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A Finite Equational Axiomatization of the Functional Algebras for the Lambda Calculus

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A lambda theory satisfies an equation between contexts, where a context is a λ-term with some "holes" in it, if all the instances of the equation fall within the lambda theory. In the main result of this paper it is shown that the equations (between contexts) valid in every lambda theory have an explicit finite equational axiomatization. The variety of algebras determined by the above equational theory is characterized as the class of isomorphic images of functional lambda abstraction algebras. These are algebras of functions and naturally arise as the "coordinatizations" of environment models or lambda models, the natural combinatory models of the lambda calculus. The main result of this paper is also applied to obtain a completeness theorem for the infinitary lambda calculus recently introduced by Berarducci.

Key Words: lambda calculus; lambda abstraction algebras; abstract substitution; combinatory algebras; lambda algebras; lambda models.

INTRODUCTION

The untyped lambda calculus was introduced by Church [8, 9] as a foundation for logic. Although the appearance of paradoxes caused the program to fail, a consistent part of the theory turned out to be successful as a theory of "functions as rules" (formalized as terms of the lambda calculus) that stresses the computational process of going from the argument to value. Every object is at the same time a function and an argument; in particular a function can be applied to itself contrary to the usual notion of function in set theory. Although lambda calculus has been the subject of research by logicians since the early 1930s, its model theory developed

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only much later, following the pioneering model construction made by Dana Scott in 1969. The subsequent decade saw a proliferation of such constructions, leading eventually to a consensus as to what in general a model of the lambda calculus should be (cf. [23, 28, 42]). The notion of an *environment model* (the name is due to Meyer [28]) originated with Hindley and Longo [23]. They are functional domains where λ-terms can be properly interpreted. Meyer describes them as "the natural, most general formulation of what might be meant by mathematical models of the untyped lambda calculus." The main result in [28] is a completeness theorem demonstrating that every lambda theory is the theory associated with some environment model. The drawback of environment models is that they are higher order structures.

There have been several attempts to reformulate the lambda calculus as a purely equational theory. However, the general methods that have been developed in universal algebra and category theory, for defining the semantics of an arbitrary algebraic theory, are not directly applicable to the lambda calculus. The reason is that the untyped lambda calculus is not an equational theory in the normal sense because the equations, unlike the associative and commutative laws for example, are not always preserved when arbitrary terms are substituted for variables (e.g., $\lambda x.yx = \lambda z.yz$ does not imply $\lambda x.xx = \lambda z.xz$). The earliest, and best known, algebraic models are the combinatory algebras of Curry. Combinatory algebras [11] have a simple pure equational characterization. Curry also specified (by a considerably less natural set of axioms) a pure equational subclass of combinatory algebras, the λ-*algebras* (see [3], Section 5.2.5), that he viewed as algebraic models of the lambda calculus. It was later discovered that the combinatory algebras most closely connected to environment models have an intrinsic characterization (up to isomorphism) as a special class of λ-algebras called *lambda models* ([3, Section 5.2.7]). They were first axiomatized by Meyer [28] and independently by Scott [42]; the first-order axiomatization of lambda models while elegant is not equational. It turns out however that the class of λ-algebras is the variety generated by the lambda models [4].

In [31, 34] Pigozzi and Salibra introduced the variety of lambda abstraction algebras (LAA's) which constitutes a purely algebraic theory of the untyped lambda calculus in the same spirit that Boolean algebras constitute an algebraic theory of classical propositional logic and, more to the point, cylindric and polyadic (Boolean) algebras of the first-order predicate logic. Combinatory algebras (CA's) and lambda abstraction algebras are both defined by universally quantified equations and thus form varieties in the universal algebraic sense. There are important differences however that result in theories of very different character. Functional application is taken as a fundamental operation in both CA's and LAA's. Lambda (i.e., functional) abstraction is also fundamental in LAA's but in CA's is defined in terms of the combinators **k** and **s**. A more important difference is connected with the role variables play in the lambda calculus as place holders. In an LAA this is also abstracted. It takes the form of a system of fundamental elements (nullary operations) of the algebra. This is a crucial feature of LAA's that is borrowed from cylindric and polyadic algebras and has no direct analogue in CA's. One important consequence of the abstraction of variables is the abstraction of term-for-variable substitution in

LAA's by inverting (β)-conversion in a natural way. In contrast, in CA's substitution is simulated by application of special combinators.

The combinatory models of lambda calculus constitute an elementary class that is not closed under natural algebraic operations. For example, a substructure of a lambda model is not in general a lambda model but only a λ-algebra. So, when we move from the combinatory models of the lambda calculus to LAA's, we earn "algebraic" properties, because LAA's constitute an equational class. Moreover, LAA's are closely related to the combinatory models of the lambda calculus since the most natural LAA's are algebras of functions, called functional LAA's, which arise as "expansions" of environment models or lambda models by the variables of lambda calculus in a natural way. The axiomatization of functional lambda abstraction algebras is a central issue in the algebraic approach to the model theory of lambda calculus. In the main result of this paper, solving a problem raised by Pigozzi and Salibra in [34], we show that every LAA is a suitable "expansion" of a combinatory model of lambda calculus, that is, it is isomorphic to a functional LAA. So, the class of isomorphic images of functional lambda abstraction algebras constitutes a variety of algebras axiomatized by the finite schema of identities characterizing LAA's.

Questions of the functional representation of various subclasses of lambda abstraction algebras were investigated by Pigozzi and Salibra in a series of papers [31–34].

Our main result proves useful in the lambda calculus as a way for applying the methods of universal algebra. We recall from Barendregt [3, Def. 14.4.1] that a context is a λ-term with some "holes" in it. The essential feature of a context is that a free variable in a λ-term may become bound when we substitute it for a "hole" within the context. A lambda theory satisfies an equation between contexts if all the instances of the equation, obtained by substituting λ-terms for holes in it, falls within the lambda theory. We show that an equation between contexts is true in every lambda theory if and only if it is satisfied in every LAA. So, the explicit finite equational axiomatization for the variety of LAA's provides also an explicit axiomatization of the equations between contexts valid in every lambda theory.

Recent work has been done by many authors on infinitary versions of lambda calculus. Berarducci defines in [5] a new model of $\lambda\beta$-calculus which is similar to the model of Böhm trees, but it does not identify all the unsolvable lambda terms. His method, that is based on an infinitary version of the lambda calculus, is also used in [6] to obtain Church–Rosser extensions of the finitary lambda calculus. The consistency results obtained in [6] would be very difficult to prove without this detour into the infinitary lambda calculus. Another infinitary version of lambda calculus has been independently introduced by Kenneway *et al.* in [24]. As an application of the main result of the paper, we prove a completeness theorem for the infinitary lambda calculus with a semantics given in terms of environment models. We also introduce a uniform family of models of $\lambda\beta$-calculus which includes the model of Böhm trees and the model introduced by Berarducci in [5].

Outline of paper. In the first section of this paper we review the basic definitions of the lambda calculus and summarize, without proofs, all definitions and results from [34] and [36] that will be needed in the subsequent part of the paper; in particular, we recall the formal definition of a lambda abstraction algebra and the theory of abstract substitution.

The main results of the paper are presented in Section 2. We prove that the combinatory reduct of every LAA is a λ-algebra. This result serves as a basis for the main result of the paper according to which the class of isomorphic images of functional LAA's coincides with the variety of LAA's. The identities characterizing LAA's axiomatize also the equations between contexts valid in every lambda theory.

In Sections 3–6 we prove the technical results needed for the proof of the main result presented in Section 2.

The relationship between the infinitary lambda calculus and lambda abstraction algebras is investigated in detail in Section 7. We begin by recalling the notion of infinitary λ-term. Then we show that every infinitary lambda theory determines a LAA. This is the basis for the completeness theorem for the infinitary lambda calculus.

1. BASIC NOTIONS AND NOTATION

To keep this article self-contained, we summarize, without proof, definitions and results from [34] and [36] that we will need in the subsequent part of the paper. Concerning lambda calculus we usually use notation and notions from Barendregt's book (see [3]).

Lambda Abstraction Algebras

Let I be a nonempty set. The similarity type of *lambda abstraction algebras of dimension* I is $\langle \cdot, \langle \lambda x \colon x \in I \rangle, \langle x \colon x \in I \rangle \rangle$, where "$\cdot$" is a binary operation symbol formalizing application, "λx" is a unary operation symbol for every $x \in I$, and "x" is a constant symbol (i.e., nullary operation symbol) for every $x \in I$. Note that "λx" is to be viewed as an indivisible symbol. The elements of I are to be thought of as the variables of lambda calculus although in their algebraic transformation they no longer play the role of variables in the usual sense. We will refer to them as *λ-variables*. The actual variables of the lambda abstraction theory will be referred to as *context variables* and denoted by the greek letters ξ, ν, and μ, possibly with subscripts. The terms of the language of lambda abstraction theory are called *λ-contexts*. They are constructed in the usual way: every λ-variable x and context variable ξ is a λ-context; if t and s are λ-contexts, then so are $t \cdot s$ and $\lambda x(t)$. Because of their similarity to the terms of the lambda calculus we use the standard notational conventions of the latter. The application operation symbol "$\cdot$" is normally omitted, and the application of t and s is written as juxtaposition ts. When parentheses are omitted, association to the left is assumed. The left parenthesis delimiting the scope of a lambda abstraction is replaced with a period and the right parenthesis is omitted. For example, $\lambda x(ts)$ is written $\lambda x.ts$. Successive λ-abstractions $\lambda x \lambda y \lambda z \cdots$ are written $\lambda xyz \cdots$.

A word of caution for those readers familiar with the lambda calculus. When dealing with models of the lambda calculus one often allows terms that contain constant symbols representing the elements of the models. These constants should not be confused with context variables; they play a much different role. Our notion of a λ-context coincides with the notion of *context* defined in Barendregt [3, Definition 14.4.1]; our context variables correspond to Barendregt's notion of a "hole." For example, the λ-context $(\lambda x.\xi\mu)\,y$ corresponds to Barendregt's context $(\lambda x.[\ ]_1[\ ]_2)\,y$.

An occurrence of a λ-variable x in a λ-context is *bound* if it falls within the scope of the operation symbol λx; otherwise it is *free*. The *free λ-variables* of a λ-context are the λ-variables that have at least one free occurrence. A λ-context without any context variables is said to be a λ-term. A λ-context without free λ-variables is said to be *closed*. Note that λ-terms of lambda abstraction theory coincide with ordinary terms of the lambda calculus.

We now give the formal definition of a lambda abstraction algebra. Readers unfamiliar with the notation of the lambda calculus may want to go directly to the reformulation of the axioms, in terms of the substitution operations, that is given later.

Definition 1. By a *lambda abstraction algebra of dimension I* we mean an algebraic structure of the form

$$\mathbf{A} := \langle A, \cdot^{\mathbf{A}}, \langle \lambda x^{\mathbf{A}} : x \in I \rangle, \langle x^{\mathbf{A}} : x \in I \rangle \rangle$$

satisfying the following identities and quasi-identities for all $x, y, z \in I$ (subject to the indicated conditions) and all $\xi, \mu, \nu \in A$.

(β_1) $(\lambda x.x)\,\xi = \xi$;

(β_2) $(\lambda x.y)\,\xi = y, \quad x \neq y$;

(β_3) $(\lambda x.\xi)\,x = \xi$;

(β_4) $(\lambda xx.\xi)\,\mu = \lambda x.\xi$;

(β_5) $(\lambda x.\xi\mu)\,\nu = (\lambda x.\xi)\,\nu(\lambda \mathrm{x}.\mu)\,\nu$

(β_6) $(\lambda y.\mu)\,z = \mu \rightarrow (\lambda xy.\xi)\,\mu = \lambda y.(\lambda x.\xi)\,\mu, \quad x \neq y, z \neq y$;

(α) $(\lambda y.\xi)\,z = \xi \rightarrow \lambda x.\xi = \lambda y.(\lambda x.\xi)\,y, \quad z \neq y$.

I is called the *dimension set* of $\mathbf{A}$, $\cdot^{\mathbf{A}}$ is called *application*, and $\lambda x^{\mathbf{A}}$ is called *λ-abstraction* with respect to x.

The class of lambda abstraction algebras of dimension I is denoted by LAA_I and the class of all lambda abstraction algebras of any dimension by LAA. We also use LAA_I as shorthand for the phrase "lambda abstraction algebra of dimension I," and similarly for LAA. An LAA_I is *infinite dimensional* if I is infinite.

In the sequel $\mathbf{A}$ will be an arbitrary infinite dimensional LAA_I, unless otherwise noted.

We will omit the superscript $^{\mathbf{A}}$ on $\cdot^{\mathbf{A}}$, $\lambda x^{\mathbf{A}}$, and $x^{\mathbf{A}}$ whenever we are sure we can do so without confusion. This will also apply to defined notions introduced below, such as $\triangle^{\mathbf{A}}$.

In the presence of the other axioms, (β_6) and (α) are equivalent to identities

(β_6') $(\lambda xy.\xi)((\lambda y.\mu)\, z = \lambda y.(\lambda x.\xi)((\lambda y.\mu)\, z), \quad x \neq y, z \neq y.$

(α') $\lambda x.(\lambda y.\xi)\, z = \lambda y.(\lambda x.(\lambda y.\xi)\, z)\, y, \quad z \neq y.$

Thus LAA_I is a variety for every dimension set I, and therefore is closed under the formation of subalgebras, homomorphic (in particular isomorphic) images, and Cartesian products. In symbols $\mathbb{S}\, \mathsf{LAA}_I = \mathbb{H}\, \mathsf{LAA}_I = \mathbb{I}\, \mathsf{LAA}_I = \mathbb{P}\, \mathsf{LAA}_I = \mathsf{LAA}_I$.

We note here one very useful immediate consequence of the axioms: in any LAA $\mathbf{A}$ the functions λx are always one–one, i.e., for all $x \in I$,

$$\lambda x.a = \lambda x.b \qquad \text{iff} \quad a = b, \qquad \text{for all} \quad a, b \in A.$$

An LAA with only one element is said to be *trivial*. It is interesting that there do not exist nontrivial finite models. In fact, any nontrivial LAA_I of positive dimension is infinite, since the one–one map λx is not onto.

Substitution and Dimension

When transformed into the equational language of lambda abstraction theory, (β)-conversion becomes the definition of abstract substitution. It takes the following form: For any set B, let B^* be the set of all finite strings of elements of B.

DEFINITION 2. Let $\mathbf{A}$ be an LAA_I.

(i) $S^x_b(a) = (\lambda x.a)b$ for all $x \in I$ and $a, b \in A$.

(ii) $S^{\mathbf{x}}_{\mathbf{b}}(a) = S^{x_1}_{b_1}(\ldots(S^{x_n}_{b_n}(a))\ldots)$ for all $\mathbf{x} = x_1 \cdots x_n \in I^*$, $\mathbf{b} = b_1 \cdots b_n \in A^*$, and $a \in A$.

S is called the (*abstract*) *substitution operator*.

The axioms for lambda abstraction algebras can be reformulated in the following way:

(β_1) $S^x_\xi(x) = \xi;$

(β_2) $S^x_\xi(y) = y, \quad y \neq x;$

(β_3) $S^x_x(\xi) = \xi;$

(β_4) $S^x_\mu(\lambda x.\xi) = \lambda x.\xi;$

(β_5) $S^x_\nu(\xi\mu) = S^x_\nu(\xi)\, S^x_\nu(\mu);$

(β_6) $S^y_z(\mu) = \mu \Rightarrow S^x_\mu(\lambda y.\xi) = \lambda y.S^x_\mu(\xi), \quad x \neq y, z \neq y;$

(α) $S^y_z(\xi) = \xi \Rightarrow \lambda x.\xi = \lambda y.S^x_y(\xi), \quad y \neq z.$

If a λ-term M does not admit free occurrences of a λ-variable x, then the process of substituting an arbitrary λ-variable z for x in M does not change M. This process is abstracted in this way.

DEFINITION 3. Let $\mathbf{A}$ be an LAA_I. Let $a \in A$ and $x \in I$. a is said to be *algebraically dependent on* x (*over* $\mathbf{A}$) if $(\lambda x.a)\, z \neq a$ for some $z \in I$; otherwise a is *algebraically independent of* x (*over* $\mathbf{A}$). The set of all $x \in I$ such that a is algebraically dependent on x over $\mathbf{A}$ is called the *dimension set* of a and is denoted by $\triangle^{\mathbf{A}} a$; thus

$$\triangle^{\mathbf{A}} a = \{x \in I : (\lambda x.a)\, z \neq a \text{ for some } z \in I\}.$$

a is *finite* (*infinite*) *dimensional* if $\triangle a$ is finite (infinite). An element a is called *zero-dimensional* if $\triangle a = \emptyset$. We denote the set of zero-dimensional elements by Zd $\mathbf{A}$.

It is convenient to treat algebraic dependency as a symmetric relation and speak of "x being algebraically dependent on (independent of) a". The following are two useful alternative characterizations of algebraic dependency and consequently of dimension set: $x \notin \triangle a$ iff $S^x_z(a) = a$ for some $z \in I \backslash \{x\}$ iff $S^x_b(a) = a$ for all $b \in A$.

In the following three propositions we give some basic properties of substitution and dimension set that will be used repeatedly in the sequel. The proofs of Propositions 4, 5 and 6 can be found in [34].

Proposition 4. *Let* $\mathbf{A} \in \mathsf{LAA}_I$, $a, b \in A$, *and* $x \in I$.

(i) $\triangle(ab) \subseteq \triangle a \cup \triangle b$.

(ii) $\triangle(\lambda x.a) = \triangle a \backslash \{x\}$.

(iii) $\triangle(S^x_b(a)) \subseteq (\triangle a \backslash \{x\}) \cup \triangle b$.

(iv) $\triangle x \subseteq \{x\}$, *with equality holding if* $\mathbf{A}$ *is nontrivial.*

Proposition 5. *For all* $x, y, z \in I$ *and* $a, b, c \in A$ *we have*:

(i) $x \notin \triangle c \Rightarrow S^y_c S^x_b(a) = S^x_{S^y_c(b)} S^y_c(a)$;

(ii) $y \notin \triangle b \Rightarrow S^y_b S^x_y(a) = S^y_b S^x_b(a)$;

(iii) $y \notin \triangle a \Rightarrow S^y_b(a) = a$;

(iv) $x \notin \triangle c,\ y \notin \triangle b \Rightarrow S^y_c S^x_b(a) = S^x_b S^y_c(a), \quad x \neq y$;

(v) $z \notin \triangle a \cup \triangle b \Rightarrow S^x_b(a) = S^z_b S^x_z(a)$.

For any set B, let $B^\star$ denote the set of all finite strings of elements of B without repetitions. We recall that B^* denotes the set of all finite strings of elements of B possibly with repetitions.

Proposition 6. *Let* $\mathbf{A}$ *be an* LAA_I, $\mathbf{x} = x_1 \cdots x_n \in I^\star$, *and* $\mathbf{b} = b_1 \cdots b_n \in A^*$. *If* b_i *is independent of* $x_1, ..., x_{i-1}$ *for* $i = 2, ..., n$, *in particular, if each* b_i *is independent of all the* x_j, *then*

$$S^{\mathbf{x}}_{\mathbf{b}}(a) = (\lambda x_1 \cdots x_n.a)\, b_1 \cdots b_n \qquad \textit{for all} \quad a \in A.$$

Lambda Calculus and Locally Finite LAA*'s*

The set $\Lambda_I(C)$ of ordinary terms of lambda calculus over a set I of λ-variables and a set C of constants is constructed as usual [3]: every λ-variable $x \in I$ and every constant $c \in C$ is a λ-term; if t and s are λ-terms, then so are (st) and $\lambda x.t$ for each λ-variable x. An occurrence of a λ-variable $x \in I$ in a λ-term is *bound* if it lies within the scope of a lambda abstraction λx; otherwise it is *free*. A λ-term s is *free for* x in t if no free occurrence of x in t lies within the scope of a lambda abstraction with respect to a λ-variable that occurs free in s. $t[x := s]$ is the result of substituting

s for all free occurrences of x in t subject to the usual provisos about renaming bound λ-variables in t to avoid capture of free λ-variables in s.

The axioms of the $\lambda\beta$-calculus are as follows: t and s are arbitrary λ-terms and x, y λ-variables.

(α) $\lambda x.t = \lambda y.t[x := y]$, for any λ-variable y that does not occur free in t;

(β) $(\lambda x.t)\, s = t[x := s]$, for every s free for x in t;

$t = t$;

$t = s$ implies $s = t$;

$t = s$, $s = r$ imply $t = r$;

$t = s$, $u = r$ imply $tu = sr$;

$t = s$ implies $\lambda x.t = \lambda x.s$.

(β)-conversion expresses the way of calculating a function $(\lambda x.t)$ on an argument s, while (α)-conversion says that bound λ-variables can be replaced in a term under the obvious condition. A *lambda theory* over $\Lambda_I(C)$ is any set of equations that is closed under (α)- and (β)-conversion and the five equality rules. An I-lambda theory is any lambda theory over $\Lambda_I(C)$ for some set C of constants.

There is a strong connection between the lambda theories and the subclass of LAA's whose elements are finite dimensional.

DEFINITION 7. A lambda abstraction algebra $\mathbf{A}$ is *locally finite* if it is of infinite dimension (i.e., I is infinite) and every $a \in A$ is of finite dimension (i.e., $|\triangle a| < \omega$).

The class of locally finite LAA's (LAA_I's) is denoted by LFA (LFA_I), which is also used as shorthand for the phrase "locally finite lambda abstraction algebra (of dimension I)".

A precise connection between lambda theories and locally finite LAA's (LFA's) is established in Propositions 8 and 9 below (see [36, Proposition 2.4]).

PROPOSITION 8. *Let T be a lambda theory in the language $\Lambda_I(C)$ and let*

$$\Lambda_{\mathbf{I}}(C) := < \Lambda_I(C),\ \cdot^{\Lambda_I(C)}, \langle \lambda x^{\Lambda_I(C)} \colon x \in I \rangle, \langle x^{\Lambda_I(C)} \colon x \in I \rangle >$$

be the absolutely free algebra of λ-terms. Then T is a congruence over the algebra $\Lambda_{\mathbf{I}}(C)$ making $\Lambda_{\mathbf{I}}(C)/T$ an LFA_I.

We call $\Lambda_{\mathbf{I}}(C)/T$ the *term* LAA of the lambda theory T. The local finiteness is a direct consequence of the trivial fact that every λ-term is a finite string of symbols and hence contains only finitely many λ-variables. Note that the set of zero-dimensional elements of a term LAA is the set of all those elements which are equivalence classes of closed λ-terms, i.e., terms without free λ-variables.

The following proposition is the algebraic analogous of Propositions 1 and 3 in [25, Chapter VII].

PROPOSITION 9. *An algebra $\mathbf{A}$ is (isomorphic to) the term algebra of a lambda theory if, and only if, it is an* LFA.

The class of LFA_I's is not elementary, i.e., it cannot be characterized by a set of first-order sentences. Suppose, on the contrary, there is a set L of first-order sentences such that $\mathbf{A}$ is an LFA_I if, and only if, $\mathbf{A}$ satisfies all the sentences in L. Consider the set

$$L_1 = \{(\exists \xi)\ S^x_\xi(c) \neq c : x \in I\}$$

of first-order sentences in the language of LAA's with an additional individual constant c. Let L_2 be a finite subset of L_1 and let $J = \{y_1, ..., y_k\}$ be the finite set of all λ-variables occurring in the members of L_2. It is easy to show that the term model of the lambda theory $\lambda\beta$ with c interpreted as $y_1 y_2 \cdots y_k$ satisfies all the sentences in $L \cup L_2$. By the compactness theorem for first-order logic, there is an algebra $\mathbf{B}$ that satisfies all the sentences of $L \cup L_1$. If b is an element of $\mathbf{B}$ that is the interpretation of the constant c, then the dimension set of b is all I. Hence $\mathbf{B}$ is not locally finite. This is a contradiction. It follows from this result that some LAA's are not locally finite: e.g. the set of Böhm-like trees over a set I of λ-variables, as defined in [3, Definition 10.1.12], determines a non locally finite LAA_I when the operations of application and lambda abstraction are defined as in [3, Definition 18.3.2].

One of the consequences of the main result in this paper is that LAA_I is just the equational class generated by LFA_I, so that every LAA_I with infinite-dimensional elements can be build from the locally finite ones by using only products, subalgebras and homomorphic images.

Functional Lambda Abstraction Algebras

The most natural LAA's, the algebras that the axioms are intended to characterize, are algebras of functions. Not surprisingly, they are closely related to the environment models of lambda calculus. Indeed, they are obtained by coordinatizing environment models by the λ-variables in a natural way.

We begin by giving the formal definition of functional domain; environment models turn out to be special kinds of functional domains.

DEFINITION 10. Let $\mathbf{V} = \langle V, \cdot^{\mathbf{V}}, \lambda^{\mathbf{V}} \rangle$ be a structure where V is a nonempty set, $\cdot^{\mathbf{V}}$ is a binary operation on V, and $\lambda^{\mathbf{V}} : V^V \xrightarrow{\text{p}} V$ is a partial function assigning elements of V to certain functions from V into itself. $\mathbf{V}$ is called a *functional domain* if for each f in the domain of $\lambda^{\mathbf{V}}$,

$$f(v) = \lambda^{\mathbf{V}}(f) \cdot^{\mathbf{V}} v, \qquad \text{for all} \quad v \in V.$$

This definition of functional domain differs slightly from the one in Meyer [28] where it is assumed that each function of the form $\langle u \cdot^{\mathbf{V}} v : v \in V \rangle$ is in the domain of $\lambda^{\mathbf{V}}$.

The definition of an environment model, as given in [28], originated with Hindley and Longo [23]. The notion of a *syntactical model* defined in Barendregt [3] is closely related.

A *lambda polynomial over a functional domain* $\mathbf{V}$ is a λ-term over a set of constants that includes a constant symbol $\bar{v}$ for each $v \in V$.

Let $\mathbf{V}$ be a functional domain and I be a set of λ-variables. An element p of V^I, i.e., an assignment of elements of V to the set of λ-variables, is called an *environment*. p_x is the value p assigns to x for each $x \in I$. For any $v \in V$ and $x \in I$, $p\{v/x\} \in V^I$ is the new environment such that for all $y \in I$

$$p\{v/x\}_y := \begin{cases} v, & \text{if} \quad y = x \\ p_y, & \text{otherwise.} \end{cases}$$

Define a partial mapping $[\![t]\!]^{\mathbf{V}}\colon V^I \xrightarrow{\text{p}} V$ by recursion on the structure of lambda polynomials over $\mathbf{V}$: for all $p \in V^I$,

$$[\![\bar{v}]\!]^{\mathbf{V}}(p) = v, \qquad \text{for all} \quad v \in V,$$

$$[\![x]\!]^{\mathbf{V}}(p) = p_x, \qquad \text{for all} \quad x \in I,$$

$$[\![ts]\!]^{\mathbf{V}}(p) = [\![t]\!]^{\mathbf{V}}(p) \cdot^{\mathbf{V}} [\![s]\!]^{\mathbf{V}}(p),$$
$$[\![\lambda x.t]\!]^{\mathbf{V}}(p) = \lambda^{\mathbf{V}}\langle [\![t]\!]^{\mathbf{V}}(p\{v/x\})\colon v \in V\rangle.$$

Definition 11. (Meyer [28]) $\mathbf{V}$ is an *environment model* if $[\![t]\!]^{\mathbf{V}}$ is defined for all $p \in V^I$ and all lambda polynomials t over $\mathbf{V}$.

The completeness theorem for the lambda calculus says that every lambda theory consists of precisely the equations valid in some environment model (see [28]). With aid of the precise connection between lambda theories and LFA's established in Propositions 8 and 9, the completeness theorem for the lambda calculus can be also obtained as a corollary of the functional representation theorem for LFA's (see Theorem 17).

Environment models can be also characterized in terms of functional LAA's.

Definition 12. Let $\mathbf{V} = \langle V, \cdot^{\mathbf{V}}, \lambda^{\mathbf{V}}\rangle$ be a functional domain and let I be a nonempty set. Let $V_I = \{f : f\colon V^I \xrightarrow{\text{p}} V\}$, i.e., the set of all partial functions from V^I to V. By the *I-coordinatization* of $\mathbf{V}$ we mean the algebra

$$\mathbf{V}_I = \langle V_I, \cdot^{\mathbf{V}_I}, \langle \lambda x^{\mathbf{V}_I}\colon x \in I\rangle, \langle x^{\mathbf{V}_I}\colon x \in I\rangle\rangle,$$

where for all $a, b\colon V^I \xrightarrow{\text{p}} V$, $x \in I$, and $p \in V^I$:

- $(a \cdot^{\mathbf{V}_I} b)(p) = a(p) \cdot^{\mathbf{V}} b(p)$, provided $a(p)$ and $b(p)$ are both defined; otherwise $(a \cdot^{\mathbf{V}_I} b)(p)$ is undefined.
- $(\lambda x^{\mathbf{V}_I}.a)(p) = \lambda^{\mathbf{V}}(\langle a(p\{v/x\}) : v \in V\rangle)$, provided $\langle a(p\{v/x\}) : v \in V\rangle$ is in the domain of $\lambda^{\mathbf{V}}$ (note this implies $a(p\{v/x\})$ is defined for all $v \in V$); otherwise $(\lambda x^{\mathbf{V}_I}.a)(p)$ is undefined.
- $x^{\mathbf{V}_I}(p) = p_x$.

Definition 13. Let $\mathbf{V}$ and I be as in the preceding definition. A subalgebra $\mathbf{A}$ of total functions of $\mathbf{V}_I$, i.e., a subalgebra such that $(\lambda x^{\mathbf{V}_I}.a)(p)$ is defined for all

$a \in A$ and $p \in V^I$, is called a *functional lambda abstraction algebra*. I is the *dimension set* of $\mathbf{A}$ and $\mathbf{V}$ is its *value domain*.

In the sequel a subalgebra of $\mathbf{V}_I$ of total functions will be called a *total subalgebra* of $\mathbf{V}_I$.

PROPOSITION 14 ([34]). *Every functional lambda abstraction algebra is a lambda abstraction algebra.*

The class of all functional lambda abstraction algebras of dimension I is denoted by FLA_I, and the class of functional lambda abstraction algebras of arbitrary dimension is denoted by FLA. As in the case of LAA and LFA we also use FLA (FLA_I) as shorthand for the phrase "functional lambda abstraction algebra (of dimension I)". Locally finite-dimensional FLA s are similar to the functional models of the lambda calculus developed in Krivine [25].

In [34] it was shown that, for every environment model $\mathbf{V}$, the set of all FLA_I's with value domain $\mathbf{V}$ constitutes a complete lattice under inclusion. The following result characterizing the elements of the largest element in the lattice will be repeatedly used in the sequel.

We recall that, for any set S, S^* is the set of all finite strings of elements of S, while $S^\star$ is the set of all finite strings of elements of S without repetitions.

THEOREM 15 ([34]; Theorem 5.8). *Let $\mathbf{V}$ be a functional domain. If the class of FLA_I's with value domain $\mathbf{V}$ is nonempty, then there exists a largest FLA_I, denoted by $\mathbf{V}_I^\top$, with value domain $\mathbf{V}$. For any a: $V^I \to V$ we have $a \in V_I^\top$ iff there exists a $(V^I \times I^\star)$-indexed system u_a of elements of V such that, for all $p \in V^I$, $\mathbf{x} = x_1 \cdots x_n \in I^\star$, and $\mathbf{v} = v_1 \cdots v_n \in V^*$*

$$a(p\{v_1/x_1, ..., v_n/x_n\}) = u_{a, p, \mathbf{x}} v_1 \cdots v_n.$$

FLA_I's of the form $\mathbf{V}_I^\top$ are called *full* FLA_I's. Every FLA_I over $\mathbf{V}$ is a subalgebra of $\mathbf{V}_I^\top$.

The following characterization of environment models in terms of FLA's was found by Pigozzi and Salibra [34].

THEOREM 16. *A functional domain $\mathbf{V}$ is an environment model if, and only if, there exists at least one FLA with value domain $\mathbf{V}$.*

Proof. If $\mathbf{V}$ is an environment model, then clearly the set of functions $[\![t]\!]^{\mathbf{V}}$, where t ranges over all lambda polynomials, coincides with the universe of the subalgebra $\mathbf{V}_I^\top(V)$ of $\mathbf{V}_I$ generated by the constant functions $[\![\bar{v}]\!]^{\mathbf{V}}$ for all $v \in V$. So $\mathbf{V}$ is an environment model in Meyer's sense iff $\mathbf{V}_I^\top(V)$ is a total subalgebra of $\mathbf{V}_I$, i.e., if it is an FLA with value domain $\mathbf{V}$.

In the opposite direction, the full FLA_I $\mathbf{V}_I^\top$ contains all constant functions. Thus $\mathbf{V}_I^\top$ includes $\mathbf{V}_I^\top(V)$ and hence the latter is total. ∎

The functional representation theorem is the main result of [31] (see also Theorem 3.11 in [34]) and was independently proved by Diskin and Beylin [13].

Let $\mathbf{A} = \langle A, \cdot^{\mathbf{A}}, \lambda x^{\mathbf{A}}, x^{\mathbf{A}} \rangle_{x \in I}$ be an arbitrary LAA_I. The functional domain $\mathbf{V} = \langle V, \cdot^{\mathbf{V}}, \lambda^{\mathbf{V}} \rangle$ *associated with* $\mathbf{A}$ is defined as follows: $V = A$ and $\cdot^{\mathbf{V}} = \cdot^{\mathbf{A}}$. The domain of $\lambda^{\mathbf{V}}$: $V^V \xrightarrow{\mathrm{p}} V$ is

$$dom(\lambda^{\mathbf{V}}) = \{ \langle S^x_v(a) \colon v \in V \rangle : a \in A \text{ and } x \in I \},$$

and for each function in this set we define

$$\lambda^{\mathbf{V}}(\langle S^x_v(a) \colon v \in V \rangle) := \lambda x^{\mathbf{A}}.a.$$

It can be shown that $\langle S^x_v(a) \colon v \in V \rangle = \langle S^y_v(b) \colon v \in V \rangle$ implies $\lambda x^{\mathbf{A}}.a = \lambda y^{\mathbf{A}}.b$. Thus $\lambda^{\mathbf{V}}$ is well defined. It is easily checked that $\mathbf{V}$ is a functional domain (see [34]).

THEOREM 17 (Functional Representation of LFA's). *Every locally finite lambda abstraction algebra* $\mathbf{A}$ *is isomorphic to a functional lambda abstraction algebra. More precisely,* $\mathbf{A}$ *is isomorphic to a total subalgebra of the I-coordinatization of its associated functional domain.*

Neat Reducts and Point-Relativized Functional LAA's

Reducts of LAA's $\mathbf{A}$ in which the λ-abstraction operations $\lambda x^{\mathbf{A}}$ are discarded for only some of the λ-variables of the dimension set were considered in [34]. The process corresponds exactly to that of forming the *compression* of a polyadic algebra ([20], p. 137) and the *neat reduct* of a cylindric algebra ([21], Part I, p. 401). The theory of neat reducts of LAA's proves to be more regular than that of cylindric algebras. For example it turns out that the class of all neat I-reducts of LAA_J's forms a variety for every $I \subseteq J$ such that $|J \setminus I| \geqslant \omega$ ([34] Theorem 6.16); for cylindric algebras it is the subalgebras of neat reducts that form a variety. This should be compared with the fact that the class of α-dimensional cylindric algebras that can be neatly embedded in some $(\alpha + \omega)$-dimensional cylindric algebra coincides (up to isomorphism) with the class of generalized cylindric set algebras of dimension α (i.e., the representable α-dimensional, cylindric algebras); see [21], Part II, Theorem 3.2.10.

DEFINITION 18. Let $\mathbf{A}$ be an LAA_J, $I \subseteq J$ and $\mathrm{Nr}_I \mathbf{A} = \{ a \in A : \triangle a \subseteq I \}$. By the *I-neat reduct* of $\mathbf{A}$ we mean the algebra

$$\mathbf{Nr}_I \mathbf{A} := \langle \mathrm{Nr}_I \mathbf{A}, \cdot^{\mathbf{Nr}_I \mathbf{A}}, \lambda x^{\mathbf{Nr}_I \mathbf{A}}, x^{\mathbf{Nr}_I \mathbf{A}} \rangle_{x \in I},$$

whose operations are the corresponding operations of $\mathbf{A}$ restricted to $\mathrm{Nr}_I \mathbf{A}$. For a class K of LAA_J's and $I \subseteq J$ we define $\mathrm{Nr}_I \mathsf{K} := \{ \mathbf{Nr}_I \mathbf{A} \colon \mathbf{A} \in \mathsf{K} \}$.

$\mathbf{Nr}_I \mathbf{A}$ is obviously an LAA_I.

The functional representation theorem for neat reducts requires a more general notion of functional algebra. This more general class of functional LAA's was introduced in [34] and it is analogous to the class of *weak cylindric algebras* in the theory of cylindric algebras; see [21], Definition 3.12.

Let $\mathbf{V}$ be a functional domain and let r be a fixed but arbitrary element of V^I, and let V^I_r be the set of all $p \in V^I$ that differ from r at only finitely many coordinates, i.e.,

$$V^I_r = \{p \in V^I \colon |\{p_i \neq r_i\}| < \omega\}.$$

Let $V_{I,r}$ be the set of all partial functions $f \colon V^I_r \xrightarrow{\mathrm{p}} V$. The (I, r)-*coordinatization* of V,

$$\mathbf{V}_{I,r} = \langle V_{I,r}, \cdot^{\mathbf{V}_{I,r}}, \langle \lambda x^{\mathbf{V}_{I,r}} \colon x \in I \rangle, \langle x^{\mathbf{V}_{I,r}} \colon x \in I \rangle \rangle,$$

is defined just as $\mathbf{V}_I$ except that all functions are restricted to V^I_r.

Definition 19. A subalgebra $\mathbf{A}$ of $\mathbf{V}_{I,r}$ of total functions is called a *point-relativized functional lambda abstraction algebra*. I is the *dimension set* of $\mathbf{A}$ and r is its *thread*. $\mathbf{V}$ is the *value domain* of $\mathbf{A}$.

The class of point-relativized functional lambda abstraction algebras of dimension I is denoted by RFA_I. Every RFA_I is an LAA_I.

An I-neat reduct of an LAA_J, with $|J \backslash I| \geqslant \omega$, is isomorphic to an RFA_I.

Theorem 20 (Functional Representation of $\mathrm{Nr}_I\mathsf{LAA}$'s, [34; Theorem 7.1]). $\mathrm{Nr}_I\mathsf{LAA}_J \subseteq \mathbb{I}\, \mathsf{RFA}_I$ *for every* $I \subseteq J$ *with* $|J \backslash I| \geqslant \omega$.

In [34] Pigozzi and Salibra have shown something more: the class of isomorphic images of RFA_I's coincide with the class $\mathrm{Nr}_I\mathsf{LAA}_J(|J \backslash I| \geqslant \omega)$ and forms a variety. However, this result will be not used in this paper.

Combinatory Algebras, λ-algebras and Lambda Models

In a combinatory algebra lambda abstraction can be simulated by combinators, so it is possible to interpret λ-terms in it. However, not all the equations provable in the lambda calculus are true in every combinatory algebra. λ-algebras constitute exactly the class of combinatory algebras where all the equations provable in the lambda calculus are true.

We begin with the definition of a basic notion in combinatory logic and lambda calculus.

Definition 21 (Curry [11], Schönfinkel [40]). Let $\mathbf{C} = \langle C, \cdot^{\mathbf{C}}, \mathbf{k}^{\mathbf{C}}, \mathbf{s}^{\mathbf{C}} \rangle$ be an algebra where $\cdot^{\mathbf{C}}$ is a binary operation and $\mathbf{k}^{\mathbf{C}}, \mathbf{s}^{\mathbf{C}}$ are constants. $\mathbf{C}$ is a *combinatory algebra* if it satisfies the following identities: (as usual the symbol $\cdot$ and the superscript $^{\mathbf{C}}$ are omitted, and association, when in doubt, is to the left)

$$\mathbf{k}xy = x; \qquad \mathbf{s}xyz = xz(yz).$$

$\mathbf{k}$ and $\mathbf{s}$ are called *combinators*. In the equational language of combinatory algebras the derived combinators $\mathbf{i}$ and $\mathbf{1}$ are defined as follows: $\mathbf{i} := \mathbf{ssk}$ and $\mathbf{1} := \mathbf{s}(\mathbf{ki})$.

Let $\mathbf{A}$ be an LAA_I. By the *combinatory reduct* of $\mathbf{A}$ we mean the algebra

$$\mathrm{Cr}\,\mathbf{A} = \langle A, \cdot^{\mathbf{A}}, \mathbf{k}^{\mathbf{A}}, \mathbf{s}^{\mathbf{A}} \rangle$$

where

$$\mathbf{k}^{\mathbf{A}} = (\lambda xy.x)^{\mathbf{A}} \qquad \text{and} \qquad \mathbf{s}^{\mathbf{A}} = (\lambda xyz.xz(yz))^{\mathbf{A}}.$$

The λ-variables x, y, and z are assumed to be distinct. Note that by Proposition 4.5 in [34] $\mathbf{k}^{\mathbf{A}}$ and $\mathbf{s}^{\mathbf{A}}$ are surely uniquely defined (i.e., independent of the choice of x, y, z) if I is infinite. In the sequel we will assume this is always the case. By Lemma 4.13 in [34] we have that $\mathbf{i}^{\mathbf{A}} = (\lambda x.x)^{\mathbf{A}}$ and $\mathbf{1}^{\mathbf{A}} = (\lambda xy.xy)^{\mathbf{A}}$.

A subalgebra of the combinatory reduct of an LAA_I $\mathbf{A}$ (i.e., a subset of $\mathbf{A}$ containing $\mathbf{k}^{\mathbf{A}}$ and $\mathbf{s}^{\mathbf{A}}$ and closed under $\cdot^{\mathbf{A}}$) is called a *combinatory subreduct* of $\mathbf{A}$.

The *zero-dimensional subreduct* of $\mathbf{A}$ is the combinatory subreduct

$$\mathbf{Zd}\,\mathbf{A} = \langle \mathrm{Zd}\,\mathbf{A}, \cdot^{\mathbf{A}}, \mathbf{k}^{\mathbf{A}}, \mathbf{s}^{\mathbf{A}} \rangle,$$

where $\mathrm{Zd}\,\mathbf{A} = \{a \in A\colon \triangle^{\mathbf{A}} a = \varnothing\}$, the set of zero-dimensional elements of $\mathbf{A}$.

In the equational logic of combinatory algebras it is traditional to let λ-variable's play the role of real variables. We follow this convention in the next definition. Recall that x, y, z, possibly with subscripts, denote arbitrary distinct λ-variables. By a *combinatory term* we mean a term of the equational logic of combinatory algebras in the usual sense. Thus $\mathbf{k}$, $\mathbf{s}$, and x, for every λ-variable x, are combinatory terms. If s and t are combinatory terms, so is st. A combinatory term is *closed* (or *ground*) if it contains no λ-variables. Note that context variables do not occur in combinatory terms.

Let $\mathbf{C}$ be a combinatory algebra. Let $\bar{c}$ be a new symbol for each $c \in C$. Extend the language of combinatory algebras by adjoining $\bar{c}$ as a new constant symbol for each $c \in C$. A term t in this extended language is called a *combinatory polynomial over* $\mathbf{C}$. The set all such polynomials is denoted by $P(\mathbf{C})$. If $t = t(x_1, ..., x_n)$, where $x_1, ..., x_n$ includes all the λ-variables occurring in t, and $v_1, ..., v_n \in C$, then $t^{\mathbf{C}}(v_1, ..., v_n)$ will denote the value of t in $\mathbf{C}$ when x_i is interpreted as v_i and each new constant $\bar{c}$ as c.

The following result is well known (Meyer [28], Barendregt [3, Theorem 5.1.10], Curry–Feys [11]);

Proposition 22 (Combinatory Completeness Lemma). *Let* $\mathbf{C}$ *be a combinatory algebra and let* $t(x_1, ..., x_n)$ *be a combinatory polynomial over* $\mathbf{C}$ *whose* λ*-variables all occur in the list* $x_1, ..., x_n$. *Then there exists an element* c *in* $\mathbf{C}$ *such that, for all* $v_1, ..., v_n \in C$,

$$t^{\mathbf{C}}(v_1, ..., v_n) = cv_1 \cdots v_n.$$

Its proof depends on the following definition and lemma that shows that some aspects of lambda abstraction can be simulated in combinatory algebras.

Let $\mathbf{C}$ be a combinatory algebra. For each λ-variable x define a transformation λ^*x of the set $P(\mathbf{C})$ of combinatory polynomials over $\mathbf{C}$ as follows: $\lambda^*x(x) = \mathbf{i}$. Let t be a combinatory term different from x. If x does not occur in t, define $\lambda^*x(t) = \mathbf{k}t$; in particular, $\lambda^*x(\bar{v}) = \mathbf{k}\bar{v}$ for every $v \in C$. Otherwise, t must be of the form sr where s and r are combinatory terms, at least one of which contains x; in this case define $\lambda^*x(t) = \mathbf{s}(\lambda^*x(r))(\lambda^*x(s))$. For any finite sequence $x_1, ..., x_n$ of λ-variables define

$$\lambda^*x_1 \cdots x_n(t) = \lambda^*x_1(\lambda^*x_2(\cdots(\lambda^*x_n(t)\cdots)).$$

Lemma 23. *Let* $\mathbf{C}$ *be a combinatory algebra, t a combinatory polynomial over* $\mathbf{C}$, *and x a λ-variable.*

(i) *x does not occur in $\lambda^*x(t)$. More precisely, the λ-variables that occur in $\lambda^*x(t)$ are exactly the λ-variables except x that occur in t.*

(ii) *Let $y_1, ..., y_n$ be any list of λ-variables that includes all λ-variables occurring in t except x, and write $t = t(x, y_1, ..., y_n)$ and $\lambda^*x(t) = (\lambda^*x(t))(y_1, ..., y_n)$. Then for all $v, u_1, ..., u_n \in C$,*

$$t^{\mathbf{C}}(v, u_1, ..., u_n) = ((\lambda^*x(t))^{\mathbf{C}}(u_1, ..., u_n))\,v.$$

Proof. Barendregt [3, Proposition 5.1.9]. ▮

λ^*x is an operation on combinatory terms; it does not define directly an operator on combinatory algebras.

Lamda Algebras

Those combinatory algebras for which the combinatory polynomial transformation λx^* simulates lambda abstraction form a variety. They are called *λ-algebras*; the concept is essentially due to Curry.

The following definition is taken from Meyer [28]; cf. Barendregt [3, Chapter 7]. Recall the transformation λ^*x of combinatory polynomials that was defined just before Lemma 23.

Definition 24. A combinatory algebra is a *λ-algebra* if, in addition to the defining identities of combinatory algebras, it satisfies the following identities.

(i) $\mathbf{k} = \lambda^*x(\lambda^*y(\mathbf{k}xy))$;

(ii) $\mathbf{s} = \lambda^*x(\lambda^*y(\lambda^*z(\mathbf{s}xyz)))$;

(iii) $\lambda^*x(\lambda^*y(\mathbf{s}(\mathbf{k}x)(\mathbf{k}y))) = \lambda^*x(\lambda^*y(\mathbf{k}(xy)))$;

(iv) $\lambda^*x(\lambda^*y(\mathbf{s}(\mathbf{s}(\mathbf{kk})x)\,y)) = \lambda^*x(\lambda^*y(\lambda^*z(xz)))$;

(v) $\lambda^*x(\lambda^*y(\lambda^*z(\mathbf{s}(\mathbf{s}(\mathbf{s}(\mathbf{ks})x)\,y)\,z))) = \lambda^*x(\lambda^*y(\lambda^*z(\mathbf{s}(\mathbf{s}xz)(\mathbf{s}yz))))$.

Note that all the axioms (i)–(v) are equations between closed combinatory terms (no λ-variables are involved).

The axioms of a λ-algebra are designed expressly to prove part (ii) of the next Lemma.

Let **C** be a combinatory algebra. Recall that $P(\mathbf{C})$ is the set of combinatory polynomials over **C**. Recall also that the members of $P(\mathbf{C})$ are constructed from λ-variables and constant symbols **k**, **s**, and $\bar{c}$ for all elements c of **C**.

Let $D_{\mathbf{C}}$ be the *equational diagram* of **C**, i.e., the set of all equations of the form $\bar{c}\bar{d}=\bar{e}$ for $c, d, e \in C$ such that $cd=e$; we also include the two equations $\mathbf{k}=\bar{c}$ and $\mathbf{s}=\bar{d}$, where $c=\mathbf{k}^{\mathbf{C}}$ and $d=\mathbf{s}^{\mathbf{C}}$. Let **CL** be the axioms combinatory logic, i.e., the equations $\mathbf{k}xy=x$ and $\mathbf{s}xyz=xz(yz)$. We denote by $\equiv_{\mathbf{C}}$ the equivalence relation of $P(\mathbf{C})$ such that $t \equiv_{\mathbf{C}} s$ iff the equation $t=s$ is a logical consequence of $D_{\mathbf{C}}$ together with the axioms **CL** of combinatory logic.

LEMMA 25. *Let* **C** *be a λ-algebra and let t, s be combinatory polynomials over* **C**. *Then we have for every $x \in I$*

(i) $\lambda^*x(t)x \equiv_{\mathbf{C}} t$;

(ii) $t \equiv_{\mathbf{C}} s$ *iff* $\lambda^*x(t) \equiv_{\mathbf{C}} \lambda^*x(s)$.

Proof. By Proposition 7.1.6(ii) in [3] and Lemma 7.12 in [28]. ∎

Let **C** be a λ-algebra, I be a set and $\mathbf{C}[I]$ be the free extension of **C** by I in the variety of combinatory algebras. Recall that $\mathbf{C}[I]$ is an expansion of **C** defined up to isomorphism by the following universal mapping properties: ($C[I]$ is the universe of $\mathbf{C}[I]$.) (1) $I \subseteq C[I]$; (2) $\mathbf{C}[I]$ is a combinatory algebra; (3) for every homomorphism h: $\mathbf{C} \to \mathbf{A}$ from **C** into a combinatory algebra **A** and every mapping g: $I \to A$ there exists a unique homomorphism f: $\mathbf{C}[I] \to \mathbf{A}$ extending both h and g. A concrete description of $\mathbf{C}[I]$ as a quotient of the algebra of combinatory polynomials over **C** with λ-variables from I may be found on page 109 of [28]. Let t be a combinatory polynomial over **C**. $t^{\mathbf{C}[I]}$ denotes the unique interpretation of t in $\mathbf{C}[I]$ when each λ-variable x in t is interpreted as $x^{\mathbf{C}[I]}$, each constant $\bar{c}$ as c, and the combinators **k**, **s** as $\mathbf{k}^{\mathbf{C}}$, $\mathbf{s}^{\mathbf{C}}$. It follows easily from basic principles of universal algebra that $t^{\mathbf{C}[I]}=s^{\mathbf{C}[I]}$ iff $t \equiv_{\mathbf{C}} s$.

In [36] Pigozzi and Salibra have considered an expansion of $\mathbf{C}[I]$ which makes $\mathbf{C}[I]$ a locally finite LAA_I. Lambda abstractions $\lambda x^{\mathbf{C}[I]}$ (for all $x \in I$) were defined as follows: Let $a \in C[I]$. Choose any $t \in P(C)$ such that $t^{\mathbf{C}[I]}=a$. Define

$$\lambda x^{\mathbf{C}[I]}.a=(\lambda^*x(t))^{\mathbf{C}[I]}.$$

Lemma 25 guarantees $\lambda x^{\mathbf{C}[I]}$ is well defined. The algebra obtained by adjoining these operations is also denoted by $\mathbf{C}[I]$.

THEOREM 26 ([36; Theorem 3.1]). *Let* **C** *be a λ-algebra.* $\mathbf{C}[I]$ *is an* LFA_I *whose zero-dimensional subreduct is* **C**.

In [36] Pigozzi and Salibra have shown something more: the category of λ-algebras and the category of locally finite LAA_I's are equivalent. The zero-dimensional combinatory subreduct of an LAA_I is always a λ-algebra and every λ-algebra can be obtained this way.

Lambda Models

Lambda models were introduced by Meyer [28] as an alternative first-order characterization of environment models. In fact, they form an elementary class, while the definition of environment model is higher order.

DEFINITION 27 (Hindley–Longo [23]; Meyer [28]; Scott [42]). A *lambda model* is a λ-algebra $\mathbf{C}$ satisfying the following *Meyer–Scott axiom*, for all $u, w \in C$:

$$\text{If} \quad uv = wv \quad \text{for all} \quad v \in C, \quad \text{then} \quad \mathbf{1}u = \mathbf{1}w.$$

In the first-order language of combinatory algebras it takes the form

$$\forall x\, \forall y (\forall z (xz = yz) \Rightarrow \mathbf{1}x = \mathbf{1}y).$$

The particular form of the definition of lambda model given in Definition 27 is taken from Barendregt [3, Definition 5.2.7].

The class of lambda models generates the variety of λ-algebras [4]. Lambda models are not closed under substructures contrary to LAA's. In [36] Pigozzi and Salibra have shown that the category of lambda models is equivalent to the category of rich locally finite LAA_I's, where an LAA is rich if it satisfies the abstract version of the term rule of lambda calculus ([3, Definition 4.1.10(ii)]).

Environmental models can also be given a combinatory structure in a natural way. Let $\mathbf{V} = \langle V, \cdot^{\mathbf{V}}, \lambda^{\mathbf{V}} \rangle$ be an environment model and let $\mathbf{A}$ be a FLA_I with value domain $\mathbf{V}$, i.e., a total subalgebra of $\mathbf{V}_I$. (Recall that by Theorem 16 at least one such $\mathbf{A}$ must exist.) Set

$$\mathbf{k}^{\mathbf{A}} := (\lambda xy.x)^{\mathbf{A}} \qquad \text{and} \qquad \mathbf{s}^{\mathbf{A}} := (\lambda xyz.xz(yz))^{\mathbf{A}},$$

where x, y, z are any three distinct λ-variables. If $\mathbf{V}$ is an environment model, then $\mathbf{k}^{\mathbf{A}}$ and $\mathbf{s}^{\mathbf{A}}$ are total, constant functions from V^I to V, i.e., $\mathbf{k}^{\mathbf{A}}(p) = \mathbf{k}^{\mathbf{A}}(q)$ and $\mathbf{s}^{\mathbf{A}}(p) = \mathbf{s}^{\mathbf{A}}(q)$ for all $p, q \in V^I$.

We denote the constant values of $\mathbf{k}^{\mathbf{A}}$ and $\mathbf{s}^{\mathbf{A}}$ in V respectively by $\mathbf{k}^{\mathbf{V}}$ and $\mathbf{s}^{\mathbf{V}}$. They do not depend on the particular total subalgebra $\mathbf{A}$ of $\mathbf{V}_I$ we choose.

PROPOSITION 28 ([28], [34, Corollary 5.4], [36, Theorem 5.4]). $\langle \mathbf{V}, \cdot^{\mathbf{V}}, \mathbf{k}^{\mathbf{V}}, \mathbf{s}^{\mathbf{V}} \rangle$ *is a lambda model.*

2. THE MAIN RESULTS

The main result of the paper proves the utility of applying the methods of universal algebra to the lambda calculus. The most natural LAA's are algebras of functions and, not surprisingly, they are closely related to the combinatory models of lambda calculus. Indeed, functional LAA's are obtained by coordinatizing lambda models or environment models by the λ-variables in a natural way. We know from Section 1 that the combinatory models of lambda calculus constitute an elementary class that is not closed under natural algebraic operations. For example, a substructure of a lambda model is not in general a lambda model but only a λ-algebra. In our main result we show that when we move from the combinatory models of the lambda calculus to the functional LAA's, we earn "algebraic" properties, because the (isomorphs of) functional LAA's constitute an equational class. It is axiomatized by the finite schema of identitites (between λ-contexts) characterizing LAA's. The same schema of identitites also axiomatizes the class of identities valid in all lambda theories.

Combinatory Reducts

We show that the combinatory reduct of every LAA is a λ-algebra. This result will be used in the main theorem and was obtained with the help of M. Dezani.

We have shown in Section 1 that the class of locally finite LAA_I's (i.e., the isomorphic images of the term models of the lambda theories) is not elementary. As a consequence of this fact, there do exist LAA's which are not locally finite. For example, the set of Böhm-like trees over a set I of λ-variables, as defined in [3; Definition 10.1.12], determines a nonlocally finite LAA_I.

The main technical difficulty in the proof of Theorem 29 below, which the combinatory reduct of every LAA $\mathbf{A}$ is a λ-algebra, is that one cannot derive directly from the LAA-axioms the β-rule: $(\lambda x.t)a = t[a/x]$, where t is a λ-term with parameters in $\mathbf{A}$ and $t[a/x]$ is the the result of substituting a for x in t *after renaming* the bound λ-variables of t to avoid capture (α-rule). Note that the difficulty only arises if a is not assumed to be finite dimensional. In this case its dimension set (the abstract version of its set of free λ-variables) may contain all the available λ-variables, and we have no spare λ-variable to apply the α-rule incorporated into the substitution $t[a/x]$. As a concrete example of this phenomenon consider the model of Böhm trees and assume that a is a Böhm tree which contains *all* the λ-variables. In this case one cannot prove for instance that letting $\mathbf{s} = \lambda xyz.xz(yz)$, we have $\mathbf{s}a = \lambda y'z'.az'(y'z')$ because y' and z' may belong to the dimension set of a for every possible choice of y' and z'.

On the other hand, a trick applied in the proof of Theorem 29 shows that the problem disappears in the special case in which we "saturate" the initial abstractions, namely if we apply $\mathbf{s}$ to three elements instead of one. One can then show that $(\lambda xyz.xz(yz))\,abc = ac(bc)$ is provable from the axioms LAA. (The same trick works if $\mathbf{s}$ is replaced with any term not containing other λ's besides the initial ones.)

THEOREM 29. *The combinatory reduct of every* LAA *is a λ-algebra.*

Proof. Let $\mathbf{A}$ be an LAA_I and let $\mathbf{k}^{\mathbf{A}} = (\lambda xy.x)^{\mathbf{A}}$, $\mathbf{s}^{\mathbf{A}} = (\lambda xyz.xz(yz))^{\mathbf{A}}$. We have to show that $\mathbf{A}$ is a combinatory algebra, i.e., it satisfies

$$\mathbf{k}^{\mathbf{A}}ab = a \qquad \text{and} \qquad \mathbf{s}^{\mathbf{A}}abc = ac(bc), \qquad \text{for all} \quad a, b, c \in A.$$

We observe that it is not possible to apply axiom (β_6) to $\mathbf{k}^{\mathbf{A}}a$ to obtain $\lambda y'.a$ because y' may belong to the dimension set of a for every possible choice of y'.

For the sake of clarity, we will omit the superscript $^{\mathbf{A}}$ in the remaining part of the proof when there is no danger of confusion. Let $a, b \in A$, $u = \lambda z.zxy$ and $t = [\lambda x.\mathbf{k}(x(\lambda xy.a))(x(\lambda xy.b))]u$. Then we have

$$\begin{aligned}
t &= (\lambda x.\mathbf{k})\, u((\lambda x.x(\lambda xy.a))u)((\lambda x.x(\lambda xy.b))u) && [(\beta_5)]\\
&= \mathbf{k}((\lambda x.x(\lambda xy.a))u)((\lambda x.x(\lambda xy.b))u) && [(\beta_4)]\\
&= \mathbf{k}[(\lambda x.x)\, u((\lambda xxy.a)u)][(\lambda x.x)\, u((\lambda xxy.b)u)] && [(\beta_5)]\\
&= \mathbf{k}(u(\lambda xy.a))(u(\lambda xy.b)) && [(\beta_1), (\beta_4)]\\
&= \mathbf{k}((\lambda z.zxy)(\lambda xy.a))((\lambda z.zxy)(\lambda xy.b)) && [\text{def } u]\\
&= \mathbf{k}((\lambda xy.a)\, xy)((\lambda xy.b)\, xy)) && [(\beta_5), (\beta_1), (\beta_2)]\\
&= \mathbf{k}ab && [(\beta_3)]
\end{aligned}$$

Let $v = x(\lambda xy.a)$ and $w = x(\lambda xy.b)$.

$$\begin{aligned}
\mathbf{k}vw &= (\lambda xy.x)\, vw && [\text{def } \mathbf{k}]\\
&= (\lambda y.v)w && [(\beta_6), y \notin \triangle^{\mathbf{A}} v]\\
&= (\lambda y.x(\lambda xy.a))w && [\text{def } v]\\
&= x((\lambda yxy.a)w) && [(\beta_5), (\beta_2)]\\
&= x(\lambda xy.a) && [y \notin \triangle^{\mathbf{A}}(\lambda xy.a)].
\end{aligned}$$

Finally,

$$\begin{aligned}
t &= (\lambda x.\mathbf{k}vw)u && [\text{def } t]\\
&= (\lambda x.x(\lambda xy.a))u && [\mathbf{k}vw = x(\lambda xy.a)]\\
&= u(\lambda xy.a) && [(\beta_5), (\beta_1), (\beta_4)]\\
&= (\lambda z.zxy)(\lambda xy.a) && [\text{def } u]\\
&= (\lambda xy.a)\, xy && [(\beta_5), (\beta_1), (\beta_2)]\\
&= a && [(\beta_3)]
\end{aligned}$$

By $t = \mathbf{k}ab$ and $t = a$ we have the conclusion. In a similar way starting from

$$t = [\lambda x.\mathbf{s}(x(\lambda xyz.a))(x(\lambda xyz.b))(x(\lambda xyz.c))](\lambda x'.x'xyz)$$

we obtain $t = \mathbf{s}abc$ and $t = ac(bc)$, and so the other identity follows.

We check the remaining axioms (i)–(v) of Definition 24. They are equations between closed combinatory terms (no λ-variables are involved). Equations (i)–(v) belong to every lambda theory ([3, Corollary 5.2.13]) when we interpret $\mathbf{k}$ as the lambda term $\lambda xy.x$ and $\mathbf{s}$ as $\lambda xyz.xz(yz)$. The free algebra with an empty set of free generators in the variety of LAA_I's is locally finite since it is a quotient of the absolutely free algebra of λ-terms (λ-contexts without context variables) and in every λ-term only a finite number of λ-variables occurs. Then by Proposition 9 it is the term LAA of a lambda theory. It follows that the free algebra with an empty set of free generators satisfies the equations (i)–(v) of Definition 24, so that every LAA_I does it. ∎

The Main Theorem

We recall that the notion of λ-context (i.e., a term in the similarity type of LAA's) coincides with that of context as defined in Barendregt [3, Definition 14.4.1]. So, a λ-context is a λ-term with some "holes" in it, where a "hole" is an occurrence of a context variable (i.e., algebraic variable). The main difference between Barendregt's notation and our's is that holes are denoted here by Greek letters $\xi, \mu, ...$, while in Barendregt's book by [ ], $[\ ]_1, ...$. The essential feature of a λ-context is that a free variable in a λ-term may become bound when we substitute it for a hole within the context. For example, if $C(\xi) = \lambda x.x(\lambda y.\xi)$ is a λ-context, in Barendregt's notation: $C([\ ]) = \lambda x.x(\lambda y.[\ ])$, and $t = xy$ is a λ-term, then $C(t) = \lambda x.x(\lambda y.xy)$.

Definition 30. We say that an *I-lambda theory T satisfies an identity*

$$t(\xi_1, ..., \xi_n) = u(\xi_1, ..., \xi_n)$$

in the similarity type of LAA_I's if all the instances of the identity, obtained by substituting λ-terms for context variables in it, falls within the lambda theory, i.e.,

$$t(t_1, ..., t_n) = u(t_1, ..., t_n) \in T, \qquad \text{for all } \lambda\text{-terms } t_1, ..., t_n.$$

For example, the identity $(\lambda x.x)\xi = \xi$ is satisfied in every lambda theory because, for every λ-term t, the equation $(\lambda x.x)t = t$ belongs to every lambda theory.

Proposition 31. *An identity between λ-contexts holds in every I-lambda theory if, and only if, it holds in the variety generated by the class of locally finite LAA_I's.*

The above proposition is a consequence of Proposition 9, since an identity between λ-contexts holds in an I-lambda theory T over the language $\Lambda_I(C)$ iff it holds in the term model $\Lambda_{\mathbf{I}}(C)/T$ of T.

As a matter of notation, we let

(i) $\mathrm{Md}(E)$ denote the class of algebras satisfying the set E of identities;

(ii) $\mathbf{T}_{I,\lambda}$ denote the class of all I-lambda theories;

(iii) $\mathrm{Eq}(\mathbf{K})$ denote the equational theory determined by the class $\mathbf{K}$ of algebras;

(iv) $\mathrm{Eq}(\mathbf{T}_{I,\lambda})$ denote the class of identities (between λ-contexts) satisfied by all the I-lambda theories;

(v) $\bar{E}$ denote the equational theory constituted by all the logical consequences of the set E of identities;

(vi) $\mathbb{U}_p\mathbf{K}$ denote the class of ultraproducts of members of $\mathbf{K}$ for every class $\mathbf{K}$ of algebras.

Theorem 32. *For any infinite set I, the class of isomorphic images of functional* LAA_I*'s constitutes a variety axiomatized by the finite set of identities characterizing* LAA_I*'s, in symbols,*

$$\mathbb{I}\,\mathsf{FLA}_I = \mathsf{LAA}_I.$$

Moreover, the class of identities (between λ-contexts) valid in all I-lambda theories constitutes an equational theory axiomatized by the same set of identities, in symbols,

$$\mathrm{Eq}(\mathbf{T}_{I,\lambda}) = \mathrm{Eq}(\mathsf{FLA}_I) = \mathrm{Eq}(\mathsf{LAA}_I).$$

So, an identity is valid in all I-lambda theories if and only if it is satisfied in all LAA*'s, hence in all* FLA*'s.*

Proof. The proof of this theorem uses heavy technical machinery such as neat reducts, point-relativized LAA's, and ultraproducts. The reason is that the proof works in the general setting of LAA's. In this context there exist algebras with elements whose dimension set may contain all the available λ-variables. The main achievement of the theorem is to show that we do not need to worry about the dimension set: it is not necessary to introduce any condition. Provided an algebra **A** satisfies the axioms of LAA's, we always get a lambda model **V** such that **A** is isomorphic to a suitable coordinatization of **V**. Roughly speaking, the proof consists in showing that every LAA embeds into a bigger LAA in which there are enough spare λ-variables to apply the α-rule. This is the whole point of introducing the heavy technical machinery.

For the sake of simplicity, we give here only the proof of the main result and postpone the proofs of the technical theorems to the next Sections. We recall for the reader that the notions of neat reduct, functional lambda abstraction algebra (FLA) and point-relativized functional lambda abstraction algebra (RFA) are defined in Section 1.

The following diagram provides a schema of the proof[1] that $\mathbb{I}\mathsf{FLA}_I = \mathsf{LAA}_I$ (J is an infinite set disjoint from I):

$\mathsf{LAA}_I \subseteq \mathrm{Nr}_I\, \mathsf{LAA}_{I\cup J}$	[Theorems 29, 48]
$\subseteq \mathbb{I}\, \mathsf{RFA}_I$	[34, Theorem 7.1]
$\subseteq \mathbb{I}\mathbb{S}\, \mathbb{U}_p\, \mathsf{FLA}_I$	[Theorem 49]
$\subseteq \mathbb{I}\mathbb{S}\mathbb{P}\, \mathsf{FLA}_I$	[Theorem 52]
$\subseteq \mathbb{I}\, \mathsf{FLA}_I$	[Theorem 53]
$\subseteq \mathsf{LAA}_I$	[Proposition 14]

[1] The proof that every RFA_I is isomorphic to an FLA_I was presented by the second author to the First Conference of the Centre for Discrete Mathematics and Theoretical Computer Science, Universities of Auckland and Waikato [16].

In other words, every $\mathsf{LAA}_I\,\mathbf{A}$ is isomorphic to the I-neat reduct of an $\mathsf{LAA}_{I\cup J}$ for an infinite J disjoint from I. By this result and the functional representation of neat reducts, there is a functional domain $\mathbf{V}$ and an $r \in V^I$ such that $\mathbf{A}$ is isomorphic to a point-relativized functional LAA_I with value domain $\mathbf{V}$ and thread r. Moreover, we will show that every point-relativized functional LAA_I can be embedded into an ultrapower of a functional LAA_I; since ultrapowers of FLA's are isomorphic to FLA's, we conclude that $\mathbf{A}$ is isomorphic to an FLA_I. The opposite direction follows from Proposition 14.

The following diagram provides a schema of the remaining part of the theorem:

$$
\begin{aligned}
\mathrm{Eq}(\mathbf{T}_{I,\lambda}) &= \mathrm{Eq}(\mathsf{LFA}_I) && [\text{Propositions 31, 8, 9}]\\
&= \mathrm{Eq}(\mathbb{HSP}\ \mathsf{LFA}_I) && \\
&= \mathrm{Eq}(\mathbb{HSP}\ \mathsf{FLA}_I) && [\text{Theorem 36}]\\
&= \mathrm{Eq}(\mathsf{LAA}_I) && [\mathsf{LAA}_I = \mathbb{I}\ \mathsf{FLA}_I]
\end{aligned}
$$

In other words, assume that the universally quantified equation $t=u$ is valid in every I-lambda theory. Every LFA_I is the term LAA of a lambda theory (Proposition 9); hence the equation $t=u$ is also satisfied in every LFA_I. But by Theorem 36 the class LFA_I of locally finite LAA_I's and tha class FLA_I of functional LAA's generate the same variety. This implies that the equation $t=u$ is satisfied in every LAA_I. In the opposite direction, assume that the universally quantified equation $t=u$ is valid in every LAA_I, so it is also valid in the class LFA_I. But, for every lambda theory T, the term LAA of T is a locally finite LAA, so that the equation $t=u$ is satisfied by T. ∎

The following immediate corollary characterizes in many ways the class of LAA_I's. We recall from [34] that an $\mathsf{LAA}_I\,\mathbf{A}$ is *dimension complemented* (in symbols, $\mathbf{A} \in \mathsf{DCA}_I$) if the dimension set of every element is not equal to all I. In one of the main results in [34] Pigozzi and Salibra show that every dimension-complemented LAA_I is isomorphic to an RFA_I.

COROLLARY 33. *Let I, J be infinite disjoint sets. Then*

$$\mathsf{LAA}_I = \mathbb{I}\ \mathsf{FLA}_I = \mathbb{I}\ \mathsf{RFA}_I = \mathrm{Nr}_I \mathsf{LAA}_{I\cup J} = \mathbb{HSP}\ \mathsf{LFA}_I = \mathbb{HSP}\ \mathsf{DCA}_I.$$

We know from Theorem 29 that the combinatory reduct of any LAA is a λ-algebra. We do not know in general if the combinatory reduct of every LAA_I is a lambda model. We conjecture that it is true. The best we have obtained is summarized in the following theorem.

THEOREM 34. (i) *The combinatory reduct of every dimension-complemented LAA_I is a lambda model.* (*This includes all the locally finite LAA_I's.*)

(ii) *The combinatory reduct of an $\mathsf{FLA}_I\,\mathbf{A}$ with value domain $\mathbf{V}$ is a lambda model if, for every environment $p \in V^I$, the set $\{a(p) : a \in A\}$ is all V.* (*This includes all the full FLA_I's.*)

Proof. (i) The proof can be found in [36] for the locally finite case. It can be extended without any change to the dimension-complemented case.

(ii) We know from Theorem 29 that the combinatory reduct Cr **A** of **A** is λ-algebra. We have to show that Cr **A** satisfies the Meyer–Scott axiom of Definition 27. Let $a, b \in A$ and

$$a \cdot^{\mathbf{V}_I} c = b \cdot^{\mathbf{V}_I} c, \qquad \text{for all} \quad c \in A.$$

First, we show that for every environment $p \in V^I$

$$a(p) \cdot^{\mathbf{V}} v = b(p) \cdot^{\mathbf{V}} v, \qquad \text{for all} \quad v \in V.$$

By hypothesis there exists at least an element $d \in A$ such that $d(p) = v$.

$$\begin{aligned} a(p) \cdot^{\mathbf{V}} v &= a(p) \cdot^{\mathbf{V}} d(p) \\ &= (a \cdot^{\mathbf{V}_I} d)(p) \\ &= (b \cdot^{\mathbf{V}_I} d)(p) \qquad [\text{hypothesis}] \\ &= b(p) \cdot^{\mathbf{V}} v. \end{aligned}$$

Recalling from Proposition 28 that $(V, \cdot^{\mathbf{V}}, \mathbf{k}^{\mathbf{V}}, \mathbf{s}^{\mathbf{V}})$ is a lambda model, we obtain

$$\mathbf{1}^{\mathbf{V}} \cdot^{\mathbf{V}} a(p) = \mathbf{1}^{\mathbf{V}} \cdot^{\mathbf{V}} b(p), \qquad \text{for all} \quad p \in V^I.$$

On the other hand, for any environment p we have

$$\begin{aligned} (\mathbf{1}^{\mathbf{A}} \cdot^{\mathbf{V}_I} a)(p) &= \mathbf{1}^{\mathbf{A}}(p) \cdot^{\mathbf{V}} a(p) \\ &= \mathbf{1}^{\mathbf{V}} \cdot^{\mathbf{V}} a(p) \qquad [\text{def } \mathbf{1}^{\mathbf{V}}] \\ &= \mathbf{1}^{\mathbf{V}} \cdot^{\mathbf{V}} b(p) \\ &= (\mathbf{1}^{\mathbf{A}} \cdot^{\mathbf{V}_I} b)(p) \end{aligned}$$

So, Cr **A** is a lambda model. ∎

3. FLA_I AND LFA_I GENERATE THE SAME VARIETY

In this section we start the proof of the technical results stated but not proved in Section 2. We now prove that functional LAA_I's and locally finite LAA_I's generate the same variety. We need some preliminaries.

We recall that an element a of an LAA_I **A** is finite dimensional if the dimension set of a is finite. By using Proposition 4 it is simple to prove that the set

$$\text{Fi } A = \{a \in A : |\triangle a| < \omega\}$$

is a subuniverse of $\mathbf{A}$. In the following Fi $\mathbf{A}$ denotes the subalgebra of all finite dimensional elements of $\mathbf{A}$, while for a class K of LAA_I's we define Fi $\mathsf{K} := \{\text{Fi } \mathbf{A} \colon \mathbf{A} \in \mathsf{K}\}$.

Let $J \subseteq I$. By the J-reduct of an LAA_I $\mathbf{A}$ we mean the algebra

$$Rd_J\,\mathbf{A} := \langle A, \cdot^{\mathbf{A}}, \langle \lambda x^{\mathbf{A}} \colon x \in J\rangle, \langle x^{\mathbf{A}} \colon x \in J\rangle\rangle.$$

As a matter of notation, $J \subseteq_\omega I$ means that J is a finite subset of I.

Lemma 35. *Let* $\mathbf{A}$ *be either an* FLA_I *or an* RFA_I *with value domain* $\mathbf{V}$, *and let* $j = \{x_1, ..., x_k\} \subseteq_\omega I$ *be a finite subset of* I. *Consider an environment* $p \in V^I$ *that is in the domain of every map* $a \in A$ (*this is always the case if* $\mathbf{A}$ *is an* FLA). *Define a map* $f_{p,j} \colon A \to V_I$ *as follows*:

$$f_{p,j}(a)(q) := a(p\{q_{x_1}/x_1, ..., q_{x_k}/x_k\}) \qquad \text{for all} \quad a \in A \quad \text{and} \quad q \in V^I.$$

Map $f_{p,j}$ *is a homomorphism from* $Rd_j(\mathbf{A})$ *into the* j-*reduct of the locally finite* FLA_I Fi $\mathbf{V}_I^\top$ *that is the subalgebra of all finite dimensional elements of the full* FLA_I $\mathbf{V}_I^\top$.

Proof. Let $b = f_{p,j}(a)$. Consider the full FLA_I $\mathbf{V}_I^\top$ with value domain $\mathbf{V}$ as defined in Theorem 15. From the same theorem we have that $b \in V_I^\top$ if there exists a $(V^I \times I^\star)$-indexed system u_b of elements of V such that, for all $q \in V^I$, $\mathbf{y} = y_1 \cdots y_n \in I^\star$, and $\mathbf{v} = v_1 \cdots v_n \in V^*$

$$b(q\{v_1/y_1, ..., v_n/y_n\}) = u_{b,q,\mathbf{y}} v_1 \cdots v_n.$$

Now we prove that there exists an indexed system u_b satisfying the above condition. Assume that $j \cap \{y_1, ..., y_n\} = \{x_{i_1}, ..., x_{i_r}\}$. Let $\mathbf{x}' = x_{i_1} \cdots x_{i_r}$ and let

$$q' = q\{v_1/y_1, ..., v_n/y_n\} \qquad \text{and} \qquad p' = p\{q_{x_1}/x_1, ..., q_{x_k}/x_k\}.$$

Then

$$\begin{aligned} b(q\{v_1/y_1, ..., v_n/y_n\}) &= f_{p,j}(a)(q') && [\text{def } b] \\ &= a(p\{q'_{x_1}/x_1, ..., q'_{x_k}/x_k\}) && [\text{def } f_{p,j}] \\ &= a(p'\{v_{i_1}/x_{i_1}, ..., v_{i_r}/x_{i_r}\}) \\ &= u_{a,p',\mathbf{x}'} v_{i_1} \cdots v_{i_r} && [a \in A \subseteq V_I^\top, \text{Theorem 15}] \end{aligned}$$

We recall from Proposition 28 that the environment model $\mathbf{V}$ can be transformed in a lambda model by defining $\mathbf{k}^{\mathbf{V}}$ and $\mathbf{s}^{\mathbf{V}}$ to be the constant values in V of the functions $(\lambda xy.x)^{\mathbf{A}}$ and $(\lambda xyz.xz(yz))^{\mathbf{A}}$. Lambda models are λ-algebras satisfying the Scott–Meyer axiom (see Definition 27). If we consider the standard translation $(-)_{CL}$ of λ-terms in combinatory terms as defined in [3, Definition 7.3.1], we have by Theorem 7.3.10 in [3]

$$(\lambda z z_1 \cdots z_n . z z_{i_1} \cdots z_{i_r})_{CL}^{\mathbf{V}}\, u_{a,p',\mathbf{x}'} v_1 \cdots v_n = u_{a,p',\mathbf{x}'} v_{i_1} \cdots v_{i_r}.$$

Hence, we obtain the conclusion if we define

$$u_{b,q,\mathbf{y}} = (\lambda z z_1 \cdots z_n . z z_{i_1} \cdots z_{i_r})^{\mathbf{V}}_{CL}\, u_{a,p',\mathbf{x}'}.$$

We now show that $f_{p,j}(a)$ is finite dimensional for every $a \in A$. By Proposition 3.6 in [34] an element a of an FLA_I is algebraically independent of x if, for all $q, r \in V^I$, $q_y = r_y$ for all $y \in I \backslash \{x\}$ implies $a(q) = a(r)$. So, for every $a \in A$, the element $f_{p,j}(a)$ is finite dimensional since it is algebraically independent of every λ-variable in $I \backslash j$.

Map $f_{p,j}$ is a homomorphism. Let $a, b \in A$ and $x_i \in j$.

$$\begin{aligned}
f_{p,j}(x_i^{\mathbf{V}_I})(q) &= x_i^{\mathbf{V}_I}(p\{q_{x_1}/x_1, ..., q_{x_n}/x_n\}) \\
&= q_{x_i} \\
&= x_i^{\mathbf{V}_I}(q) \\
f_{p,j}(a \cdot^{\mathbf{V}_I} b)(q) &= (a \cdot^{\mathbf{V}_I} b)(p\{q_{x_1}/x_1, ..., q_{x_n}/x_n\}) \\
&= a(p\{q_{x_1}/x_1, ..., q_{x_n}/x_n\}) \cdot^{\mathbf{V}} b(p\{q_{x_1}/x_1, ..., q_{x_n}/x_n\}) \\
&= f_{p,j}(a)(q) \cdot^{\mathbf{V}} f_{p,j}(b)(q) \\
&= [f_{p,j}(a) \cdot^{\mathbf{V}_I} f_{p,j}(b)](q) \\
f_{p,j}(\lambda x_i^{\mathbf{V}_I}.b)(q) &= (\lambda x_i^{\mathbf{V}_I}.b)(p\{q_{x_1}/x_1, ..., q_{x_i}/x_i, ..., q_{x_n}/x_n\}) \\
&= \lambda^{\mathbf{V}} \langle b(p\{q_{x_1}/x_1, ..., v/x_i, ..., q_{x_n}/x_n\}) \colon v \in V \rangle \\
&= \lambda^{\mathbf{V}} \langle f_{p,j}(b)(q\{v/x_i\}) \colon v \in V \rangle \\
&= [\lambda x_i^{\mathbf{V}_I}. f_{p,j}(b)](q) \quad \blacksquare
\end{aligned}$$

The variety generated by a class K of algebras is the smallest class of algebras, including K, closed under homomorphic images, Cartesian products and subalgebras. By Birkhoff's theorem ([27, Theorem 4.131]) a class of algebras is a variety if and only if it is an equational class.

THEOREM 36. $\mathbb{HSP}\,\mathsf{FLA}_I = \mathbb{HSP}\,\mathsf{LFA}_I$.

Proof. From the functional representation theorem of LFA's (see Theorem 17) it follows that every LFA_I is isomorphic to an FLA_I. By $\mathsf{LFA}_I \subseteq \mathbb{I}\,\mathsf{FLA}_I$ every identity valid in the class FLA_I holds also in the class LFA_I. For the opposite direction, let $t(\xi_1, ..., \xi_n) = u(\xi_1, ..., \xi_n)$ be an identity valid in the class LFA_I, where t, u are λ-contexts and $\xi_1, ..., \xi_n$ are all context variables occurring in t, u. Then, for every environment model $\mathbf{V}$, the identity $t = u$ is valid in the locally finite FLA_I Fi $\mathbf{V}_I^{\top}$ that is the subalgebra of all finite dimensional elements of the full $\mathsf{FLA}_I\,\mathbf{V}_I^{\top}$. If we prove that every identity satisfied in the algebra Fi $\mathbf{V}_I^{\top}$ holds also in the full $\mathsf{FLA}_I\,\mathbf{V}_I^{\top}$, then we obtain the conclusion of the theorem if we recall that every FLA_I is a subalgebra of a full FLA_I for a suitable value domain $\mathbf{V}$.

We now show that the identity $t(\xi_1, ..., \xi_n) = u(\xi_1, ..., \xi_n)$ holds in $\mathbf{V}_I^\top$, that is

$$[t^{\mathbf{V}_I}(a_1, ..., a_n)](q) = [u^{\mathbf{V}_I}(a_1, ..., a_n)](q)$$

for all $a_1, ..., a_n \in V_I^\top$ and all $q \in V^I$.

Let $j = \{x_1, ..., x_k\} \subseteq_\omega I$ be all the λ-variables occurring in t, u either as constants x_i or as λ-abstractions λx_i. Then t, u belong to the minimum class of λ-contexts constructed from $x_1, ..., x_k$ and $\xi_1, ..., \xi_n$ by using the application operator and the λ-abstractions λx_i $(i = 1, ..., k)$. Since an $\mathsf{LAA}_I\,\mathbf{A}$ satisfies the equation $t(\xi_1, ..., \xi_n) = u(\xi_1, ..., \xi_n)$ if and only if its reduct $Rd_j\,\mathbf{A}$ satisfies it, then we can use the homomorphism $f_{q,j}\colon Rd_j(\mathbf{V}_I^\top) \to Rd_j(\mathrm{Fi}\,\mathbf{V}_I^\top)$ defined in Lemma 35 above:

$$\begin{aligned} t^{\mathbf{V}_I}(a_1, ..., a_n)(q) &= t^{\mathbf{V}_I}(a_1, ..., a_n)(q\{q_{x_1}/x_1, ..., q_{x_k}/x_k\}) \\ &= f_{q,j}(t^{\mathbf{V}_I}(a_1, ..., a_n))(q) \\ &= t^{\mathbf{V}_I}(f_{q,j}(a_1), ..., f_{q,j}(a_n))(q) \\ &= u^{\mathbf{V}_I}(f_{q,j}(a_1), ..., f_{q,j}(a_n))(q), \qquad [t = u \text{ holds in } \mathrm{Fi}\,\mathbf{V}_I^\top] \\ &= f_{q,j}(u^{\mathbf{V}_I}(a_1, ..., a_n))(q) \\ &= u^{\mathbf{V}_I}(a_1, ..., a_n)(q) \quad \blacksquare \end{aligned}$$

4. LAA'S ARE NEAT REDUCTS

In this section we will show that every LAA embeds into a bigger LAA in which there are enough spare λ-variables to apply the α-rule.

A will be an arbitrary lambda abstraction algebra of dimension I. Recall that the combinatory reduct Cr **A** of **A** is a λ-algebra from Theorem 29. We start the section by defining an $\mathsf{LAA}_{I \cup J}$ which is a candidate for having **A** as I-neat reduct.

Recall from Section 1 that

(i) By Theorem 26 the free extension $(\mathrm{Cr}\,\mathbf{A})[J]$ of the λ-algebra Cr **A** by an infinite set J in the variety of combinatory algebras can be turned in a locally finite LAA_J whose zero-dimensional subreduct is Cr **A**.

(ii) The members of the set $P_J(\mathrm{Cr}\,\mathbf{A})$ of combinatory polynomials are constructed from λ-variables in J, constant symbols **k**, **s** and $\bar{a}$ for all elements a of A.

(iii) For every element $b \in (\mathrm{Cr}\,\mathbf{A})[J]$, there exists at least one combinatory polynomial $t \in P_J(\mathrm{Cr}\,\mathbf{A})$ such that $b = t^{(\mathrm{Cr}\,\mathbf{A})[J]}$.

(iv) The lambda abstractions λx $(x \in J)$ are defined as $\lambda x^{(\mathrm{Cr}\,\mathbf{A})[J]}.b = \lambda^* x(t)^{(\mathrm{Cr}\,\mathbf{A})[J]}$.

For all $x \in I$ and all $t, u \in P_J(\mathrm{Cr}\,\mathbf{A})$, we define by induction over the complexity of t the combinatory polynomial t_x^u as follows: $\bar{a}_x^u = (\overline{\lambda x^{\mathbf{A}}.a})u$, for every $a \in A$. Let t be a combinatory polynomial different from $\bar{a}$ for every $a \in A$. If each constant $\bar{a}$ $(a \in A)$ does not occur in t, in particular t can be **k** or **s**, define $t_x^u = t$; otherwise,

t must be of the form vw, where w and v are combinatory polynomials, at least one of which contains a constant $\bar{a}$; in this case define $(vw)^u_x = v^u_x w^u_x$.

Recall the definition of J-reduct from Section 3.

DEFINITION 37. Let $\mathbf{A}$ be an LAA_I and suppose J is an infinite set disjoint from I. The *J-expansion* of $\mathbf{A}$ is an algebra

$$\mathbf{B} = (B, \cdot^{\mathbf{B}}, \langle \lambda x^{\mathbf{B}} \colon x \in I \cup J \rangle, \langle x^{\mathbf{B}} \colon x \in I \cup J \rangle)$$

in the similarity type of $\mathsf{LAA}_{I \cup J}$ satisfying the following conditions:

(i) $Rd_J\,\mathbf{B} = (\mathrm{Cr}\ \mathbf{A})[J]$, i.e.,

$$B = (\mathrm{Cr}\ A)[J], \qquad \cdot^{\mathbf{B}} = \cdot^{(\mathrm{Cr}\ \mathbf{A})[J]}, \qquad x^{\mathbf{B}} = x^{(\mathrm{Cr}\ \mathbf{A})[J]}$$

and

$$\lambda x^{\mathbf{B}} = \lambda x^{(\mathrm{Cr}\ \mathbf{A})[J]} \qquad \text{for all} \quad x \in J.$$

(ii) $x^{\mathbf{B}} := x^{\mathbf{A}}$, for all $x \in I$.

(iii) Let $b \in B$ and $x \in I$. If $b = t^{\mathbf{B}}$ for a combinatory polynomial t and $z \in J$ is a λ-variable not occurring in t, we define

$$\lambda x^{\mathbf{B}}.b := (\lambda^* z(t^z_x))^{\mathbf{B}}.$$

Note that $t^{\mathbf{B}} = t^{(\mathrm{Cr}\ \mathbf{A})[J]}$ for every combinatory polynomial $t \in P_J(\mathrm{Cr}\ \mathbf{A})$ since the J-reduct of $\mathbf{B}$ is $(\mathrm{Cr}\ \mathbf{A})[J]$.

We are going to show that the J-expansion of $\mathbf{A}$ is well defined; i.e., the definition of $\lambda x^{\mathbf{B}}.b$ $(x \in I)$ is independent of the choice of the λ-variable z and of the combinatory polynomial t such that $b = t^{\mathbf{B}}$.

If t, u are combinatory polynomials, $t[z := u]$ denotes the combinatory polynomial obtained from t substituting u for λ-variable z in t.

LEMMA 38. *Let t, u, w be combinatory polynomials, $x \in I$ and $z \in J$. If z does not occur in t, then*

$$t^{w[z := u]}_x = t^w_x[z := u].$$

Proof. The proof is by induction over the complexity of t. ∎

Recall from Section 1 that $\equiv_{\mathrm{Cr}\ \mathbf{A}}$ is an equivalence relation on $P_J(\mathrm{Cr}\ \mathbf{A})$ and that $t \equiv_{\mathrm{Cr}\ \mathbf{A}} u$ iff the equation $t = u$ is a logical consequence of the equational diagram of $\mathrm{Cr}\ \mathbf{A}$ together with the axioms of combinatory logic.

The following lemma is Proposition 7.1.6(iii) in [3]. Its easy proof is given by induction over the complexity of t.

LEMMA 39.

$$(\lambda^* z(t))u \equiv_{\mathbf{Cr\,A}} t[z := u],$$

for all combinatory polynomials t, u and λ-variables $z \in J$.

PROPOSITION 40. *Let $x \in I$, $t \in P_J(\mathrm{Cr}\,\mathbf{A})$ be a combinatory polynomial, and $z, y \in J$ be two λ-variables not occurring in t. Then*

$$\lambda^* z(t^z_x) \equiv_{\mathbf{Cr\,A}} \lambda^* y(t^y_x).$$

So, for every $b = t^{\mathbf{B}} \in B$, the definition of $\lambda x^{\mathbf{B}}.b$ as $(\lambda^* z(t^z_x))^{\mathbf{B}}$ is independent of the choice of the λ-variable z that does not occur in t.

Proof.

$$\begin{aligned} \lambda z^*(t^z_x) &\equiv_{\mathbf{Cr\,A}} \lambda y^*[\lambda z^*(t^z_x)\, y] && [(\alpha)\text{-axiom in } (\mathrm{Cr}\,\mathbf{A})[J]] \\ &\equiv_{\mathbf{Cr\,A}} \lambda y^*(t^z_x[z := y]) && [\text{Lemma } 39] \\ &= \lambda y^*(t^y_x) && [\text{Lemma } 38] \quad \blacksquare \end{aligned}$$

We now start the proof that the definition of $\lambda x^{\mathbf{B}}.b$ $(x \in I)$ is independent of the choice of t such that $b = t^{\mathbf{B}}$.

LEMMA 41. *Let $\mathbf{s} = \lambda xyz.xz(yz)$. Then every LAA_I satisfies the following identity*

$$\mathbf{s}(\lambda x.\xi)(\lambda x.\mu) = \lambda x.\xi\mu.$$

Proof. Let $\mathbf{A}$ be an LAA_I and $a, b \in A$. Let

$$t = [\lambda y.\mathbf{s}(y(\lambda yzx.a))(y(\lambda yzx.b))](\lambda x'.x'yz).$$

Then we have

$$\begin{aligned} t &= \mathbf{s}((\lambda x'.x'yz)(\lambda yzx.a))((\lambda x'.x'yz)(\lambda yzx.b)) && [(\beta_5), (\beta_1), (\beta_4)] \\ &= \mathbf{s}((\lambda yzx.a)\, yz)((\lambda yzx.b)\, yz) && [(\beta_5), (\beta_1), (\beta_2)] \\ &= \mathbf{s}(\lambda x.a)(\lambda x.b) && [(\beta_3)] \end{aligned}$$

Let $u = \lambda yzx.a$ and $v = \lambda yzx.b$. From Proposition 4 it follows that yu and yv are independent of z, x.

$$\begin{aligned} \mathbf{s}(yu)(yv) &= (\lambda yzx.yx(zx))(yu)(yv) && [\text{def } \mathbf{s}] \\ &= [\lambda zx.(yu)\, x(zx)](yv) && [(\beta_6), \text{z,x} \notin \triangle^{\mathbf{A}}(yu)] \\ &= \lambda x.(yu)\, x((yv)\, x) && [(\beta_6), x \notin \triangle^{\mathbf{A}}(yv)] \end{aligned}$$

Then

$$
\begin{aligned}
t &= (\lambda y.\mathbf{s}(yu)(yv))(\lambda x'.x'yz) && [\text{def } t, u, v]\\
&= [\lambda y.\lambda x.(yu)\,x((yv)\,x)](\lambda x'.x'yz) && [\text{above}]\\
&= \lambda x.((\lambda x'.x'yz)u)\,x(((\lambda x'.x'yz)v)\,x) && [(\beta_6)]\\
&= \lambda x.(uyz)\,x((vyz)\,x) && [(\beta_5), (\beta_1), (\beta_2)]\\
&= \lambda x.(\lambda x.a)\,x((\lambda x.b)\,x) && [(\beta_3), \text{def } u, v]\\
&= \lambda x.ab.
\end{aligned}
$$

By $t = \lambda x.ab$ and $t = \mathbf{s}(\lambda x.a)(\lambda x.b)$ we obtain the conclusion. ∎

COROLLARY 42. *Let* $\mathbf{A}$ *be an* LAA_I, $a \in A$ *and* $x \in I$. *Then* $\mathbf{s}(\mathbf{k}\,\overline{\lambda x^{\mathbf{A}}.a})\mathbf{i} \equiv_{\mathrm{Cr}\,\mathbf{A}} \overline{\lambda x^{\mathbf{A}}.a}$.

Proof. By using the axioms $\mathbf{k} = \lambda xy.x$ and $\mathbf{s} = \lambda xyz.xz(yz)$, it is sufficient to prove that the algebra $\mathbf{A}$ satisfies the equation $\mathbf{s}(\mathbf{k}(\lambda x.a))\mathbf{i} = \lambda x.a$, that is a particular case of the previous Lemma because $\mathbf{k}(\lambda x.a) = \lambda xx.a$. ∎

LEMMA 43. *Let* $t, u, w \in P_J(\mathrm{Cr}\,\mathbf{A})$ *be combinatory polynomials such that* w *and* tu *have no common* λ*-variables. Then* $t \equiv_{\mathrm{Cr}\,\mathbf{A}} u$ *implies that* $t^w_x \equiv_{\mathrm{Cr}\,\mathbf{A}} u^w_x$ *for all* $x \in I$.

Proof. The proof is by induction on the length of the derivation of the equation $t = u$ from the equational diagram $D_{\mathrm{Cr}\,\mathbf{A}}$ of $\mathrm{Cr}\,\mathbf{A}$ together with the axioms **CL** of combinatory logic. Recall from Section 1 that the equational diagram $D_{\mathrm{Cr}\,\mathbf{A}}$ of the combinatory reduct $\mathrm{Cr}\,\mathbf{A}$ is constituted by the set of all equations of the form $\bar{c}\bar{d} = \bar{e}$ for $c, d, e \in A$ such that $cd = e$ in $\mathbf{A}$ plus the two equations $\mathbf{k} = \overline{\lambda xy^{\mathbf{A}}.x}$ $(x, y \in I)$ and $\mathbf{s} = \overline{\lambda xyz^{\mathbf{A}}.xz(yz)}$ $(x, y, z \in I)$.

We omit the superscript $^{\mathbf{A}}$ when there is no possibility of confusion.

(*Axiom*: $\bar{a} = \bar{c}\bar{d}$ $(a, c, d \in A)$): The conclusion derives from the following chain of equalities:

$$
\begin{aligned}
(\bar{c}\bar{d})^w_x &= \bar{c}^w_x \bar{d}^w_x && [\text{def}(-)^w_x]\\
&= (\overline{\lambda x.c})\,w((\overline{\lambda x.d})\,w)\\
&\equiv_{\mathrm{Cr}\,\mathbf{A}} \mathbf{s}(\overline{\lambda x.c})(\overline{\lambda x.d})\,w && [\text{axioms } \mathbf{CL}]\\
&\equiv_{\mathrm{Cr}\,\mathbf{A}} (\overline{\lambda yzx.yx(zx)})(\overline{\lambda x.c})(\overline{\lambda x.d})\,w && [\text{def } \mathbf{s}]\\
&\equiv_{\mathrm{Cr}\,\mathbf{A}} \overline{(\lambda yzx.yx(zx))(\lambda x.c)(\lambda x.d)}\,w\\
&= (\overline{\lambda x.cd})\,w && [\text{Lemma } 41]\\
&= (\overline{\lambda x.a})\,w && [a = cd]\\
&= \bar{a}^w_x
\end{aligned}
$$

(*Axioms* $\mathbf{k} = \overline{\lambda xy.x}$ and $\mathbf{s} = \overline{\lambda xyz.xz(yz)}$ $(x, y, z \in I)$): We have

$$\begin{aligned} (\overline{\lambda xy.x})^w_x &= (\overline{\lambda xxy.x})\,w \\ &= \overline{(\lambda yx.y)(\lambda xy.x)}\,w \\ &\equiv_{\mathbf{Cr\,A}} \mathbf{kk}w && [\mathbf{k} = \overline{\lambda xy.x}] \\ &\equiv_{\mathbf{Cr\,A}} \mathbf{k} && [\text{axioms } \mathbf{CL}] \\ &= \mathbf{k}^w_x. && [\text{def}(-)^w_x] \end{aligned}$$

Similarly for the other axiom.

(*Axioms*: $\mathbf{k}xy = x$ and $\mathbf{s}xyz = xz(yz)$ $(x, y, z \in J)$: There is no constant $\bar{a}$ $(a \in A)$ occurring in $\mathbf{k}xy$ and $\mathbf{s}xyz$, so the result is trivial.

(*Reflexivity*, *Symmetry*): Obvious.

(*Transitivity*): Suppose that $t \equiv_{\mathbf{Cr\,A}} s$ is derived from $t \equiv_{\mathbf{Cr\,A}} u$ and $u \equiv_{\mathbf{Cr\,A}} s$ by transitivity. If w and u have no common variables, then the result follows by induction hypothesis and transitivity. Otherwise, let $w' = w[y_1 := z_1, ..., y_k := z_k]$ be obtained by renaming all the variables $y_1, ..., y_k$ in w by new λ-variables $z_1, ..., z_k$ not occurring in t, s, u. By applying the induction hypothesis and the transitivity rule we derive that $t^{w'}_x \equiv_{\mathbf{Cr\,A}} s^{w'}_x$. Then we apply the substitution rule of the equational calculus and Lemma 38:

$$t^w_x = t^{w'}_x[z_1 := y_1, ..., z_k := y_k] \equiv_{\mathbf{Cr\,A}} s^{w'}_x[z_1 := y_1, ..., z_k := y_k] = s^w_x.$$

(*Replacement rule*): Suppose that $tu \equiv_{\mathbf{Cr\,A}} ts$ is derived from $u \equiv_{\mathbf{Cr\,A}} s$ and $tu \equiv_{\mathbf{Cr\,A}} tu$ by replacement. By induction hypothesis we have $u^w_x \equiv_{\mathbf{Cr\,A}} s^w_x$, so that

$$(tu)^w_x = t^w_x u^w_x \equiv_{\mathbf{Cr\,A}} t^w_x s^w_x = (ts)^w_x.$$

(*Substitution rule*): Let $t[y := u] \equiv_{\mathbf{Cr\,A}} s[y := u]$ obtained from $t \equiv_{\mathbf{Cr\,A}} s$ by applying the substitution rule. If y does not occur in ts then the result follows by induction hypothesis. Otherwise, let $z' \neq y \in J$ be a variable that does not occur in t, s, u, w and let $w' = w[y := z']$. Since w' and t, s have no common variables $t \equiv_{\mathbf{Cr\,A}} s$ implies $t^{w'}_x \equiv_{\mathbf{Cr\,A}} s^{w'}_x$, by induction hypothesis. An application of the substitution rule provides

$$t^{w'}_x[y := u^{w'}_x] \equiv_{\mathbf{Cr\,A}} s^{w'}_x[y := u^{w'}_x]$$

that is equal to

$$(t[y := u])^{w'}_x \equiv_{\mathbf{Cr\,A}} (s[y := u])^{w'}_x$$

since y does not occur in w'. Another application of the substitution rule gives

$$(t[y := u])^{w'}_x\,[z' := y] \equiv_{\mathbf{Cr\,A}} (s[y := u])^{w'}_x\,[z' := y].$$

Then Lemma 38 and the hypothesis that z' does not occur in $t[y:=u]$ and $s[y:=u]$ give the conclusion

$$(t[y:=u])^w_x \equiv_{\mathrm{Cr}\,\mathbf{A}} (s[y:=u])^w_x. \quad \blacksquare$$

Proposition 44. *The definition of $\lambda x^{\mathbf{B}}.b$ is independent of the choice of the combinatory polynomial t such that $b = t^{\mathbf{B}}$.*

Proof. Let $b = t^{\mathbf{B}} = u^{\mathbf{B}}$, so that $t \equiv_{\mathrm{Cr}\,\mathbf{A}} u$. We must show that $\lambda^* z(t^z_x) \equiv_{\mathrm{Cr}\,\mathbf{A}} \lambda^* y(u^y_x)$ for every variable z not occurring in t and every variable y not occurring in u. Let v be a variable not occurring in t, u. By Lemma 43 we have that the hypothesis $t \equiv_{\mathrm{Cr}\,\mathbf{A}} u$ implies $t^v_x \equiv_{\mathrm{Cr}\,\mathbf{A}} u^v_x$. Moreover, by applying Lemma 25 we obtain

$$\lambda^* v(t^v_x) \equiv_{\mathrm{Cr}\,\mathbf{A}} \lambda^* v(u^v_x).$$

Finally, by Proposition 40 we have

$$\lambda^* z(t^z_x) \equiv_{\mathrm{Cr}\,\mathbf{A}} \lambda^* v(t^v_x) \equiv_{\mathrm{Cr}\,\mathbf{A}} \lambda^* v(u^v_x) \equiv_{\mathrm{Cr}\,\mathbf{A}} \lambda^* y(u^y_x). \quad \blacksquare$$

Proposition 45. *If b is an element of A and $x \in I$, we have that*

$$\lambda x^{\mathbf{B}}.b = \lambda x^{\mathbf{A}}.b.$$

Proof. Let $b \in A$. Then b is a zero-dimensional element of $(\mathrm{Cr}\,\mathbf{A})[J] = Rd_J\,\mathbf{B}$, and $b = \bar{b}^{\mathbf{B}}$ for the combinatory polynomial $\bar{b}$. Then we have

$$\begin{aligned}
\lambda x^{\mathbf{B}}.b &= (\lambda^* z(\bar{b}^z_x))^{\mathbf{B}} && [\mathrm{def}\ \lambda x^{\mathbf{B}}]\\
&= (\lambda^* z((\overline{\lambda x^{\mathbf{A}}.b})z))^{\mathbf{B}} && [\mathrm{def}(-)^z_x]\\
&= [(\mathbf{s}(\mathbf{k}(\overline{\lambda x^{\mathbf{A}}.b}))\mathbf{i}]^{\mathbf{B}} && [\mathrm{def}\ \lambda z^*]\\
&= (\overline{\lambda x^{\mathbf{A}}.b})^{\mathbf{B}} && [\text{Corollary 42}]\\
&= \lambda x^{\mathbf{A}}.b. && \blacksquare
\end{aligned}$$

Theorem 46. *Let $\mathbf{A}$ be an LAA_I and J be an infinite set disjoint from I. Then the J-expansion $\mathbf{B}$ of A is an $\mathsf{LAA}_{I \cup J}$.*

Proof. *(Axiom β_1)*: $(\lambda x^{\mathbf{B}}.x^{\mathbf{B}})a = a$ for all $a \in B$.

Assume $x \in I$ and let $a = t^{\mathbf{B}}$ for a combinatory polynomial t. Then

$$\begin{aligned}
(\lambda x^{\mathbf{B}}.x^{\mathbf{B}}) \cdot^{\mathbf{B}} a &= (\lambda^* z(\bar{x}^z_x)t)^{\mathbf{B}} && [\mathrm{def}\ \lambda x^{\mathbf{B}}]\\
&= (\lambda^* z((\overline{\lambda x^{\mathbf{A}}.x})z)t)^{\mathbf{B}}\\
&= ((\overline{\lambda x^{\mathbf{A}}.x})t)^{\mathbf{B}}\\
&= (\overline{\mathbf{s}^{\mathbf{A}}\mathbf{k}^{\mathbf{A}}\mathbf{k}^{\mathbf{A}}}t)^{\mathbf{B}}\\
&= (\mathbf{skk}t)^{\mathbf{B}}\\
&= t^{\mathbf{B}}\\
&= a.
\end{aligned}$$

($Axiom\ \beta_2$): $(\lambda x^{\mathbf{B}}.y^{\mathbf{B}})a = y^{\mathbf{B}}$ for all $a \in B$ $(x \neq y)$.
Let $a = t^{\mathbf{B}}$ for a combinatory polynomial t. We have three nontrivial subcases.
$(x,\ y \in I)$:

$$\begin{aligned}(\lambda x^{\mathbf{B}}.y^{\mathbf{B}}) \cdot^{\mathbf{B}} a &= (\lambda^* z(\bar{y}^z_x)\, t)^{\mathbf{B}} \\ &= (\lambda^* z((\overline{\lambda x^{\mathbf{A}}.y})\, z)\, t)^{\mathbf{B}} \\ &= ((\overline{\lambda x^{\mathbf{A}}.y})\, t)^{\mathbf{B}} \\ &= (\overline{\mathbf{k}^{\mathbf{A}} y}\, t)^{\mathbf{B}} \\ &= (\mathbf{k}\,\bar{y} t)^{\mathbf{B}} \\ &= \bar{y}^{\mathbf{B}} \\ &= y^{\mathbf{B}}\end{aligned}$$

$(x \in I,\ y \in J)$:

$$\begin{aligned}(\lambda x^{\mathbf{B}}.y^{\mathbf{B}}) \cdot^{\mathbf{B}} a &= (\lambda^* z(y^z_x))^{\mathbf{B}} \cdot^{\mathbf{B}} t^{\mathbf{B}} && [\text{def}\ \lambda x^{\mathbf{B}}] \\ &= (\lambda^* z(y)\, t)^{\mathbf{B}} \\ &= (\mathbf{k}\, y t)^{\mathbf{B}} && [z \neq y] \\ &= y^{\mathbf{B}}.\end{aligned}$$

$(x \in J,\ y \in I)$:

$$\begin{aligned}(\lambda x^{\mathbf{B}}.y^{\mathbf{B}})a &= (\lambda x^{\mathbf{B}}.y^{\mathbf{A}})a \\ &= [(\lambda^* x(\bar{y})\, t]^{\mathbf{B}} && [Rd_J\,\mathbf{B} = (\mathrm{Cr}\ \mathbf{A})[J]] \\ &= [\mathbf{k}\,\bar{y} t]^{\mathbf{B}} \\ &= \bar{y}^{\mathbf{B}} \\ &= y^{\mathbf{B}}.\end{aligned}$$

($Axiom\ \beta_3$): $(\lambda x^{\mathbf{B}}.a)\, x^{\mathbf{B}} = a$ for all $a \in B$.
Assume $x \in I$ and $a = t^{\mathbf{B}}$ for a combinatory polynomial t.

$$\begin{aligned}(\lambda x^{\mathbf{B}}.a)\, x^{\mathbf{B}} &= (\lambda^* z(t^z_x)\, \bar{x})^{\mathbf{B}} \\ &= (t^z_x[z := \bar{x}])^{\mathbf{B}} && [\text{Lemma } 39] \\ &= (t^{\bar{x}}_x)^{\mathbf{B}} && [\text{Lemma } 38] \\ &= t^{\mathbf{B}} \\ &= a\end{aligned}$$

since it is possible to prove by induction that $t^{\bar{x}}_x \equiv_{\mathbf{Cr\ A}} t$.

($Axiom\ \beta_4$): $(\lambda x^{\mathbf{B}}.\lambda x^{\mathbf{B}}.a)b = \lambda x^{\mathbf{B}}.a$ for all $a, b \in B$.
Let $x \in I$, $a = t^{\mathbf{B}}$, $b = u^{\mathbf{B}}$ with $t, u \in P_J(\mathrm{Cr}\ \mathbf{A})$. We have $\lambda x^{\mathbf{B}}.a = (\lambda^* z(t^z_x))^{\mathbf{B}}$ and

$$\begin{aligned}(\lambda x^{\mathbf{B}}.\lambda x^{\mathbf{B}}.a)b &= [\lambda^* z[\lambda^* z(t^z_x)^z_x]u]^{\mathbf{B}} \\ &= ((\lambda^* z(t^z_x)^z_x)[z := u])^{\mathbf{B}} && [\text{Lemma 39}] \\ &= (\lambda^* z(t^z_x)^u_x)^{\mathbf{B}} && [\text{Lemma 38}]\end{aligned}$$

We can use two times the same variable z because z does not occur in $\lambda^* z(t^z_x)$. The conclusion follows if we show that

$$\lambda^* z(t^z_x) \equiv_{\mathrm{Cr}\ \mathbf{A}} \lambda^* z(t^z_x)^u_x.$$

The proof is by induction over the complexity of the combinatory polynomial t.
($t = \bar{c}$ with $c \in A$): We omit the superscript $^{\mathbf{A}}$. Then

$$\begin{aligned}\lambda^* z(\bar{c}^z_x)^u_x &= \lambda^* z((\overline{\lambda x.c})z)^u_x \\ &= (\mathbf{s}(\mathbf{k}(\overline{\lambda x.c}))\mathbf{i})^u_x && [\text{def } \lambda^*] \\ &\equiv_{\mathrm{Cr}\ \mathbf{A}} (\overline{\lambda x.c})^u_x && [\text{Corollary 42, Lemma 43}] \\ &= (\overline{\lambda xx.c})u && [\text{def}(-)^u_x] \\ &= (\overline{(\lambda yx.\,y)\ \lambda x.c})u \\ &\equiv_{\mathrm{Cr}\ \mathbf{A}} \mathbf{k}\overline{\lambda x.c}u && [\mathbf{k} = \overline{\lambda yx.y}] \\ &\equiv_{\mathrm{Cr}\ \mathbf{A}} \overline{\lambda x.c} \\ &\equiv_{\mathrm{Cr}\ \mathbf{A}} \mathbf{s}(\mathbf{k}(\overline{\lambda x.c}))\mathbf{i} && [\text{Corollary 42}] \\ &= \lambda^* z((\overline{\lambda x.c})z) && [\text{def } \lambda^*] \\ &= \lambda^* z(\bar{c}^z_x)\end{aligned}$$

(t does not contain constants $\bar{c}$ with $c \in A$): We have

$$\begin{aligned}(\lambda^* z(t^z_x))^u_x &= (\lambda^* z(t))^u_x \\ &= (\mathbf{k}t)^u_x && [z \text{ not in } t] \\ &= \mathbf{k}t \\ &= \lambda^* z(t) \\ &= \lambda^* z(t^z_x)\end{aligned}$$

($t = wv$): There is at least one constant $\bar{c}$ occurring in t, so that z occurs in t^z_x. Then,

$$\begin{aligned}\lambda^* z((wv)^z_x) &= \lambda^* z(w^z_x v^z_x) && [\text{def}(-)^z_x] \\ &= \mathbf{s}\ \lambda^* z(w^z_x)\ \lambda^* z(v^z_x) && [\text{def } \lambda^*]\end{aligned}$$

On the other hand,

$$\begin{aligned}(\lambda^*z((wv)^z_x))^u_x &= (\mathbf{s}\,\lambda^*z(w^z_x)\,\lambda^*z(v^z_x))^u_x\\ &= \mathbf{s}(\lambda^*z(w^z_x))^u_x\,(\lambda^*z(v^z_x))^u_x\\ &= \mathbf{s}\,\lambda^*z(w^z_x)\,\lambda^*z(v^z_x). \qquad [\text{induction}]\end{aligned}$$

(*Axiom* β_5): $(\lambda x^{\mathbf{B}}.ab)c = (\lambda x^{\mathbf{B}}.a)\,c((\lambda x^{\mathbf{B}}.b)c)$.

Assume that $x \in I$ and that $a = t^{\mathbf{B}}$, $b = u^{\mathbf{B}}$ and $c = w^{\mathbf{B}}$ for some combinatory polynomials t, u, w.

$$\begin{aligned}(\lambda x^{\mathbf{B}}.ab)c &= ((\lambda^*z((tu)^z_x))w)^{\mathbf{B}}\\ &= ((tu)^z_x\,[z := w])^{\mathbf{B}} \qquad [\text{Lemma 39}]\\ &= (t^z_x[z := w]\;u^z_x[z := w])^{\mathbf{B}}\\ &= [\lambda^*z(t^z_x)\;w((\lambda^*z(u^z_x))w)]^{\mathbf{B}} \qquad [\text{Lemma 39}]\\ &= (\lambda x^{\mathbf{B}}.a)\;c((\lambda x^{\mathbf{B}}.b)c).\end{aligned}$$

(*Axiom* α): $\lambda x^{\mathbf{B}}.a = \lambda y^{\mathbf{B}}.(\lambda x^{\mathbf{B}}.a)\,y^{\mathbf{B}}$ if $(\lambda y^{\mathbf{B}}.a)\,z' = a$ $(z' \neq y)$.

Since the algebra **B** satisfies (β_1)–(β_5), then from Lemmas 1.6 and 1.4 in [34] it follows that $(\lambda y^{\mathbf{B}}.a)z' = a$ if and only if $(\lambda y^{\mathbf{B}}.a)c = a$ for all $c \in B$. Then we can assume, when it is convenient, that either z' is in J or z' is in I.

$(x \in I,\ y \in J)$: Since $(\lambda y^{\mathbf{B}}.a)b = a$ for all $b \in B$, then we can find a combinatory polynomial t such that $a = t^{\mathbf{B}}$ and y does not occur in t. Since by Lemma 38 we have

$$(\lambda x^{\mathbf{B}}.a)\;y^{\mathbf{B}} = (t^z_x[z := y])^{\mathbf{B}} = (t^y_x)^{\mathbf{B}}$$

then

$$\lambda y^{\mathbf{B}}.(\lambda x^{\mathbf{B}}.a)\;y^{\mathbf{B}} = (\lambda^*y(t^y_x))^{\mathbf{B}},$$

while

$$\lambda x^{\mathbf{B}}.a = (\lambda^*y(t^y_x))^{\mathbf{B}}$$

because by assumption y does not occur in t.

$(x \in J,\ y \in I)$: Let $a = t^{\mathbf{B}}$. Then,

$$\lambda x^{\mathbf{B}}.a = (\lambda^*x(t))^{\mathbf{B}} \qquad \text{and} \qquad \lambda y^{\mathbf{B}}.(\lambda x^{\mathbf{B}}.a)\;y^{\mathbf{B}} = (\lambda^*x(t[x := \bar{y}]^x_y))^{\mathbf{B}}$$

because the variable $x \in J$ does not occur in $t[x := \bar{y}]$. The conclusion follows from Lemma 25 if we show that

$$t[x := \bar{y}]^x_y \equiv_{\mathbf{Cr\,A}} t.$$

First, we have

$$\begin{aligned} t^{\mathbf{B}} &= a \\ &= (\lambda y^{\mathbf{B}}.a)\, x^{\mathbf{B}} && [\text{hypothesis}] \\ &= [\lambda^* z(t^z_y)x]^{\mathbf{B}} && [\text{def } \lambda y^{\mathbf{B}}] \\ &= (t^z_y[z := x])^{\mathbf{B}} && [\text{Lemma 39}] \\ &= (t^x_y)^{\mathbf{B}}. && [\text{Lemma 38}] \end{aligned}$$

The proof that $t[x := \bar{y}]^x_y \equiv_{\mathrm{Cr}\,\mathbf{A}} t^x_y$ is by induction over the complexity of t. If $t = \bar{b}$ with $b \in A$ then $\bar{b}[x := \bar{y}]^x_y = \bar{b}^x_y$, while the case $t = x$ is treated as follows:

$$x[x := \bar{y}]^x_y = \bar{y}^x_y = (\overline{\lambda y^{\mathbf{A}}.y})x \equiv_{\mathrm{Cr}\,\mathbf{A}} x = x^x_y.$$

If x does not occur in t then $t[x := \bar{y}]^x_y = t^x_y = t$. Otherwise, $t = uv$ and

$$(uv)[x := \bar{y}]^x_y = u[x := \bar{y}]^x_y\, v[x := \bar{y}]^x_y \equiv_{\mathrm{Cr}\,\mathbf{A}} u^x_y v^x_y.$$

($x, y \in I$): The algebra $\mathbf{B}$ satisfies (β_1)–(β_5). Then from Proposition 1.5 in [34] it follows that the quasi-identity (α) is equivalent to the identity (α'): $\lambda x.(\lambda y.a)\, j = \lambda y.(\lambda x.(\lambda y.a)\, j)\, y$, for some λ-variable $j \neq y$.

Let $a = t^{\mathbf{B}}$ and $j, z \in J$ be two distinct variables not occurring in t. Then by applying Lemma 38 we have

$$(\lambda y^{\mathbf{B}}.a)\, j^{\mathbf{B}} = [\lambda^* z(t^z_y)\, j]^{\mathbf{B}} = (t^j_y)^{\mathbf{B}}$$

from which we derive

$$\lambda x^{\mathbf{B}}.(\lambda y^{\mathbf{B}}.a)\, j^{\mathbf{B}} = (\lambda^* z(t^{j\,z}_{y\,x}))^{\mathbf{B}}$$

and

$$\lambda y^{\mathbf{B}}.(\lambda x^{\mathbf{B}}.(\lambda y^{\mathbf{B}}.a)\, j^{\mathbf{B}})\, y^{\mathbf{B}} = \lambda y^{\mathbf{B}}.(t^{j\bar{y}}_{y_x})^{\mathbf{B}} = (\lambda^* z(t^{j^{\bar{y}^z}}_{y_{x_y}}))^{\mathbf{B}}.$$

We can utilize the variable z two times because z does not occur in $t^{j^{\bar{y}}}_{y_x}$. Note that we can assume that $t = \bar{c} j_1 \cdots j_n$ with $c \in A$ and $j_1, ..., j_n \in J$. Otherwise, by considering that $Rd_J\, \mathbf{B} = (\mathrm{Cr}\,\mathbf{A})[J]$ is a locally finite LAA_J, we have that $\triangle^{(\mathrm{Cr}\,\mathbf{A})[J]}(a) = \{j_1, ..., j_n\}$ is a finite set; so, $a = (\lambda j_1 \cdots j_n.a)\, j_1 \cdots j_n$ and $c = (\lambda j_1 \cdots j_n.a)$ is an element of $A = \mathrm{Zd}\ (\mathrm{Cr}\,\mathbf{A})[J]$, from which we derive the conclusion. Then,

$$t^{j^z}_{y_x} = (\overline{\lambda x y^{\mathbf{A}}.c})\, z j j_1 \cdots j_n$$

and

$$\begin{aligned} t^{j^{\bar{y}z}}_{y_{x_y}} &= [(\overline{\lambda xy^{\mathbf{A}}.c})\ \bar{y}jj_1 \cdots j_n]^z_y \\ &\equiv_{\mathbf{Cr\,A}} [(\overline{(\lambda xy^{\mathbf{A}}.c)\ y})\ jj_1 \cdots j_n]^z_y \qquad [\text{Lemma } 43] \\ &= (\overline{\lambda y^{\mathbf{A}}.(\lambda xy^{\mathbf{A}}.c)\ y})\ zjj_1 \cdots j_n. \end{aligned}$$

But $\mathbf{A}$ is an LAA_I and by (α) we have

$$\lambda xy^{\mathbf{A}}.c = \lambda y^{\mathbf{A}}.(\lambda xy^{\mathbf{A}}.c)\ y^{\mathbf{A}}.$$

The conclusion is now obvious.

(*Axiom* β_6): $(\lambda xy^{\mathbf{B}}.a)b = \lambda y^{\mathbf{B}}.(\lambda x^{\mathbf{B}}.a)b$ if $(\lambda y^{\mathbf{B}}.b)z' = b$ $(z' \neq y, x \neq y)$.

Let $a = t^{\mathbf{B}}$ and $b = u^{\mathbf{B}}$ for some combinatory polynomials t, u.

$(y \in J,\ x \in I)$: Since $(\lambda y^{\mathbf{B}}.b)\ z' = b$ and $y \in J$ we can choose the polynomial u in such a way the variable y does not occur in it. Then we have

$$\begin{aligned} (\lambda xy^{\mathbf{B}}.a)b &= (\lambda x^{\mathbf{B}}.\lambda^* y(t)^{\mathbf{B}})b \\ &= [(\lambda^* y(\lambda^* y(t)^y_x))u]^{\mathbf{B}}, \qquad [y \text{ not in } \lambda^* y(t), b = u^{\mathbf{B}}] \\ &= (\lambda^* y(t)^y_x\ [y := u])^{\mathbf{B}} \qquad [\text{Lemma } 39] \\ &= (\lambda^* y(t)^u_x)^{\mathbf{B}}. \qquad [\text{Lemma } 38] \end{aligned}$$

On the other hand, we have for a variable $z \in J$ not occurring in t, u:

$$\begin{aligned} \lambda y^{\mathbf{B}}.(\lambda x^{\mathbf{B}}.a)b &= \lambda y^{\mathbf{B}}.(\lambda^* z(t^z_x)u)^{\mathbf{B}} \qquad [\text{def } \lambda^{\mathbf{B}}] \\ &= \lambda y^{\mathbf{B}}.(t^z_x[z := u])^{\mathbf{B}} \qquad [\text{Lemma } 39] \\ &= (\lambda^* y(t^u_x))^{\mathbf{B}}. \qquad [\text{Lemma } 38] \end{aligned}$$

The conclusion follows if we show that

$$\lambda^* y(t)^u_x \equiv_{\mathbf{Cr\,A}} \lambda^* y(t^u_x).$$

If t does not contain occurrences of constants $\bar{c}$ $(c \in A)$, we have $t^u_x = t$. But $\lambda^* y(t)^u_x = \lambda^* y(t)$ since $\lambda^* y(t)$ does not also contain constants.

If y does not occur in t we have

$$\begin{aligned} \lambda^* y(t)^u_x &= (\mathbf{k}t)^u_x \qquad [\text{def } \lambda^*] \\ &= \mathbf{k}t^u_x \qquad [\text{def}(-)^u] \\ &= \lambda^* y(t^u_x). \qquad [y \text{ not in } u, t] \end{aligned}$$

Otherwise, the proof is by induction over the complexity of t. The only non-trivial case is $t = wv$ and t contains occurrences of y and of constants.

$$\begin{aligned} \lambda^*y(wv)^u_x &= \mathbf{s}\,\lambda^*y(w)^u_x\,\lambda^*y(v)^u_x && [\text{def } \lambda^*] \\ &= \mathbf{s}\,\lambda^*y(w^u_x)\,\lambda^*y(v^u_x) && [\text{induction}] \\ &= \lambda^*y((wv)^u_x). && [\text{def } \lambda^*] \end{aligned}$$

$(x \in J,\ y \in I)$: Let $z \in J$ be a variable different from x and not occurring in t, u. Since the algebra $\mathbf{B}$ satisfies (β_1)–(β_5), from Lemmas 1.6 and 1.4 in [34] it follows that $(\lambda y^{\mathbf{B}}.b)\, z' = b$ if and only if $(\lambda y^{\mathbf{B}}.b) c = b$ for all $c \in B$. In particular we have that $(\lambda y^{\mathbf{B}}.b)\, z^{\mathbf{B}} = b$, that implies

$$u^z_y[z := z] = u^z_y \equiv_{\mathrm{Cr}\,\mathbf{A}} u.$$

Then we have

$$\begin{aligned} (\lambda x y^{\mathbf{B}}.a) b &= (\lambda^*x(\lambda^*z(t^z_y))u)^{\mathbf{B}} && [\text{def } \lambda x^{\mathbf{B}}, \lambda y^{\mathbf{B}}] \\ &= (\lambda^*z(t^z_y)[x := u]))^{\mathbf{B}}. && [\text{Lemma 39}] \end{aligned}$$

Moreover,

$$\begin{aligned} \lambda y^{\mathbf{B}}.(\lambda x^{\mathbf{B}}.a) b &= \lambda y^{\mathbf{B}}.(\lambda^*x(t) u)^{\mathbf{B}} && [\text{def } \lambda^{\mathbf{B}}] \\ &= \lambda y^{\mathbf{B}}.(t[x := u])^{\mathbf{B}} && [\text{Lemma 39}] \\ &= (\lambda^*z((t[x := u])^z_y))^{\mathbf{B}} && [\text{def } \lambda y^{\mathbf{B}}] \\ &= (\lambda^*z(t^z_y[x := u^z_y]))^{\mathbf{B}} \\ &= (\lambda^*z(t^z_y[x := u]))^{\mathbf{B}} && [u^z_y \equiv_{\mathrm{Cr}\,\mathbf{A}} u] \end{aligned}$$

The conclusion

$$\lambda^*z(t^z_y)[x := u] \equiv_{\mathrm{Cr}\,\mathbf{A}} \lambda^*z(t^z_y[x := u])$$

follows from Lemma 7.1.7 p. 153 in [3] since z does not occur in u.

$(x \in I,\ y \in I)$: The algebra $\mathbf{B}$ satisfies (β_1)–(β_5). Then from Proposition 1.5 in [34] it follows that the quasi-identity (β_6) is equivalent to the identity

$$(\beta_6'){:}\ (\lambda x y.a)((\lambda y.b)\, j) = \lambda y.(\lambda x.a)((\lambda y.b)\, j), \qquad x \neq y, \quad j \neq y.$$

Without lost of generality we may assume $j \in J$. Let $z \neq j$ be not occurring in t, u. Then

$$\begin{aligned} (\lambda y^{\mathbf{B}}.b)\, j^{\mathbf{B}} &= (\lambda^*z(u^z_y)\, j)^{\mathbf{B}} && [\text{def } \lambda y^{\mathbf{B}}] \\ &= u^z_y[z := j] && [\text{Lemma 39}] \\ &= u^j_y. && [\text{Lemma 38}] \end{aligned}$$

Let $w = u^j_y$. We have

$$\begin{aligned}\lambda y^{\mathbf{B}}.(\lambda x^{\mathbf{B}}.a)((\lambda y^{\mathbf{B}}.b)\ j^{\mathbf{B}}) &= \lambda y^{\mathbf{B}}.(\lambda x^{\mathbf{B}}.a)\ w^{\mathbf{B}} && [\text{above}]\\ &= \lambda y^{\mathbf{B}}.(\lambda^* z(t^z_x) w)^{\mathbf{B}}\\ &= \lambda y^{\mathbf{B}}.(t^w_x)^{\mathbf{B}} && [\text{Lemmas 39, 38}]\\ &= (\lambda^* z(t^{w\ z}_{x\ y}))^{\mathbf{B}}\end{aligned}$$

and

$$\begin{aligned}(\lambda xy^{\mathbf{B}}.a)((\lambda y^{\mathbf{B}}.b)\ j^{\mathbf{B}}) &= (\lambda xy^{\mathbf{B}}.a)\ w^{\mathbf{B}} && [\text{above}]\\ &= (\lambda x^{\mathbf{B}}.(\lambda^* z(t^z_y))^{\mathbf{B}})\ w^{\mathbf{B}} && [\text{def } \lambda y^{\mathbf{B}}]\\ &= (\lambda^* z(\lambda^* z(t^z_y)^z_x) w)^{\mathbf{B}} && [\text{def } \lambda x^{\mathbf{B}}]\\ &= [(\lambda^* z(t^z_y)^z_x)[z := w]]^{\mathbf{B}} && [\text{Lemma 39}]\\ &= [\lambda^* z(t^z_y)^w_x]^{\mathbf{B}} && [\text{Lemma 38}]\\ &= [\lambda^* z(t^{z^w}_{y_x})]^{\mathbf{B}}. && [\text{Lemma 47 below}]\end{aligned}$$

LEMMA 47. *Assume that z does not occur in the combinatory polynomial w. Then*

$$\lambda^* z(t^z_y)^w_x = \lambda^* z(t^{z^w}_{y_x}).$$

Proof. The proof is by induction over the complexity of t.
($t = \bar{c}$ with $c \in A$):

$$\begin{aligned}\lambda^* z(\bar{c}^z_y)^w_x &= \lambda^* z((\overline{\lambda y.c})z)^w_x\\ &= (\mathbf{s}(\mathbf{k}\ \overline{\lambda y.c})\mathbf{i})^w_x\\ &= \mathbf{s}(\mathbf{k}((\overline{\lambda xy.c})w))\mathbf{i}\\ &= \lambda^* z((\overline{\lambda xy.c})\ wz) && [z \text{ not in } w]\\ &= \lambda^* z((\overline{\lambda y.cz})^w_x)\\ &= \lambda^* z(\bar{c}^{z^w}_{y_x}).\end{aligned}$$

(t does not contain constants $\bar{c}$ ($c \in A$)):

$$\begin{aligned}\lambda^* z(t^z_y)^w_x &= \lambda^* z(t)^w_x\\ &= (\mathbf{k}t)^w_x\\ &= \mathbf{k}t\\ &= \lambda^* z(t)\\ &= \lambda^* z(t^{z^w}_{y_x}).\end{aligned}$$

($t = vp$ with occurrences of constants):

$$\begin{aligned}\lambda^* z((vp)^z_y)^w_x &= \lambda^* z(v^z_y p^z_y)^w_x \\ &= (\mathbf{s}\,\lambda^* z(v^z_y)\,\lambda^* z(p^z_y))^w_x \qquad [z \text{ occurs in } v^z_y\, p^z_y] \\ &= \mathbf{s}\,\lambda^* z(v^z_y)^w_x\,\lambda^* z(p^z_y)^w_x \\ &= \mathbf{s}\,\lambda^* z(v^{z^w}_{y_x})\,\lambda^* z(p^{z^w}_{y_x}) \qquad [\text{induction}] \\ &= \lambda^* z((vp)^{z^w}_{y_x}). \qquad \blacksquare\end{aligned}$$

The conclusion will be obtained by Lemma 25(ii) if we show that

$$t^{w^z}_{x_y} \equiv_{\mathrm{Cr}\,\mathbf{A}} t^{z^w}_{y_x}. \tag{+}$$

We may assume that $t = \bar{c} o_1 \cdots o_n$ and $u = \bar{d} i_1 \cdots i_r$ with $c, d \in A$ and $o_1, ..., o_n$, $i_1, ..., i_r \in J$. In fact, by considering that $Rd_J\,\mathbf{B} = (\mathrm{Cr}\,\mathbf{A})[J]$ is a locally finite LAA_J, we have that $\triangle^{(\mathrm{Cr}\,\mathbf{A})[J]}(t^{\mathbf{B}}) = \{o_1, ..., o_n\}$ is a finite set; so, $t^{\mathbf{B}} = (\lambda o_1 \cdots o_n{}^{\mathbf{B}}.t^{\mathbf{B}})\, o_1 \cdots o_n$ and $c = (\lambda o_1 \cdots o_n{}^{\mathbf{B}}.t^{\mathbf{B}})$ is an element of $A = \mathrm{Zd}\ (\mathrm{Cr}\,\mathbf{A})[J]$. A similar argument works for u. Recalling that $w = u^j_y$ then the equivalence (+) assumes the form

$$(\overline{\lambda yx.c})\, z[(\overline{\lambda yy.d})\, zji_1 \cdots i_r]\, o_1 \cdots o_n \equiv_{\mathrm{Cr}\,\mathbf{A}} (\overline{\lambda xy.c})[(\overline{\lambda y.d})\, ji_1 \cdots i_r]\, zo_1 \cdots o_n \tag{*}$$

We start the proof of this equivalence. Assume for a moment that, for every $c, d \in A$, we have

$$\lambda^* zw[(\overline{\lambda yx^{\mathbf{A}}.c})\, zw] \equiv_{\mathrm{Cr}\,\mathbf{A}} \lambda^* zw[(\overline{\lambda xy^{\mathbf{A}}.c})\, wz] \tag{π_1}$$

$$\lambda^* z[(\overline{\lambda yy^{\mathbf{A}}.d}) z] \equiv_{\mathrm{Cr}\,\mathbf{A}} \lambda^* z(\overline{\lambda y^{\mathbf{A}}.d}) \tag{π_2}$$

Hence by Lemma 25(ii) we get

$$(\overline{\lambda yx^{\mathbf{A}}.c})\, zw \equiv_{\mathrm{Cr}\,\mathbf{A}} (\overline{\lambda xy^{\mathbf{A}}.c})\, wz. \tag{i}$$

and

$$(\overline{\lambda yy^{\mathbf{A}}.d}) z \equiv_{\mathrm{Cr}\,\mathbf{A}} (\overline{\lambda y^{\mathbf{A}}.d}). \tag{ii}$$

By a repeated application of the **CL** congruence rule [3, Definition 7.1.2]

$$\vdash P = Q \quad \Rightarrow \quad \vdash Pj = Qj$$

from (ii) it follows that

$$(\overline{\lambda yy^{\mathbf{A}}.d})\, zji_1 \cdots i_r \equiv_{\mathrm{Cr}\,\mathbf{A}} (\overline{\lambda y^{\mathbf{A}}.d})\, ji_1 \cdots i_r. \tag{iii}$$

But **CL** has the substitution rule [3, Proposition7.1.3]

$$\vdash P(w) = P'(w), \vdash R = R' \quad \Rightarrow \quad \vdash P(R) = P'(R')$$

so applying this rule, substituting (iii) for w in (i), yields

$$(\overline{\lambda yx^{\mathbf{A}}.c})\, z[(\overline{\lambda yy^{\mathbf{A}}.d})\, zji_1 \cdots i_r] \equiv_{\mathrm{Cr}\,\mathbf{A}} (\overline{\lambda xy^{\mathbf{A}}.c})[(\overline{\lambda y^{\mathbf{A}}.d})\, ji_1 \cdots i_r]\, z.$$

Then further use of the congruence rule gives

$$(\overline{\lambda yx^{\mathbf{A}}.c})\, z[(\overline{\lambda yy^{\mathbf{A}}.d})\, zji_1 \cdots i_r]\, o_1 \cdots o_n \equiv_{\mathrm{Cr}\,\mathbf{A}} (\overline{\lambda xy^{\mathbf{A}}.c})[(\overline{\lambda y^{\mathbf{A}}.d})\, ji_1 \cdots i_r]\, zo_1 \cdots o_n.$$

But this is just $(*)$ above. We conclude the proof by showing that Cr **A** satisfies the identities π_1 and π_2. To prove π_2, we apply the following chain of equalities

$$\begin{aligned}
[\lambda^* z[(\overline{\lambda yy^{\mathbf{A}}.d})z]]^{\mathrm{Cr}\,\mathbf{A}} &= [\mathbf{s}(\mathbf{k}(\overline{\lambda yy^{\mathbf{A}}.d}))\mathbf{i}]^{\mathrm{Cr}\,\mathbf{A}} && [\text{def } \lambda^*]\\
&= \mathbf{s}^{\mathbf{A}}(\mathbf{k}^{\mathbf{A}}(\lambda yy^{\mathbf{A}}.d))\, \mathbf{i}^{\mathbf{A}}\\
&= \lambda yy^{\mathbf{A}}.d && [\text{Corollary 42}]\\
&= (\lambda xy^{\mathbf{A}}.x)(\lambda y^{\mathbf{A}}.d) && [(\beta_6)]\\
&= \mathbf{k}^{\mathbf{A}}(\lambda y^{\mathbf{A}}.d) && [\text{def } \mathbf{k}^{\mathbf{A}}]\\
&= [\lambda^* z(\overline{\lambda y^{\mathbf{A}}.d})]^{\mathrm{Cr}\,\mathbf{A}} && [\text{def } \lambda^*]
\end{aligned}$$

while π_1 follows by reducing the right-side and the left-side of the identity π_1 to $\lambda yx^{\mathbf{A}}.c$.

$$\begin{aligned}
&[\lambda^* zw[(\overline{\lambda xy^{\mathbf{A}}.c})\, wz]]^{\mathrm{Cr}\,\mathbf{A}}\\
&= [\lambda^* z[\mathbf{s}(\mathbf{s}[\mathbf{k}(\overline{\lambda xy^{\mathbf{A}}.c})]\mathbf{i})(\mathbf{k}z)]]^{\mathrm{Cr}\,\mathbf{A}} && [\text{def } \lambda^*]\\
&= [\mathbf{s}[\mathbf{k}[\mathbf{s}(\mathbf{s}[\mathbf{k}(\overline{\lambda xy^{\mathbf{A}}.c})]\mathbf{i})]](\mathbf{s}(\mathbf{kk})\mathbf{i})]^{\mathrm{Cr}\,\mathbf{A}} && [\text{def } \lambda^*]\\
&= [\mathbf{s}[\mathbf{k}[\mathbf{s}(\overline{\lambda xy^{\mathbf{A}}.c})]](\mathbf{s}(\mathbf{kk})\mathbf{i})]^{\mathrm{Cr}\,\mathbf{A}} && [\text{Corollary 42}]\\
&= [\mathbf{s}[\mathbf{k}[\mathbf{s}(\overline{\lambda xy^{\mathbf{A}}.c})]]\,\mathbf{k}]^{\mathrm{Cr}\,\mathbf{A}} && [\text{Corollary 42, } \mathbf{k} \equiv_{\mathrm{Cr}\,\mathbf{A}} \overline{\lambda xy.x}]\\
&= \mathbf{s}^{\mathbf{A}}[\mathbf{k}^{\mathbf{A}}[\mathbf{s}^{\mathbf{A}}(\lambda xy^{\mathbf{A}}.c)]]\, \mathbf{k}^{\mathbf{A}}\\
&= (\lambda zxy^{\mathbf{A}}.zy(xy))[\mathbf{k}^{\mathbf{A}}[\mathbf{s}^{\mathbf{A}}(\lambda xy^{\mathbf{A}}.c)]]\, \mathbf{k}^{\mathbf{A}}\\
&= \lambda y^{\mathbf{A}}.\mathbf{k}^{\mathbf{A}}[\mathbf{s}^{\mathbf{A}}(\lambda xy^{\mathbf{A}}.c)]\; y(\mathbf{k}^{\mathbf{A}}y) && [(\beta_6)]\\
&= \lambda y^{\mathbf{A}}.\mathbf{s}^{\mathbf{A}}(\lambda xy^{\mathbf{A}}.c)(\lambda x^{\mathbf{A}}.y) && [\mathbf{k}^{\mathbf{A}}ab = a,\, (\beta_3)]\\
&= \lambda y^{\mathbf{A}}.\lambda x^{\mathbf{A}}.(\lambda xy^{\mathbf{A}}.c)\, x((\lambda x^{\mathbf{A}}.y)x) && [\text{as above}]\\
&= \lambda y^{\mathbf{A}}.\lambda x^{\mathbf{A}}.(\lambda y^{\mathbf{A}}.c)\, y && [(\beta_3)]\\
&= \lambda yx^{\mathbf{A}}.c. && [(\beta_3)]
\end{aligned}$$

We omit the superscript $^{\mathbf{A}}$ in the following chain of equalities.

$$
\begin{aligned}
&[\lambda^* zw[(\overline{\lambda yx.c})\,zw]]^{\mathrm{Cr}\,\mathbf{A}} \\
&= [\lambda^* z[\mathbf{s}[\mathbf{k}((\overline{\lambda yx.c})z)]\mathbf{i}]]^{\mathrm{Cr}\,\mathbf{A}} && [\text{def } \lambda^*] \\
&= [\mathbf{s}\{\mathbf{s}(\mathbf{ks})[\mathbf{s}(\mathbf{kk})(\mathbf{s}(\mathbf{k}(\overline{\lambda yx.c}))\mathbf{i})]\}(\mathbf{ki})]^{\mathrm{Cr}\,\mathbf{A}} && [\text{def } \lambda^*] \\
&= [\mathbf{s}\{\mathbf{s}(\mathbf{ks})[\mathbf{s}(\mathbf{kk})(\overline{\lambda yx.c})]\}(\mathbf{ki})]^{\mathrm{Cr}\,\mathbf{A}} && [\text{Corollary 42}] \\
&= [\mathbf{s}\{\mathbf{s}(\mathbf{ks})[\mathbf{s}(\mathbf{kk})(\lambda yx.c)]\}(\mathbf{ki})]^{\mathbf{A}} \\
&= \mathbf{s}\{\mathbf{s}(\mathbf{ks})[(\lambda zxy.zy(xy))(\mathbf{kk})(\lambda yx.c)]\}(\mathbf{ki}) \\
&= \mathbf{s}\{\mathbf{s}(\mathbf{ks})[(\lambda y.\mathbf{kk}y((\lambda yx.c)\,y))]\}(\mathbf{ki}) && [(\beta_6)] \\
&= \mathbf{s}\{\mathbf{s}(\mathbf{ks})[(\lambda y.(\lambda yx.\,y)(\lambda x.c)]\}(\mathbf{ki}) && [\text{def } \mathbf{k}, (\beta_3)] \\
&= \mathbf{s}\{\mathbf{s}(\mathbf{ks})(\lambda yxx.c)\}(\mathbf{ki}) && [(\beta_6)] \\
&= (\lambda zxy.zy(xy))\{\mathbf{s}(\mathbf{ks})(\lambda yxx.c)\}(\mathbf{ki}) \\
&= \lambda y.\mathbf{s}(\mathbf{ks})(\lambda yxx.c)\;y[(\mathbf{ki})\;y] && [(\beta_6)] \\
&= \lambda y.\mathbf{s}(\mathbf{ks})(\lambda yxx.c)\;y\mathbf{i} \\
&= \lambda y.\mathbf{ks}y[(\lambda yxx.c)\;y]\mathbf{i} \\
&= \lambda y.\mathbf{s}(\lambda xx.c)\mathbf{i}
\end{aligned}
$$

We cannot directly apply axiom (β_6) to $\mathbf{s}(\lambda xx.c)\mathbf{i}$ to obtain $\lambda x.c$ because $\lambda xx.c$ may be algebraically dependent on $I\backslash\{x\}$. So, we have the conclusion in the following way

$$
\begin{aligned}
\mathbf{s}(\lambda xx.c)\mathbf{i} &= \mathbf{s}((\lambda zyxx.c)\;zy)\mathbf{i} \\
&= \mathbf{s}((\lambda u.uzy)(\lambda zyxx.c))\mathbf{i} \\
&= [\lambda z.\mathbf{s}(z(\lambda zyxx.c))\mathbf{i}](\lambda u.uzy) \\
&= [\lambda z.(\lambda zyx.zx(yx))(z(\lambda zyxx.c))\mathbf{i}](\lambda u.uzy) \\
&= [\lambda zx.(z(\lambda zyxx.c))\;x(\mathbf{i}x)](\lambda u.uzy) \\
&= \lambda x.((\lambda u.uzy)(\lambda zyxx.c))\;x(\mathbf{i}x) \\
&= \lambda x.(\lambda zyxx.c)\;zyx(\mathbf{i}x) \\
&= \lambda x.(\lambda xx.c)\;xx \\
&= \lambda x.c. \quad \blacksquare
\end{aligned}
$$

THEOREM 48. *Let J be an infinite set disjoint from I. Then* $\mathbf{A}$ *is an* LAA_I *if and only if it is the I-neat reduct of an* $\mathsf{LAA}_{I\cup J}$.

Proof. The I-neat reduct of an $\mathsf{LAA}_{I\cup J}$ is always an LAA_I. In the opposite direction, let $\mathbf{B}$ be the J-expansion of $\mathbf{A}$. We recall that $\mathbf{Nr}_I\mathbf{B}$ denotes the LAA_I that is the I-neat reduct of $\mathbf{B}$, and that $Nr_I\mathbf{B}$ is the universe of this algebra. Recall also that $(\mathrm{Cr}\,\mathbf{A})[J]$ is the J-reduct of $\mathbf{B}$.

Let $a \in A$. Since a is a zero-dimensional element of $(\mathrm{Cr}\,\mathbf{A})[J]$, then $\triangle^{\mathbf{B}}(a) \subseteq I$, that is, $a \in Nr_I\mathbf{B}$. If $a \in B \backslash A$, then a depends on some λ-variable $z \in J$. Then $Nr_I\mathbf{B} = A$. Moreover, the operation $\cdot^{\mathbf{B}}$ restricted to A is the operation $\cdot^{\mathbf{A}}$ because the combinatory reduct of $\mathbf{A}$ is the zero-dimensional reduct of $(\mathrm{Cr}\,\mathbf{A})[J]$. By definition $x^{\mathbf{B}} = x^{\mathbf{A}}$ for all $x \in I$, and finally, $\lambda x^{\mathbf{B}}(x \in I)$ restricted to A is equal to $\lambda x^{\mathbf{A}}$ by Proposition 45. ▮

5. RFA'S CAN BE EMBEDDED INTO ULTRAPOWERS OF FLA'S

Recall from Lemma 35 that, for every $\mathsf{RFA}_I\,\mathbf{A}$ with value domain $\mathbf{V}$ and thread r and every $j = \{x_1, ..., x_k\} \subseteq_\omega I$, the map $f_{r,j}$, defined by

$$f_{r,j}(a)(q) := a(r\{q_{x_1}/x_1, ..., q_{x_k}/x_k\}), \qquad \text{for all} \quad a \in A \quad \text{and} \quad q \in V^I$$

is a homomorphism from $Rd_j(\mathbf{A})$ into $Rd_j(\mathbf{V}_I^{\top})$.

Let

$$K := \{j \subseteq I: j \text{ is finite}\}$$

and F is a nonprincipal ultrafilter on K that contains the set

$$K_x = \{j: x \in j\}, \qquad \text{for each} \quad x \in I.$$

Hence F contains

$$K_e = \{j: e \subseteq j\} \qquad \text{for each} \quad e \subseteq_\omega I.$$

THEOREM 49. *Every* RFA_I *is isomorphic to a subalgebra of an ultrapower of an* FLA_I. *More precisely, if* $\mathbf{A}$ *is an* RFA_I *with value domain* $\mathbf{V}$ *and thread* r, *then the map* f, *defined by*

$$f(a) := \langle f_{r,j}(a): j \subseteq_\omega I \rangle / F, \qquad \textit{for all} \quad a \in A,$$

is a monomorphism from the $\mathsf{RFA}_I\,\mathbf{A}$ *into the ultrapower* $(\mathbf{V}_I^{\top})^K/F$ *of the full* FLA_I *with value domain* $\mathbf{V}$.

Proof. (f *is injective*): If $f(a) = f(b)$ then $\{j \subseteq_\omega I: f_{r,j}(a) = f_{r,j}(b)\} \in F$. This implies that for every finite subset e of I we have that

$$K_e \cap \{j \subseteq_\omega I: f_{r,j}(a) = f_{r,j}(b)\} \neq \emptyset.$$

Let $e=\{x_1, ..., x_n\}$ and let $j=\{x_1, ..., x_n, y_1, ..., y_k\}$ be an element in the previous intersection set. Let $v_1, ..., v_n \in V$ and let $q \in V^I$ be an environment such that

$$q_y := \begin{cases} v_i, & \text{if} \quad y = x_i \quad \text{for } i=1, ..., n \\ r_y, & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} a(r\{v_1/x_1, ..., v_n/x_n\}) &= a(r\{v_1/x_1, ..., v_n/x_n, r_{y_1}/y_1, ..., r_{y_k}/y_k\}) \\ &= f_{r,j}(a)(q) \\ &= f_{r,j}(b)(q) \\ &= b(r\{v_1/x_1, ..., v_n/x_n, r_{y_1}/y_1, ..., r_{y_k}/y_k\}) \\ &= b(r\{v_1/x_1, ..., v_n/x_n\}). \end{aligned}$$

Since the above equality holds for every $x_1 \cdots x_n \in I^\star$ and every $v_1 \cdots v_n \in V^*$, then $a=b$.

(*f is a homomorphism*): We know from Lemma 35 that

(i) $f_{r,j}(x^{\mathbf{V}_{I,r}}) = x^{\mathbf{V}_I}$, for every j such that $x \in j$;

(ii) $f_{r,j}(a \cdot^{\mathbf{V}_{I,r}} b) = f_{r,j}(a) \cdot^{\mathbf{V}_I} f_{r,j}(b)$, for all $a, b \in A$;

(iii) $f_{r,j}(\lambda x^{\mathbf{V}_{I,r}}.a) = \lambda x^{\mathbf{V}_I}.f_{r,j}(a)$, for all $a \in A$ and all j such that $x \in j$.

Then we have

$$\begin{aligned} f(x^{\mathbf{V}_{I,r}}) &= \langle f_{r,j}(x^{\mathbf{V}_{I,r}}) \colon j \subseteq_\omega I \rangle / F \\ &= \langle x^{\mathbf{V}_I} \colon j \subseteq_\omega I \rangle / F \qquad [K_x \subseteq \{ j \colon f_{r,j}(x^{\mathbf{V}_{I,r}}) = x^{\mathbf{V}_I}\} \in F] \\ &= (x^{\mathbf{V}_I})^K / F \\ f(a \cdot^{\mathbf{V}_{I,r}} b) &= \langle f_{r,j}(a \cdot^{\mathbf{V}_{I,r}} b) \colon j \subseteq_\omega I \rangle / F \\ &= \langle f_{r,j}(a) \cdot^{\mathbf{V}_I} f_{r,j}(b) \colon j \subseteq_\omega I \rangle / F \\ &= (\langle f_{r,j}(a) \colon j \subseteq_\omega I \rangle / F) \cdot^{(\mathbf{V}_I)^{K/F}} (\langle f_{r,j}(b) \colon j \subseteq_\omega I \rangle / F) \\ f(\lambda x^{\mathbf{V}_{I,r}}.a) &= \langle f_{r,j}(\lambda x^{\mathbf{V}_{I,r}}.a) \colon j \subseteq_\omega I \rangle / F \\ &= \langle \lambda x^{\mathbf{V}_I}.f_{r,j}(a) \colon j \subseteq_\omega I \rangle / F \\ &= \lambda x^{(\mathbf{V}_I)^{K/F}}.f(a). \quad \blacksquare \end{aligned}$$

6. ULTRAPOWERS OF FLA'S ARE ISOMORPHIC TO FLA'S

Let $\mathbf{A}$ be an FLA_I with value domain $\mathbf{V} = (V, \cdot^{\mathbf{V}}, \lambda^{\mathbf{V}})$. Thus $\lambda^{\mathbf{V}} \colon V^V \xrightarrow{\text{p}} V$ is a partial function such that, for each f in the domain of $\lambda^{\mathbf{V}}$,

$$f(v) = (\lambda^{\mathbf{V}}(f)) \cdot^{\mathbf{V}} v, \qquad \text{for all} \quad v \in V.$$

Let K be any set and F be any ultrafilter on K. Write V^* for the ultrapower V^K/F. Make this into a functional domain

$$\mathbf{V}^* = (V^*, \cdot^*, \lambda^*).$$

Put

$$(a/F) \cdot^* (b/F) = \langle a(j) \cdot^{\mathbf{V}} b(j) \colon j \in K \rangle / F, \qquad \text{for} \quad a, b \in V^K.$$

Use $\lambda^{\mathbf{V}}$ to define λ^* as follows: $f\colon V^* \to V^* \in dom\, \lambda^*$ if there exists a K-indexed family of functions $f_j\colon V \to V \in dom\, \lambda^{\mathbf{V}}$ such that

$$f(u/F) = \langle f_j(u_j) \colon j \in K \rangle / F, \qquad \text{for all} \quad u \in V^K.$$

It is an easy matter to check that the definition of $f(u/F)$ is independent of the choice of the representative of the equivalence class u/F as, for all $v \in u/F$,

$$\{ j \colon u_j = v_j \} \subseteq \{ j \colon f_j(u_j) = f_j(v_j) \}.$$

Put

$$\lambda^*(f) = \langle \lambda^{\mathbf{V}}(f_j) \colon j \in K \rangle / F.$$

Finally, the equalities

$$\begin{aligned} f(u/F) &= \langle f_j(u_j) \colon j \in K \rangle / F && [\text{def f}] \\ &= \langle \lambda^{\mathbf{V}}(f_j) \cdot^{\mathbf{V}} u_j \colon j \in K \rangle / F && [\text{def } \mathbf{V}] \\ &= \langle \lambda^{\mathbf{V}}(f_j) \colon j \in K \rangle / F \cdot^* u/F && [\text{def } \cdot^*] \\ &= \lambda^*(f) \cdot^* u/F && [\text{def } \lambda^*] \end{aligned}$$

ensure that we do indeed have a functional domain.

We can therefore consider total subalgebras of $\mathbf{V}_I^*$ relative to this domain.

Let F be an ultrafilter on K. Any function of the form

$$ch\colon V^K/F \to V^K$$

assigns to each F-equivalence class $A \in V^K/F$ a K-function. ch is called a *choice function* if $ch(A) \in A$.

Note that in the theory of cylindric algebras (see [21, 3.1.89]) a choice function has the form $I \times (V^K/F) \to V^K$. The choice function used here is a "special case" of the more general one used in cylindric algebras, with the value of the function being independent of the first argument from I.

Observe that

$$ch(A)/F = A$$

since $ch(A) \in A$. Any choice function ch induces a function

$$ch^+ : (V^K/F)^I \to (V^I)^K$$

where, if $p = \langle p_x : x \in I \rangle \in (V^K/F)^I$, $ch^+(p)$ is the K-function given by

$$ch^+(p)(j) = \langle ch(p_x)(j) : x \in I \rangle \in V^I.$$

As an explanation, since $p_x \in V^K/F$, $ch(p_x) \in V^K$, so we are defining $ch^+(p)(j)_x = ch(p_x)(j) \in V$.

Consider again the $\mathsf{FLA}_I\,\mathbf{A}$ with value domain $\mathbf{V}$. Let ch: $V^K/F \to V^K$ be a choice function. Define a map ϑ_{ch}: $A^K/F \to V_I^*$ as follows. For all $a \in A^K$ and all $q \in (V^K/F)^I$:

$$\vartheta_{ch}(a/F)(q) = \langle a_j(\langle ch(q_x)(j) : x \in I \rangle) : j \in K \rangle/F = \langle a_j(ch^+(q)(j)) : j \in K \rangle/F.$$

Theorem 50. *Let* $\mathbf{A}$ *be an* FLA_I *with value domain* $\mathbf{V}$ *and* ch: $V^K/F \to V^K$ *be a choice function. Then the map* ϑ_{ch}: $A^K/F \to V_I^*$ *is a homomorphism from the ultrapower* $\mathbf{A}^K/F$ *into a total subalgebra of* $\mathbf{V}_I^*$. *Hence a homomorphic image of the ultrapower* $\mathbf{A}^K/F$ *is isomorphic to the* $\mathsf{FLA}_I\,\vartheta_{ch}(\mathbf{A}^K/F)$.

Proof. Let $\mathbf{B} := \mathbf{A}^K/F$ and $\vartheta := \vartheta_{ch}$ in this proof.

ϑ preserves the λ-variables x:

$$\begin{aligned}
\vartheta(x^{\mathbf{B}})(q) &= \vartheta(\langle x^{\mathbf{A}} : j \in K \rangle/F)(q) && [\text{def } x^{\mathbf{B}}]\\
&= \langle x^{\mathbf{A}}(\langle ch(q_y)(j) : y \in I \rangle) : j \in K \rangle/F && [\text{def } \vartheta]\\
&= \langle ch(q_x)(j) : j \in K \rangle/F && [\text{def } x^{\mathbf{A}}]\\
&= ch(q_x)/F\\
&= q_x && [\text{ch}(q_x) \in q_x]\\
&= x^{\mathbf{V}_I^*}(q). && [\text{def } x^{\mathbf{V}_I^*}]
\end{aligned}$$

Hence $\vartheta(x^{\mathbf{B}}) = x^{\mathbf{V}_I^*}$.

ϑ preserves the application operator "$\cdot$": Let $a, b \in \mathbf{A}^K$.

$$\begin{aligned}
\vartheta(a/F \cdot^{\mathbf{B}} b/F)(q) &= \langle (a_j \cdot^{\mathbf{A}} b_j)(ch^+(q)(j)) : j \in K \rangle/F && [\text{def } \vartheta]\\
&= \langle a_j(ch^+(q)(j)) \cdot^{\mathbf{V}} b_j(ch^+(q)(j)) : j \in K \rangle/F\\
&= \langle a_j(ch^+(q)(j)) : j \in K \rangle/F \cdot^* \langle b_j(ch^+(q)(j)) : j \in K \rangle/F\\
&= \vartheta(a/F)(q) \cdot^* \vartheta(b/F)(q)\\
&= [\vartheta(a/F) \cdot^{\mathbf{V}_I^*} \vartheta(b/F)](q)
\end{aligned}$$

Hence $\vartheta(a/F \cdot^{\mathbf{B}} b/F) = \vartheta(a/F) \cdot^{\mathbf{V}_I^*} \vartheta(b/F)$.

It remains to show ϑ preserves λx. This requires some preliminary analysis.

LEMMA 51. *Let $u \in V^K$ and a: $V^I \to V$ be a map.*

(i) $ch^+(q\{(u/F)/x\})(j) = (ch^+(q)(j))\{u_j/x\}$ *for F-almost all* $j \in K$.

(ii) $\langle a(ch^+(q)(j)\{u_j/x\})\colon j \in K\rangle/F = \langle a(ch^+(q\{(u/F)/x\})_j)\colon j \in K\rangle/F$.

Proof of the Lemma. Let $s(j) = ch^+(q\{(u/F)/x\})(j)$ and $t(j) = (ch^+(q)(j))\{u_j/x\}$ *for all* $j \in K$. *If* $y \neq x$, *then*

$$\begin{aligned} s(j)_y &= ch^+(q\{(u/F)/x\})(j)_y && [\text{def } s(j)] \\ &= ch(q\{(u/F)/x\}_y)(j) && [\text{def } ch^+] \\ &= ch(q_y)(j) && [y \neq x] \\ &= ch^+(q)(j)_y && [\text{def } ch^+] \\ &= [ch^+(q)(j)\{u_j/x\}]_y && [y \neq x] \\ &= t(j)_y. \end{aligned}$$

Hence for any j, the sequences $s(j)$ and $t(j)$ differ at most at x, so $s(j) = t(j)$ iff $s(j)_x = t(j)_x$. But

$$\begin{aligned} s(j)_x &= ch^+(q\{(u/F)/x\})(j)_x \\ &= ch(q\{(u/F)/x\}_x)(j) && [\text{def } ch^+] \\ &= ch(u/F)(j) \end{aligned}$$

and

$$\begin{aligned} t(j)_x &= (ch^+(q)(j))\{u_j/x\}_x \\ &= u_j, \end{aligned}$$

so

$$s(j) = t(j) \qquad \text{iff} \quad ch(u/F)(j) = u_j.$$

Now $ch(u/F)$ and u both belong to u/F, so $M = \{j\colon ch(u/F)(j) = u_j\} \in F$. From above, $M = \{j\colon s(j) = t(j)\} \in F$, so $M \in F$ proves (i). Next, $M \subseteq \{j\colon a(s(j)) = a(t(j))\}$, which establish (ii). ▌

We now are ready to show that ϑ preserves the lambda abstractions. Let $q \in (V^K/F)^I$ and $a \in V^K$. Then for $x \in I$, the environment $q\{(a/F)/x\}$ also belongs to $(V^K/F)^I$. This differs from q only in assigning (a/F) to x. Let $f_j = \langle a_j(ch^+(q)(j)\{v/x\})\colon v \in V\rangle$ for every $j \in K$.

$$\begin{aligned} \vartheta(\lambda x^{\mathbf{B}}.a/F)(q) &= \langle(\lambda x^{\mathbf{A}}.a_j)(ch^+(q)(j))\colon j \in K\rangle/F && [\text{def } \vartheta] \\ &= \langle \lambda^{\mathbf{V}} \langle a_j(ch^+(q)(j)\{v_j/x\})\colon v_j \in V\rangle\colon j \in K\rangle/F && [\text{def } \lambda x^{\mathbf{A}}] \\ &= \langle \lambda^{\mathbf{V}}(f_j)\colon j \in K\rangle/F && [\text{def } f_j] \\ &= \lambda^*(f) && [\text{def } \lambda^*] \end{aligned}$$

where $f\colon V^* \to V^* \in dom\ \lambda^*$ is the map defined by

$$f(u/F) = \langle f_j(u_j)\colon j \in K\rangle/F, \qquad \text{for all} \quad u \in V^K.$$

Then we have

$$\begin{aligned} f(u/F) &= \langle f_j(u_j)\colon j \in K\rangle/F \\ &= \langle a_j(ch^+(q)(j)\{u_j/x\})\colon j \in K\rangle/F \\ &= \langle a_j(ch^+(q\{(u/F)/x\})_j)\colon j \in K\rangle/F \qquad \text{by Lemma 51(i)} \\ &= \vartheta(a/F)(q\{(u/F)/x\}) \end{aligned}$$

So,

$$\begin{aligned} \vartheta(\lambda x^{\mathbf{B}}.a/F)(q) &= \lambda^*(f) \\ &= \lambda^*(\langle \vartheta(a/F)(q\{(u/F)/x\})\colon u/F \in V^K/F\rangle) \\ &= [\lambda x^{\mathbf{B}}.\vartheta(a/F)](q). \quad \blacksquare \end{aligned}$$

THEOREM 52. *Every ultrapower of an* FLA_I *is isomorphic to a subdirect product of* FLA_I*'s.*

Proof. Let $\mathbf{A}$ be an FLA_I with value domain $\mathbf{V}$, K be a set, and F be an ultrafilter on K. We know from Theorem 50 that for every choice function $ch\colon V^K/F \to V^K$, the map $\vartheta_{ch}\colon A^K/F \to V_I^*$ is a homomorphism from the ultrapower $\mathbf{A}^K/F$ into a total subalgebra of $\mathbf{V}_I^*$. By Lemma 8.2 in [7] we have the conclusion of the theorem if the family of maps ϑ_{ch} (indexed by choice functions) satisfies the following property: for all distinct a/F, $b/F \in A^K/F$ there exists a choice function ch for which $\vartheta_{ch}(a/F) \neq \vartheta_{ch}(b/F)$.

For every $j \in K$, let $p_j \in V^I$ such that $a_j(p_j) \neq b_j(p_j)$ whenever $a_j \neq b_j$. For every $x \in I$, let $r_x \in V^K$ such that $r_x(j) = p_j(x)$ for all $j \in K$. Define $q \in (V^K/F)^I$ as $q_x = r_x/F$ and consider any choice function ch such that $ch(q_x) = r_x$. Then we have

$$\begin{aligned} \vartheta_{ch}(a/F)(q) &= \langle a_j(\langle ch(q_x)(j)\colon x \in I\rangle)\colon j \in K\rangle/F \\ &= \langle a_j(\langle r_x(j)\colon x \in I\rangle)\colon j \in K\rangle/F \\ &= \langle a_j(\langle p_j(x)\colon x \in I\rangle)\colon j \in K\rangle/F \\ &= \langle a_j(p_j)\colon j \in K\rangle/F \end{aligned}$$

and similarly for b we have

$$\vartheta_{ch}(b/F)(q) = \langle b_j(p_j)\colon j \in K\rangle/F.$$

But

$$\{j\colon a_j(p_j) \neq b_j(p_j)\} = \{j\colon a_j \neq b_j\} \in F$$

since $a/F \neq b/F$. It follows that $\vartheta_{ch}(a/F)(q) \neq \vartheta_{ch}(b/F)(q)$ and then $\vartheta_{ch}(a/F) \neq \vartheta_{ch}(a/F)$. ∎

THEOREM 53. *The class of* FLA_I*'s is closed under products and subalgebras.*

Proof. For subalgebras the result holds directly from the definition of FLA. We prove the closure under products.

Let $\langle \mathbf{A}_j\colon j \in J\rangle$ be a family of FLA_I's with each $\mathbf{A}_j$ based on a functional domain $\mathbf{V}_j$. Define a new functional domain $\mathbf{W} = \langle W, \cdot^{\mathbf{W}}, \lambda^{\mathbf{W}}\rangle$ as follows: The carrier set W is the cartesian product of the carrier sets V_j. The operation $\cdot^{\mathbf{W}}$ is the usual product of the operations $\cdot^{\mathbf{V}_j}$. Moreover, a function $f\colon W \to W$ is in the domain of $\lambda^{\mathbf{W}}$ if and only if there exists a J-indexed family of functions $f_j\colon V_j \to V_j \in dom(\lambda^{\mathbf{V}_j})$ such that

$$f(\langle v_j \in V_j\colon j \in J\rangle) = \langle f_j(v_j)\colon j \in J\rangle.$$

Moreover, for each such a function $f \in dom(\lambda^{\mathbf{W}})$ we define

$$\lambda^{\mathbf{W}}(f) = \langle \lambda^{\mathbf{V}_j}(f_j)\colon j \in J\rangle.$$

$\mathbf{W}$ is a functional domain since

$$\begin{aligned}\lambda^{\mathbf{W}}(f) \cdot^{\mathbf{W}} \langle v_j\colon j \in J\rangle &= \langle \lambda^{\mathbf{V}_j}(f_j)\colon j \in J\rangle \cdot^{\mathbf{W}} \langle v_j\colon j \in J\rangle \\ &= \langle \lambda^{V_j}(f_j) \cdot^{V_j} v_j\colon j \in J\rangle \\ &= \langle f_j(v_j)\colon j \in J\rangle \\ &= f(\langle v_j\colon j \in J\rangle).\end{aligned}$$

Denote the product of the family $\langle \mathbf{A}_j\colon j \in J\rangle$ by $\mathbf{B}$. Then we define a map $h\colon B \to W_I$ as

$$h(\langle a_j\colon j \in J\rangle)(p) = \langle a_j(p_j)\colon j \in J\rangle, \qquad \text{for all} \quad p \in W^I$$

where $p_j \in (V_j)^I$ is defined as $(p_j)_x = (p_x)_j$ for all $x \in I$. h is a monomorphism. The only nontrivial case is the homomorphism property for the lambda abstraction operators.

$$\begin{aligned}h(\lambda x^B.\langle a_j\colon j \in J\rangle)(p) &= h(\langle \lambda x^{\mathbf{A}_j}.a_j\colon j \in J\rangle)(p) \\ &= \langle [\lambda x^{\mathbf{A}_j}.a_j](p_j)\colon j \in J\rangle \qquad [\text{def } h] \\ &= \langle \lambda^{\mathbf{V}_j}\langle a_j(p_j\{v_j/x\})\colon v_j \in V_j\rangle\colon j \in J\rangle.\end{aligned}$$

Let $f: W \to W$ be the function defined by

$$f(\langle v_j: j \in J \rangle) = \langle a_j(p_j\{v_j/x\}): j \in J \rangle.$$

Then f is in the domain of λ^W because $f_j = \langle a_j(p_j\{v_j/x\}): v_j \in V_j \rangle$ is in the domain of $\lambda^{\mathbf{V}_j}$ for every $j \in J$. Moreover, let $a: W^I \to W$ be the map defined by

$$a(p) = \langle a_j(p_j): j \in J \rangle.$$

Then, for every $v = \langle v_j: j \in J \rangle \in W$ (i.e., in the Cartesian product of V_j), we have

$$\begin{aligned} a(p\{v/x\}) &= \langle a_j(p\{v/x\}_j): j \in J \rangle \\ &= \langle a_j(p_j\{v_j/x\}): j \in J \rangle \end{aligned}$$

so that

$$\begin{aligned} h(\lambda x^B . \langle a_j: j \in J \rangle)(p) &= \langle \lambda^{\mathbf{V}_j} \langle a_j(p_j\{v_j/x\}): v_j \in V_j \rangle: j \in J \rangle \\ &= \lambda^{\mathbf{W}}(f) \end{aligned}$$

and

$$\begin{aligned} [\lambda x^{\mathbf{W}_I} . h(\langle a_j: j \in J \rangle)](p) &= \lambda^{\mathbf{W}} \langle h(\langle a_j: j \in J \rangle)(p\{v/x\}): v = \langle v_j: j \in J \rangle \in W \rangle \\ &= \lambda^{\mathbf{W}} \langle \langle a_j(p\{v/x\}_j): j \in J \rangle: v \in W \rangle \\ &= \lambda^{\mathbf{W}} \langle \langle a_j(p_j\{v_j/x\}): j \in J \rangle: v \in W \rangle \\ &= \lambda^{\mathbf{W}} \langle f(v): v \in W \rangle \\ &= \lambda^{\mathbf{W}}(f). \end{aligned}$$

So, we have the conclusion. ▮

7. A COMPLETENESS THEOREM FOR THE INFINITARY LAMBDA CALCULUS

Recent work has been done by many authors on infinitary versions of lambda calculus. Berarducci defines in [5] a new model of lambda calculus which is similar to the model of Böhm trees, but it does not identify all the unsolvable lambda terms. His method, which is based on an infinitary version of lambda calculus, is also used in [6] to obtain Church–Rosser extensions of the finitary lambda calculus. Another infinitary version of lambda calculus has been independently introduced by Kenneway *et al.* in [24]. The two approaches are different since Berarducci does not equate all the unsolvable closed terms and allows infinitary terms of the form $(((\ldots)t_2)t_1)t_0$ (infinitely many parenthesis).

In this section, as an application of the main results of the paper, we will provide

(i) A completeness theorem for the infinitary lambda calculus with a semantics given in terms of environment models;

(ii) A uniform family of models of $\lambda\beta$-calculus which includes the model of Böhm trees and the model introduced by Berarducci in [5].

We advise the reader that in this section we heavily use notation and definitions from Barendregt's book [3].

Let I be a set of λ-variables and $\perp$ a new symbol. An *infinitary λ-term over I* (see [5]) is defined as a finite or infinite rooted tree such that each leaf is either labeled by a λ-variable in I or by the constant $\perp$, and the inner nodes are either binary "application nodes," or unary "abstraction nodes," in which case they have a label of the form λx where $x \in I$ is a λ-variable. The set of infinitary λ-terms is denoted by Λ_I^i. The infinitary λ-terms include as special cases the finitary λ-terms. The notion of free and bound occurrences of a λ-variable is easily extended to infinitary λ-terms. We write infinitary λ-terms in their linear form. Unless otherwise stated λ-term means "finitary λ-term."

Infinitary λ-terms arise as "limits" of infinite sequences of β-reductions. For example, let $\omega_3 = \lambda x.xxx$ and $\Omega_3 = \omega_3\omega_3$. If we start with Ω_3 we can generate the infinite sequence of β-reductions

$$\Omega_3 \to \Omega_3\omega_3 \to (\Omega_3\omega_3)\,\omega_3 \to \cdots \to ((((\Omega_3\omega_3)\omega_3)\omega_3)\omega_3)\omega_3 \to \cdots.$$

Then it is natural to consider the infinitary λ-term

$$\Omega := ((((\ldots\omega_3)\omega_3)\omega_3)\omega_3)\omega_3 \qquad \text{(infinitely many } \omega_3\text{'s)}$$

as the limit of the above sequence of reductions. In [5] Berarducci defines a new model of the lambda calculus that identifies two λ-terms if they have the same "asympotic behavior," namely they approach the same limit by repeated β-reductions. Such an idea is already present in the notion of Böhm tree. However, Böhm trees give no information on the inner structure of the unsolvable λ-terms, i.e., the Böhm tree of an unsolvable λ-term is defined to be $\perp$. Berarducci applies the idea of infinite unfolding also to the unsolvable λ-terms. For example, the infinite unfolding of the unsolvable λ-term Ω_3 is just the infinitary λ-term Ω_3^∞.

We observe here that some operations on Böhm-like trees as introduced in [3] and on infinitary λ-terms as introduced in [5] make sense only with the usual assumption that with respect to every infinitary λ-term, infinitely many fresh λ-variables are available. Berarducci observes in [5] that a notion of β-reduction is defined for infinitary λ-terms in exactly the same way as for finite ones, namely $(\lambda x.A)B \to A[x := B]$ (with renaming of bound λ-variables to avoid conflicts). However, if we fix the set I of λ-variables, for example $I = \{x_0, x_1, \ldots, x_n, \ldots\}$ is a countably infinite set of λ-variables, it is not clear how the substitution $[x_1 := (x_0(x_1(x_2(\ldots))))]$ is applied to the λ-term $(\lambda x_0.x_1)$.

If A is an infinitary λ-term, $Var(A)$ will denote the set of λ-variables x occurring either free/bound or as "λx" in A. $Var(A_1, \ldots, A_n)$ will denote the set $Var(A_1) \cup \cdots \cup Var(A_n)$ for infinitary λ-terms $A_1, \ldots, A_n$. Note that $Var(A)$ may be all the dimension set I. For example, if $I = \{x_0, x_1, \ldots, x_n, \ldots\}$ and $A = (x_0(x_1(x_2(\ldots))))$, then $Var(A) = I$.

If A is an infinitary λ-term, then the λ-term A^n is obtained by cutting off the tree A at level n; in other word, $A^0 = \bot$; if $A = BC$ then $A^{n+1} = B^n C^n$; if $A = \lambda x.B$ then $A^{n+1} = \lambda x.B^n$.

For $A, B \in A^i_I$ we let

$$A \subseteq B \qquad \text{iff} \quad A \text{ results from } B \text{ by cutting off some subtrees.}$$

For every infinitary λ-term C and all variables x, z with $z \notin Var(C)$, denote by $C\{z/x\}$ the infinitary λ-term obtained as the result of the replacement of every free occurrence of x in C by z.

We provide a formal definition of the substitution operator for infinitary lambda terms. Assume I is a well-ordered infinite set of λ-variables. Let A, B be infinitary λ-terms over I and let t be a λ-term such that $t \subseteq A$. Define the substitution operator $t[x :=_A B]$ by induction over the complexity of the λ-term t as follows.

(i) $x[x :=_x B] = B$;

(ii) $y[x :=_y B] = y \ (y \neq x)$;

(iii) $\bot[x :=_A B] = \bot$;

(iv) $(t_1 t_2)[x :=_{A_1 A_2} B] = (t_1[x :=_{A_1} B])(t_2[x :=_{A_2} B])$, where $t_1 \subseteq A_1$ and $t_2 \subseteq A_2$;

(v) $(\lambda x.t)[x :=_{\lambda x.A} B] = \lambda x.t$, where $t \subseteq A$;

(vi) $(\lambda y.t)[x :=_{\lambda y.A} B] = \lambda y.t[x :=_A B]$ if $y \neq x$, y is not free in B and $t \subseteq A$;

(vii) Let y be free in B, $y \neq x$, $t \subseteq A$ and $I \setminus Var(A, B)$ be nonempty. Let z be the first variable in $I \setminus Var(A, B)$. Then we define

$$(\lambda y.t)[x :=_{\lambda y.A} B] = \lambda z.t\{z/y\}[x :=_{A\{z/y\}} B]$$

(viii) $(\lambda y.t)[x :=_{\lambda y.A} B] = (\lambda xy.t)B$ if y is free in B, $y \neq x$, $t \subseteq A$ and $I \setminus Var(A, B) = \emptyset$.

The above definition is well given because $t\{z/y\}$ in item (vii) has the same complexity of t and $t\{z/y\} \subseteq A\{z/y\}$ by the hypothesis $t \subseteq A$.

A variable z as in (vii) exists always in the following two cases: (i) I uncountable; (ii) I countably infinite, $I = \bigcup_{n \geqslant 0} I_n$ with $|I_{n+1} \setminus I_n| = \omega$ and every infinitary λ-term A is over I_n for some n. Assuming (i) and/or (ii), for every finite sequence $A_1, ..., A_n$ of infinitary λ-terms over I, there exists an infinite set of λ-variables not occurring (free and/or bound) in $A_1, ..., A_n$.

Extend the definition of substitution to infinitary λ-terms A, B as

$$A[x := B] = \bigcup_{n \geqslant 0} A^n[x :=_A B].$$

The above definition is well given as proven in the following lemma.

Lemma 54. *Let $t \subseteq u$ be λ-terms and let A be an infinitary λ-term such that $u \subseteq A$. Then*

$$t[x :=_A B] \subseteq u[x :=_A B].$$

Proof. The proof is by induction over the complexity of t. The only non trivial case is $t = \lambda y.t'$. Then $u = \lambda y.u'$, $A = \lambda y.D$, $t' \subseteq u'$ and $t', u' \subseteq D$. We have three subcases. If y is not free in B, then

$$\begin{aligned}(\lambda y.t')[x :=_A B] &= \lambda y.t'[x :=_D B] && [\text{(vi) above}]\\ &\subseteq \lambda y.u'[x :=_D B] && [\text{induction}]\\ &= (\lambda y.u')[x :=_A B] && [\text{(vi) above}]\end{aligned}$$

Let y be free in B. Assume that $I \backslash Var(A, B)$ is nonempty and z is the first variable in it. Then by the definition of the substitution operator we have

$$(\lambda y.t')[x :=_A B] = \lambda z.t'\{z/y\}[x :=_{D\{z/y\}} B]$$

and

$$(\lambda y.u')[x :=_A B] = \lambda z.u'\{z/y\}[x :=_{D\{z/y\}} B].$$

The λ-terms $t'\{z/y\}$ and $u'\{z/y\}$ have the same complexity of t' and u', respectively. Moreover, $t'\{z/y\} \subseteq u'\{z/y\}$ since $t' \subseteq u'$ by hypothesis. Then, another application of the induction hypothesis gives

$$t'\{z/y\}[x :=_{D\{z/y\}} B] \subseteq u'\{z/y\}[x :=_{D\{z/y\}} B].$$

So, we have

$$\begin{aligned}(\lambda y.t')[x :=_A B] &= \lambda z.t'\{z/y\}[x :=_{D\{z/y\}} B] && [\text{(vii) above}]\\ &\subseteq \lambda z.u'\{z/y\}[x :=_{D\{z/y\}} B]\\ &= (\lambda y.u')[x :=_A B]. && [\text{(vii) above}]\end{aligned}$$

Finally, assume that $I \backslash Var(A, B) = \emptyset$. Then,

$$(\lambda y.t')[x :=_A B] = (\lambda xy.t')B \subseteq (\lambda xy.u')B = (\lambda y.u')[x :=_A B]. \quad \blacksquare$$

The axioms of the infinitary lambda calculus are as follows: A and B are arbitrary infinitary λ-terms.

($\alpha\iota$) $\lambda x.A = \lambda y.A[x := y]$, for any variable y that does not occur free in A;

($\beta\iota$) $(\lambda x.A)B = A[x := B]$;

(1) $A = A$;

(2) $A = B$ implies $B = A$;

(3) $A = B$, $B = C$ imply $A = C$;

(4) $A = B$, $C = D$ imply $AC = BD$;

(5) $A = B$ implies $\lambda x.A = \lambda x.B$.

An *infinitary lambda theory* is any set of equations between infinitary λ-terms that is closed under ($\alpha\iota$)- and ($\beta\iota$)-conversion and the five equality rules. The minimal infinitary lambda theory is denoted by $\lambda\beta\iota$.

Let

$$\Lambda^{\mathbf{i}}_{\mathbf{I}} := \langle \Lambda^{i}_{I}, \cdot^{\Lambda^{\mathbf{i}}_{\mathbf{I}}}, \langle \lambda x^{\Lambda^{\mathbf{i}}_{\mathbf{I}}} : x \in I \rangle, \langle x^{\Lambda^{\mathbf{i}}_{\mathbf{I}}} : x \in I \rangle \rangle$$

be the absolutely free algebra of infinitary λ-terms.

THEOREM 55. *The minimal infinitary lambda theory $\lambda\beta\iota$ is a congruence over $\Lambda^{\mathbf{i}}_{\mathbf{I}}$ making $\Lambda^{\mathbf{i}}_{\mathbf{I}}/\lambda\beta\iota$ isomorphic to a functional* LAA.

Proof. By Theorem 32 it is sufficient to prove that $\Lambda^{\mathbf{i}}_{\mathbf{I}}/\lambda\beta\iota$ satisfies axioms (β_1)–(β_5), (β'_6), and (α'). These last two axioms are the equational versions of the quasi-identities (β_6) and (α) (see Section 1).

(β_1):

$$\begin{aligned}(\lambda x.x)B &= x[x := B] \qquad [(\beta\iota)]\\ &= \bigcup_{n \geqslant 0} x^n[x :=_x B]\\ &= \bigcup_{n > 0} x[x :=_x B]\\ &= \bigcup_{n > 0} B\\ &= B.\end{aligned}$$

(β_2):

$$\begin{aligned}(\lambda x.y)B &= y[x := B] \qquad [(\beta\iota)]\\ &= \bigcup_{n \geqslant 0} y^n[x :=_y B]\\ &= \bigcup_{n > 0} y[x :=_y B]\\ &= \bigcup_{n > 0} y\\ &= y.\end{aligned}$$

(β_3):

$$\begin{aligned}(\lambda x.A)x &= A[x:=x] \qquad [(\beta\iota)]\\ &= \bigcup_{n\geqslant 0} A^n[x:=_A x]\\ &= \bigcup_{n\geqslant 0} A^n\\ &= A\end{aligned}$$

since it is possible to prove by induction over the complexity of the λ-terms that $t[x:=_A x]=t$ for every t such that $t\subseteq A$.

(β_4):

$$\begin{aligned}(\lambda xx.A)B &= (\lambda x.A)[x:=B] && [(\beta\iota)]\\ &= \bigcup_{n\geqslant 0} (\lambda x.A)^n\,[x:=_{\lambda x.A} B] && [\mathrm{def}[x:=B]]\\ &= \bigcup_{n>0} (\lambda x.A^{n-1})[x:=_{\lambda x.A} B] && [\mathrm{def}(\lambda x.A)^n]\\ &= \bigcup_{n>0} \lambda x.A^{n-1} && [\mathrm{def}[x:=_{\lambda x.A} B]]\\ &= \lambda x.\bigcup_{n\geqslant 0} A^n\\ &= \lambda x.A.\end{aligned}$$

(β_5):

$$\begin{aligned}(\lambda x.AB)C &= (AB)[x:=C] && [(\beta\iota)]\\ &= \bigcup_{n\geqslant 0} (AB)^n\,[x:=_{AB} C] && [\mathrm{def}[x:=C]]\\ &= \bigcup_{n>0} (A^{n-1}B^{n-1})[x:=_{AB} C]\\ &= \bigcup_{n>0} (A^{n-1}[x:=_A C])(B^{n-1}[x:=_B C]) && [\mathrm{def}[x:=_{AB} C]]\\ &= \left(\bigcup_{n\geqslant 0} A^n[x:=_A C]\right)\left(\bigcup_{n\geqslant 0} B^n[x:=_A C]\right)\\ &= (A[x:=C])(B[x:=C])\\ &= (\lambda x.A)\,C((\lambda x.B)C). && [(\beta\iota)]\end{aligned}$$

(β_6'): Let $C = (\lambda y.B)z$. Then

$$\begin{aligned}
(\lambda xy.A)\,C &= (\lambda y.A)[x := C] && [(\beta\iota)]\\
&= \bigcup_{n\geqslant 0} (\lambda y.A)^n\,[x :=_{\lambda y.A} C] && [\mathrm{def}[x := C]]\\
&= \bigcup_{n>0} (\lambda y.A^{n-1})[x :=_{\lambda y.A} C] && [\mathrm{def}(\lambda y.A)^n]\\
&= \bigcup_{n>0} \lambda y.A^{n-1}[x :=_A C] && [\mathrm{def}[x :=_{\lambda y.A} C]]\\
&= \lambda y.\bigcup_{n\geqslant 0} A^n[x :=_A C]\\
&= \lambda y.A[x := C] && [\mathrm{def}[x := C]]\\
&= \lambda y.(\lambda x.A)\,C. && [(\beta\iota)]
\end{aligned}$$

(α'): Let $A = (\lambda y.B)z$. Then

$$\begin{aligned}
\lambda x.A &= \lambda y.A[x := y] && [(\alpha\iota)]\\
&= \lambda y.(\lambda x.A)\,y. && [(\beta\iota)]
\end{aligned}$$

So, $\Lambda_{\mathbf{I}}^{\mathbf{i}}/\lambda\beta\iota$ is an LAA, hence it is isomorphic to a FLA. ∎

Theorem 56. *For every infinitary lambda theory T, the algebra $\Lambda_{\mathbf{I}}^{\mathbf{i}}/T$ is isomorphic to a functional* LAA.

Proof. T is a congruence over the LAA_I $\Lambda_{\mathbf{I}}^{\mathbf{i}}/\lambda\beta\iota$. The conclusion follows because the class LAA_I is a variety, so it is closed under homomorphic image. ∎

Let $\mathbf{V}$ be a functional domain. We say that a Λ_I^i-indexed family of (total) mapping $[\![B]\!]^{\mathbf{V}}$: $V^I \to V$ is an *interpretation* if the following conditions are satisfied for all $A, B \in \Lambda_I^i$ and all $p \in V^I$:

$[\![x]\!]^{\mathbf{V}}(p) = p_x$, for all $x \in I$,

$[\![AB]\!]^{\mathbf{V}}(p) = [\![A]\!]^{\mathbf{V}}(p) \cdot^{\mathbf{V}} [\![B]\!]^{\mathbf{V}}(p)$,

$[\![\lambda x.B]\!]^{\mathbf{V}}(p) = \lambda^{\mathbf{V}} \langle [\![B]\!]^{\mathbf{V}}(p\{v/x\}) : v \in V\rangle$.

An equation $A = B$ between infinitary λ-terms is satisfied by an interpretation if $[\![A]\!]^{\mathbf{V}}(p) = [\![B]\!]^{\mathbf{V}}(p)$ for all $p \in V^I$.

The completeness theorem for the infinitary lambda calculus says that every infinitary lambda theory consists of precisely the equations between infinitary λ-terms valid in some environment model.

Theorem 57 (The Infinitary Completeness Theorem). *Let T be an infinitary lambda theory. Then there exists an environment model $\mathbf{V}$ and an interpretation $[\![-]\!]^{\mathbf{V}}$ of Λ_I^i into $\mathbf{V}$ such that*

$$(A, B) \in T \quad \textit{iff} \quad [\![A]\!]^{\mathbf{V}}(p) = [\![B]\!]^{\mathbf{V}}(p) \quad \textit{for all } p \in V^I.$$

Proof. By Theorem 56 there exists an environment model $\mathbf{V}$ such that Λ_I^i/T is isomorphic to an FLA_I with value domain $\mathbf{V}$. The conclusion of the theorem is now immediate. ∎

We conclude the section by defining a class of models of the lambda calculus including the model of Böhm trees and Berarducci's model. In the remaining part of this section we assume that the dimension set I is countably infinite, $I = \bigcup_{n \geqslant 0} I_n$ with $|I_{n+1} \setminus I_n| = \omega$ and every infinitary λ-term A is over I_n for some n.

COROLLARY 58. *For every infinitary lambda theory T, the combinatory reduct* $\mathrm{Cr}\,\Lambda_{\mathbf{I}}^{\mathbf{i}}/T$ *of* $\Lambda_{\mathbf{I}}^{\mathbf{i}}/T$ *is a lambda model.*

Proof. By the assumption on the dimension set I, the $\mathsf{LAA}_I\,\Lambda_{\mathbf{I}}^{\mathbf{i}}/T$ is dimension-complemented, so that the conclusion follows from Theorem 34(i). ∎

We now show that Berarducci's model determines an infinitary lambda theory.

We introduce the notion of infinitary reduction following the presentation in [6]. β-reduction is defined for infinitary λ-terms as the process of replacing a subterm of the form $(\lambda x.A)B$ (called a redex) with $A[x := B]$. An infinitary λ-term A is a *zero term* if there is no finite β-reduction from A to an abstraction term, i.e., to an infinitary λ-term of the form $\lambda x.B$. For example, $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$, $\bot$, Ω_3 and Ω_3^∞ are zero terms. We say that an infinitary λ-term *is a top normal form* if it is either a λ-variable, or an abstraction term, or an application term of the form BC with B a zero term. An infinitary λ-term *has a top normal form* if it can be reduced to a top normal form by a finite β-reduction. Infinitary λ-terms which do not have a top normal form are called *mute*. For example, Ω is mute, while Ω_3 is not. Mute infinitary λ-terms have a totally undefined operational behavior. In other words, the asympotic behavior of an infinitary λ-term is well defined only for nonmute infinitary λ-terms.

Define now $\rightarrow_\bot$ as the least notion of reduction between infinitary λ-terms which contains β-reduction and sends all the mute terms (different from $\bot$) to $\bot$. The reflexive and transitive closure of $\rightarrow_\bot$ will be denoted by $\rightarrow_\bot^*$.

Given two infinitary λ-terms A and B, we say $A \equiv_n B$ if A and B coincide up to the nth level of their tree representation. The depth of an occurrence of a subterm B in an infinitary λ-term A is defined as the length of the path connecting the root of A to the root of B.

We define $A \rightarrow_{\bot\infty} B$ (*infinite $\beta\bot$-reduction*) either $A \rightarrow_\bot^* B$ or there is an infinite sequence

$$s\colon A \equiv A_0 \rightarrow_\bot A_1 \rightarrow_\bot A_2 \rightarrow_\bot \cdots$$

of $\beta\bot$-reductions such that

(i) $(\forall k)(\exists n)(\forall m \geqslant n)\ A_m \equiv_k B$;

(ii) the depth of the redex reduced in $A_i \rightarrow_\bot A_{i+1}$ tends to infinity with i.

One of the main results in [5] is the following.

THEOREM 59. (Berarducci [5])

(i) $\rightarrow_{\perp\infty}$ *is transitive;*

(ii) $\rightarrow_{\perp\infty}$ *is Church–Rosser;*

(iii) *Every infinitary λ-term A can be reduced to a unique normal form A^{∞} via $\rightarrow_{\perp\infty}$.*

Consider the $\perp\infty$-conversion defined by

$$A =_{\perp\infty} B \qquad \text{iff} \quad (\exists C)\, A \rightarrow_{\perp\infty} C \quad \textit{and} \quad B \rightarrow_{\perp\infty} C.$$

COROLLARY 60.

(i) *The relation $=_{\perp\infty}$ is an infinitary lambda theory.*

(ii) *The combinatory reduct of $\Lambda^{\mathbf{i}}_{\mathbf{I}}/=_{\perp\infty}$ is a lambda model.*

Proof. The transitivity of $=_{\perp\infty}$ follows from Theorem 59(i)–(ii). The remaining part of the Corollary is a consequence of Theorem 58. ∎

A similar approach can be also developed for the model of Böhm trees.

8. RELATED WORK

There have been several attempts to reformulate the lambda calculus as a purely algebraic theory within the context of category theory: Obtułowicz and Wieger [30] via the *algebraic theories* of Lawvere; Adachi [*e*Ada] via *monads*, and Curien [10] via *categorical combinators*. There have also been several works that present an algebraic theory of the lambda calculus very close to lambda calculus in spirit. Locally finite functional LAA's are very similar to the functional models of the lambda calculus developed in Krivine [25]. However, Krivine's models do not have an explicit algebraic structure. An abstractly defined class of algebras, called lambda term systems, that is even closer in spirit to LAA has been introduced by Diskin [12, 13].

Lambda abstraction algebras do for lambda calculus what cylindric (and polyadic) algebras do for first-order predicate calculus. The theories of cylindric and polyadic algebras are two early contributions to the algebraization of quantifier logics. The main references for cylindric algebras are [21] and [22]; for polyadic algebras the main reference is [20]; see in particular [19]. We also mention here Nemeti [29]. It contains an extensive survey of the various algebraic versions of quantifier logics. LAA's, like cylindric and polyadic algebras, can be also viewed as a contribution to the theory of abstract substitution. However, in lambda abstraction and cylindric algebras, abstract substitution is a defined operation, while in polyadic algebras it is one of the primitive notions. The importance of abstract substitution, and lambda abstraction, has been recognized for some time among computer scientists because it leads among other things to more natural term rewriting systems, which are useful in the analysis of processes of computations. See for example [1]. In the *transformation* *algebras* and *substitution algebras* of LeBlanc [26] and Pinter [37] substitution is primitive and abstract quantification is defined in terms of it. A pure theory of abstract substitution has been developed by Feldman [14, 15].

Methods from nonstandard analysis were used by the second author in [17] to give a new proof that every RFA_I is isomorphic to an FLA_I. Finally, we mention that some recent work of the first author connecting a theory of substitution in combination with abstract variable-binding operators has been recently done (see [35, 38]).

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