND} C$. The complexity of of T_1 and T_2 is less than the complexity of T. By induction hypothesis there exist two proof trees T_1^* and T_2^* proving $\Gamma \vdash_H B$ and $\Gamma \vdash_H C$. We add to these trees two applications of MP by using the axiom $B \to C \to B \wedge C$.

If the last rule applied is $\vee_e$, then we have three subtrees:

(a) T_1 proving $\Gamma, B \vdash_{ND} A$

(b) T_2 proving $\Gamma, C \vdash_{ND} A$

(c) T_3 proving $\Gamma \vdash_{ND} B \vee C$

By induction hypothesis there exist three trees T_1^*, T_2^* and T_3^* proving respectively $\Gamma, B \vdash_H A$, $\Gamma, C \vdash_H A$ and $\Gamma \vdash_H B \vee C$. By applying to T_1^* and T_2^* the following proposition:

Proposition. If $\Gamma, A \vdash C$ (by using MP as unique rule), then there exists a proof of $\Gamma \vdash A \to C$ (by using MP as unique rule) if the axiomatic system contains the following axioms:

(I) $A \to A$

(K) $C \to (A \to C)$

(S) $(A \to (B \to C)) \to (A \to B) \to (A \to C)$.

we get two other proofs S_1 and S_2 proving respectively $\Gamma \vdash_H B \to A$ and $\Gamma \vdash_H C \to A$. Finally, three applications of MP to the axiom $(B \to A) \to (C \to A) \to (B \vee C) \to A$ provide a proof of A in the Hilbert system. In a similar way for the other rules.

($\Leftarrow$) It is sufficient to prove that Hilbert's axioms are provable in ND. This is trivial. For example, consider the or introduction: $(A \to C) \to (B \to C) \to (A \vee B) \to C$. By ($\to_i$) it is sufficient to prove that $\Gamma \vdash C$, where $\Gamma = A \to C, B \to C, A \vee B$. By $\vee_e$ this is equivalent to prove $\Gamma, A \vdash C$ and $\Gamma, B \vdash C$.

(a) $\Gamma, A \vdash A$ (axiom)

(b) $\Gamma, A \vdash A \to C$ (axiom)

(c) $\Gamma, A \vdash C$ by MP