The Sensible Graph Theories of Lambda Calculus

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Abstract

Sensible λ-theories are equational extensions of the untyped lambda calculus that equate all the unsolvable λ-terms and are closed under derivation. A longstanding open problem in lambda calculus is whether there exists a non-syntactic model whose equational theory is the least sensible λ-theory $\mathcal{H}$ (generated by equating all the unsolvable terms). A related question is whether, given a class of models, there exist a minimal and maximal sensible λ-theory represented by it. In this paper we give a positive answer to this question for the semantics of lambda calculus given in terms of graph models. We conjecture that the least sensible graph theory, where "graph theory" means "λ-theory of a graph model", is equal to $\mathcal{H}$, while in the main result of the paper we characterize the greatest sensible graph theory as the λ-theory $\mathcal{B}$ generated by equating λ-terms with the same Böhm tree. This result is a consequence of the fact that all the equations between solvable λ-terms, which have different Böhm trees, fail in every sensible graph model. Further results of the paper are: (i) the existence of a continuum of different sensible graph theories strictly included in $\mathcal{B}$ (this result positively answers Question 2 in [7, Section 6.3]); (ii) the non-existence of a graph model whose equational theory is exactly the minimal lambda theory $\lambda\beta$ (this result negatively answers Question 1 in [7, Section 6.2] for the restricted class of graph models).

1. Introduction

Lambda theories are compatible equivalence relations on λ-terms closed under (α)- and (β)-conversion. They arise by syntactical or semantic considerations. Indeed, a λ-theory may correspond to a possible operational (observational) semantics of the lambda calculus, as well as it may be induced by a model of lambda calculus through the kernel congruence relation of the interpretation function. Although researchers have mainly focused their interest on a limited number of them, the class of λ-theories constitutes a very rich and complex structure (see e.g. [4, 7]). Syntactical techniques are usually difficult to use in the study of λ-theories. Therefore, semantic methods have been extensively investigated.

Topology is at the center of the known approaches to giving models of the untyped lambda calculus. The first model, found by Scott in 1969 in the category of complete lattices and Scott continuous functions, was successfully used to show that all the unsolvable λ-terms can be consistently equated. After Scott, a large number of mathematical models for lambda calculus, arising from syntax-free constructions, have been introduced in various categories of domains and were classified into semantics according to the nature of their representable functions, see e.g. [1, 4, 7, 21]. Scott's continuous semantics [24] is given in the category whose objects are complete partial orders and morphisms are Scott continuous functions. The stable semantics (Berry [8]) and the strongly stable semantics (Bucciarelli-Ehrhard [9]) are a strengthening of the continuous semantics, introduced to capture the notion of "sequential" Scott continuous function. All these semantics are structurally and equationally rich in the sense that it is possible to build up $2^{\aleph_0}$ models in each of them inducing pairwise distinct λ-theories [18, 19]. Nevertheless, the above denotational semantics are *equationally incomplete*: they do not match all the possible operational semantics of lambda calculus. The problem of the equational incompleteness was positively solved by Honsell-Ronchi della Rocca [16] for the continuous semantics, and by Bastonero-Gouy [15, 6] for the stable semantics. Salibra [22, 23] has recently shown in a uniform way that all the semantics, which involve monotonicity with respect to some partial order and have a bottom element, fail to induce a continuum of λ-theories. From this it follows the incompleteness of the strongly stable semantics, which had been conjectured by Bastonero-Gouy [6] and by Berline [7].

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If a semantics is incomplete, then there exists a λ-theory T that is not induced by any model in the semantics. In such a case we say that the semantics *omits* the λ-theory T. More generally, a semantics *omits* (*forces*, respectively) an equation if the equation fails (holds) in every model of the semantics. The set of equations forced by a semantics $\mathcal{C}$ constitutes a λ-theory. It is the minimal λ-theory of $\mathcal{C}$ if it is induced by a model of $\mathcal{C}$.

The following natural question arises (see Berline [7]): given a class of models in a semantics $\mathcal{C}$, is there a minimal λ-theory represented in it? Di Gianantonio et al. [14] have shown that the above question admits a positive answer for Scott's continuous semantics, at least if we restrict to extensional models. However, the proofs of [14] use logical relations, and since logical relations do not allow to distinguish terms with the same applicative behavior, the proofs do not carry out to non-extensional models. The authors [10] have recently shown that the same question admits a positive answer for the *graph semantics*, that is, the semantics of lambda calculus given in terms of *graph models*. These models, isolated in the seventies by Scott and Engeler [4] within the continuous semantics, have been proved useful for giving proofs of consistency of extensions of lambda calculus and for studying operational features of lambda calculus. For example, the simplest graph model, namely Engeler's model, has been used by Berline [7] to give concise proofs of the head-normalization theorem and of the left-normalization theorem of lambda calculus, while a semantical proof of the "easiness" of $(\lambda x.xx)(\lambda x.xx)$ was obtained by Baeten and Boerboom in [3]. Kerth has recently shown in [18] that there exists a continuum of different (sensible) graph theories (where "graph theory" means "λ-theory of a graph model"). However, it is well known that the graph semantics is incomplete, since it trivially omits the axiom of extensionality (i.e., the equation $\lambda x.x = \lambda xy.xy$).

Sensible λ-theories are equational extensions of the untyped lambda calculus that equate all the unsolvable λ-terms and are closed under derivation. The least sensible λ-theory is the λ-theory $\mathcal{H}$ (generated by equating all the unsolvable terms), while the greatest sensible λ-theory is the λ-theory $\mathcal{H}^*$ (generated by equating λ-terms with the same Böhm tree up to possibly infinite η-equivalence). A longstanding open problem in lambda calculus is whether there exists a non-syntactic model whose equational theory is the least sensible λ-theory $\mathcal{H}$. A related question is whether, given a class of models, there is a minimal and maximal sensible λ-theory represented by it. In this paper we give a positive answer to this question for the graph semantics. Two further questions arise: what equations between λ-terms belong to the least sensible graph theory? And to the greatest one? The answer to the first difficult question is still unknown; we conjecture that the λ-theory $\mathcal{H}$ is the least sensible graph theory. In this paper we answer the second question: the λ-theory $\mathcal{B}$ (generated by equating λ-terms with the same Böhm tree) is the greatest sensible graph theory. This result is a consequence of the main technical theorem of the paper: the graph semantics omits all the equations $M = N$ between λ-terms satisfying the following conditions:

$$\mathcal{H}^* \vdash M = N \text{ and } \mathcal{B} \not\vdash M = N. \tag{1}$$

The following are other consequences of the main result of the paper.

(i) There exists a continuum of different sensible graph theories strictly included in $\mathcal{B}$ (this result positively answers Question 2 in [7, Section 6.3]);

(ii) For every closed term P, the λ-theory generated by $(\lambda x.xx)(\lambda x.xx) = P$ contains no equation satisfying condition (1).

A longstanding open problem in lambda calculus is whether there exists a non-syntactic model whose equational theory is equal to the least λ-theory $\lambda\beta$. In this paper we show that this model cannot be found within graph semantics (this result negatively answers Question 1 in [7, Section 6.2] for the restricted class of graph models). From the above result it follows that the minimal graph theory, whose existence was shown in [10], is not equal to $\lambda\beta$, so that graph semantics forces equations between non-β-equivalent λ-terms.

The paper is organized as follows. In Section 2 we review the basic definitions of lambda calculus and graph models. In particular, we recall the formal definition of the Engeler completion of a partial pair. The proof of the existence of a minimal sensible graph theory is presented in Section 3, while in Section 4 it is shown that the least graph theory is not equal to $\lambda\beta$. Sections 5 and 6 are devoted to the characterization of the maximal sensible graph theory.

2. Preliminaries

To keep this article self-contained, we summarize some definitions and results concerning lambda calculus and graph models that we need in the subsequent part of the paper. With regard to the lambda calculus we follow the notation and terminology of [4].

2.1. Lambda calculus

The set Λ of λ-terms over an infinite set of variables is constructed as usual: every variable is a λ-term; if t and s are λ-terms, then so are (st) and $\lambda x.t$ for each variable x. The symbol $\equiv$ denotes syntactic equality. The following are some well-known λ-terms:

$$\Omega \equiv (\lambda x.xx)(\lambda x.xx); \quad \Omega_3 \equiv (\lambda x.xxx)(\lambda x.xxx);$$

$$\mathbf{i} \equiv \lambda x.x; \quad \mathbf{k} \equiv \lambda xy.x; \quad \mathbf{1} \equiv \lambda xy.xy.$$

A *compatible λ-relation* T is any set of equations between λ-terms that is closed under the following two rules:

(i) If $t = u \in T$ and $w = v \in T$, then $tw = uv \in T$;

(ii) If $t = u \in T$ then $\lambda x.t = \lambda x.u \in T$ for every variable x.

We will write either $T \vdash t = u$ or $t =_T u$ for $t = u \in T$.

A *λ-theory* T is any compatible λ-relation which is an equivalence relation and includes (α)- and (β)-conversion. The set of all λ-theories is naturally equipped with a lattice structure, with meet defined as set theoretical intersection. The join of two λ-theories T and S is the least equivalence relation including $T \cup S$. $\lambda\beta$ denotes the minimal λ-theory, while $\lambda\beta\eta$ denotes the minimal extensional λ-theory (axiomatized by $\mathbf{i} = \mathbf{1}$).

The λ-theory $\mathcal{H}$, generated by equating all the unsolvable λ-terms, is consistent by [4, Thm. 16.1.3] and admits a unique maximal consistent extension $\mathcal{H}^*$ [4, Thm. 16.2.6]. A λ-theory T is called *sensible* [4, Def. 4.1.7(ii)] if $\mathcal{H} \subseteq T$. The set of all sensible λ-theories is naturally equipped with a structure of bounded lattice. $\mathcal{H}$ is the least sensible λ-theory, while $\mathcal{H}^*$ is the greatest one. $\mathcal{H}^*$ is an extensional λ-theory.

2.2. Böhm trees

The Böhm tree $BT(M)$ of a λ-term M is a finite or infinite labelled tree. If M is unsolvable, then $BT(M) = \bot$, that is, $BT(M)$ is a tree with a unique node labelled by $\bot$. If M is solvable and $\lambda x_1 \dots x_n.yM_1 \dots M_k$ is the principal head normal form of M [4, Def. 8.3.20] then we have

$$BT(M) = \lambda x_1 \dots x_n.y$$

$$BT(M_1) \dots\dots\dots BT(M_k)$$

The λ-theory $\mathcal{B}$, generated by equating λ-terms with the same Böhm tree, is sensible and non-extensional. $\mathcal{B}$ is distinct from $\mathcal{H}$ and $\mathcal{H}^*$, so that $\mathcal{H} \subset \mathcal{B} \subset \mathcal{H}^*$. Notice that not all the λ-theories T satisfying the condition $\mathcal{B} \subset T \subset \mathcal{H}^*$ are extensional (see the remark after Thm. 5.2).

In the remaining part of this subsection we characterize the λ-theory $\mathcal{H}^*$ in terms of Böhm trees.

For all λ-terms M and N, we write $M \leq_\eta N$ if $BT(N)$ is a (possibly infinite) *η-expansion* of $BT(M)$ (see [4, Def. 10.2.10]). For example, let $J \equiv \Theta(\lambda jxy.x(jy))$, where Θ is the Turing's fixpoint combinator. Then, $x \leq_\eta Jx$ (see [4, Example 10.2.9]), since

$$Jx =_{\lambda\beta} \lambda z_0.x(Jz_0) =_{\lambda\beta} \lambda z_0.x(\lambda z_1.z_0(Jz_1))$$

$$=_{\lambda\beta} \lambda z_0.x(\lambda z_1.z_0(\lambda z_2.z_1(Jz_2))) =_{\lambda\beta} \dots$$

We write $N =_\eta M$ if there exists a Böhm-like tree A such that $BT(M) \leq_\eta A$ and $BT(N) \leq_\eta A$ (see [4, Def. 10.2.25, Thm. 10.2.31]). It is well known that

$$M =_{\mathcal{H}^*} N \Leftrightarrow M =_\eta N \quad \text{(see [4, Thm. 19.2.9]).}$$

2.3. Graph models

The class of graph models belongs to Scott's continuous semantics. Historically, the first graph model was Scott's P_ω, which is also known in the literature as "the graph model". "Graph" referred to the fact that the continuous functions were encoded in the model via (a sufficient fragment of) their graph.

As a matter of notation, for every set D, D^* is the set of all finite subsets of D, while $\mathcal{P}(D)$ is the powerset of D. If C is a complete partial ordering (cpo, for short), then $[C \to C]$ denotes the cpo of all the Scott continuous functions from C into C.

Definition 2.1 *A* graph model *is a pair (D, p), where D is an infinite set and $p : D^* \times D \to D$ is an injective total function.*

As a matter of notation, we write $a \to_p \alpha$, or also simply $a \to \alpha$, for $p(a, \alpha)$. When parenthesis are omitted, then association to the right is assumed. For example, $a \to b \to \alpha$ stands for $p(a, p(b, \alpha))$. If $\overline{a} = a_1 \dots a_n$ is a sequence of finite subsets of D, then we write $\overline{a}_n \to \alpha$ for $a_1 \to a_2 \to \dots \to a_n \to \alpha$.

The function p is useful to encode a fragment of the graph of a Scott continuous function $f : \mathcal{P}(D) \to \mathcal{P}(D)$ as a subset $G(f)$ of D:

$$G(f) = \{a \to_p \alpha \mid \alpha \in f(a) \text{ and } a \in D^*\}. \qquad (2)$$

Any graph model (D, p) is used to define a model of lambda calculus through the reflexive cpo $(\mathcal{P}(D), \subseteq)$ determined by two Scott continuous mappings $G : [\mathcal{P}(D) \to \mathcal{P}(D)] \to \mathcal{P}(D)$ and $F : \mathcal{P}(D) \to [\mathcal{P}(D) \to \mathcal{P}(D)]$. The function G is defined in (2), while F is defined as follows:

$$F(X)(Y) = \{\alpha \in D : (\exists a \subseteq Y)\ a \to_p \alpha \in X\}.$$

For more details we refer the reader to Berline [7] and to Chapter 5 of Barendregt's book [4].

Let Env_D be the set of D-environments ρ mapping the set of the variables of lambda calculus into $\mathcal{P}(D)$. The interpretation $M^p : Env_D \to \mathcal{P}(D)$ of a λ-term M is defined as follows:

- $x^p_\rho = \rho(x)$
- $(MN)^p_\rho = \{\alpha \in D : (\exists a \subseteq N^p_\rho)\ a \to_p \alpha \in M^p_\rho\}$
- $(\lambda x.M)^p_\rho = \{a \to_p \alpha : \alpha \in M^p_{\rho[x:=a]}\}$

If $\overline{x} \equiv x_1 \dots x_n$ is a sequence of variables and $\overline{a} = a_1 \dots a_n$ is a sequence of finite subsets of D, then we have

$$(\lambda \overline{x}.M)^p_\rho = \{\overline{a}_n \to \alpha : \alpha \in M^p_{\rho[x_1:=a_1]\dots[x_n:=a_n]}\}.$$

Given a graph model (D, p), we have that $M^p = N^p$ if, and only if, $M^p_\rho = N^p_\rho$ for all environments ρ. The λ-theory $Th(D, p)$ induced by (D, p) is defined as

$$Th(D, p) = \{M = N : M^p = N^p\}.$$

A λ-theory induced by a graph model will be called a *graph theory*. The graph model (D, p) is called *sensible* if $Th(D, p)$ is a sensible λ-theory. It is well known that the graph theory $Th(D, p)$ is never extensional because $(\lambda x.x)^p \neq (\lambda xy.xy)^p$. Di Gianantonio and Honsell [13] have shown that graph models are related to filter models (see Coppo-Dezani [11] and Barendregt et al. [5]), since the class of graph theories is included in the class of λ-theories induced by non-extensional filter models. Alessi et al. [2] have shown that this inclusion is strict, namely there exists an equation between λ-terms, which is omitted in graph semantics, whilst it is satisfied in some non-extensional filter model.

A graph theory T will be called

1. *the minimal graph theory* if $T \subseteq Th(D, p)$ for all graph models (D, p);
2. *the minimal sensible graph theory* if T is sensible and $T \subseteq Th(D, p)$ for all sensible graph models (D, p);
3. *the maximal sensible graph theory* if T is sensible and $Th(D, p) \subseteq T$ for all sensible graph models (D, p).

The completion method for building graph models from "partial pairs" was initiated by Longo in [20] and recently developed on a wide scale by Kerth in [18, 19]. This method is useful to build models satisfying prescribed constraints, such as domain equations and inequations, and it is particularly convenient for dealing with the equational theories of graph models.

Definition 2.2 *A* partial pair (D, p) *is given by an infinite set* D *and a partial, injective function* $p : D^* \times D \to D$.

A partial pair is a graph model if and only if p is total. We always suppose that no element of D is a pair. This is not restrictive because partial pairs can be considered up to isomorphism.

Definition 2.3 *Let* (D, p) *be a partial pair. The* Engeler completion *of* (D, p) *is the graph model* (E, i) *defined as follows:*

- $E = \bigcup_{n \in \omega} E_n$*, where* $E_0 = D$*,* $E_{n+1} = E_n \cup ((E^*_n \times E_n) - dom(p))$.
- *Given* $a \in E^*$*,* $\alpha \in E$*,*

$$i(a, \alpha) = \begin{cases} p(a, \alpha) & \text{if } p(a, \alpha) \text{ is defined} \\ (a, \alpha) & \text{otherwise} \end{cases}$$

It is easy to check that the Engeler completion of a given partial pair (D, p) is actually a graph model. The Engeler completion of a total pair (D, p) is equal to (D, p).

A notion of *rank* can be naturally defined on the Engeler completion (E, i) of a partial pair (D, p). The elements of D are the elements of rank 0, while an element $\alpha \in E - D$ has rank n if $\alpha \in E_n$ and $\alpha \notin E_{n-1}$.

Classic graph models, such as Scott's P_ω [4] and Engeler's $\mathcal{E}_A$ (where A is an arbitrary nonempty set) [7], can be viewed as the Engeler completions of suitable partial pairs. In fact, P_ω and $\mathcal{E}_A$ are respectively isomorphic to the Engeler completions of $(\{0\}, p)$ (with $p(\emptyset, 0) = 0$) and $(A, \emptyset)$.

Let $\overline{x} = x_1 \dots x_n$ be a sequence of variables and ρ be a D-environment such that $\rho(x_i)$ is a finite set. As a matter of notation, we write $\rho(\overline{x}_n) \to \alpha$ for $\rho(x_1) \to \rho(x_2) \to \dots \to \rho(x_n) \to \alpha$.

3. The minimal sensible graph theory

In this section we show that the class of sensible graph theories has a minimum element, i.e., there exists a graph model whose equational theory is the smallest sensible graph theory.

In [10] a general technique for "gluing together" the elements of a family of graph models is described.

The idea is the following: given a family $\mathcal{M} = \{(M_j, i_j)\}$ of graph models, take the partial pair given by the disjoint union of the M_j and of the i_j. The key point is that the theory of the Engeler completion of this partial pair, that we call here the *canonical product* of $\mathcal{M}$, is smaller than that of all the (M_j, i_j)'s. This is enough to conclude that the class of graph theories has a minimum element (simply take a "complete" family $\{(M_j, i_j)\}$, i.e. a family such that, for any inequation between λ-terms which holds in some graph model, there exists j such that (M_j, i_j) realises that inequation).

Here we restrict our attention to sensible models; we can use the same technique, starting from a complete family $\mathcal{S}$ of sensible graph models, but we have to be careful: it remains to show that the canonical product of $\mathcal{S}$ is sensible. This is a consequence of the property of sensible graph models expressed in Lem. 3.2 below.

The proof of the following lemma can be found in Example 5.3.7 of Kerth's thesis [17].

Lemma 3.1 *Let* (D, p) *be a graph model. If* $\alpha \in (\Omega_3)^p$*, then there exists a natural number* $k \geq 1$ *such that*

$$\alpha = b_1 \to \dots \to b_k \to \alpha$$

for suitable finite subsets b_i contained in the interpretation of $\lambda x.xxx$.

Lemma 3.2 *If all the closed unsolvable λ-terms have the same interpretation in a graph model, then it must be the empty set.*

Proof: Let (D, i) be a graph model and let X be a nonempty subset of D, that is the common interpretation of all closed unsolvables. Since Ω and $\lambda x.\Omega$ are both unsolvables, then we have that

$$X = (\lambda x.\Omega)^i = \{a \to \alpha : \alpha \in \Omega^i\} = \{a \to \alpha : \alpha \in X\}. \quad (3)$$

It follows that $a \to \alpha \in X$ for all finite subsets a of D and all $\alpha \in X$. Let γ be a fixed element of X. Then $a \to \gamma \in (\Omega_3)^i$ by (3), since Ω_3 is unsolvable and $(\Omega_3)^i = X$. From Lem. 3.1 it follows that

$$a \to \gamma = b_1 \to ... \to b_k \to a \to \gamma,$$

where $b_1, \ldots, b_k$ are finite subsets contained in the interpretation of $\lambda x.xxx$. It follows that $b_1 = a$. By the arbitrariness of a we can conclude that $(\lambda x.xxx)^i = D$. This is not possible, because, for example, $\emptyset \to \beta \notin (\lambda x.xxx)^i$. $\triangle$

We state here some definitions and lemmata, sketching the main result of [10]. We need them for proving the main result of this section, Thm. 3.7:

Definition 3.3 *Let $\mathcal{F} = \{(D_j, i_j)\}_{j\in J}$ be a family of graph models (without loss of generality, we may assume that $D_{j_1} \cap D_{j_2} = \emptyset$ for $j_1 \neq j_2 \in J$). Consider the partial pair (D_J, q_J) defined by:*

$$D_J = \bigcup_{j\in J} D_j; \quad q_J = \bigcup_{j\in J} i_j.$$

The canonical product *of $\mathcal{F}$ is the Engeler completion of (D_J, q_J).*

In the following, we denote by (D, i) the canonical product of $\mathcal{F} = \{(D_j, i_j)\}_{j\in J}$.

Definition 3.4 *Let $j \in J$. We call j-*flattening *the following function $f_j : D \to D$, defined by induction on the rank of elements of D:*

if $rank(x) = 0$ then $f_j(x) = x$
if $rank(x) = n + 1$ and $x = (\{y_1, ..., y_k\}, y)$ then

$$f_j(x) = \begin{cases} i_j(d, f_j(y)) & \text{if } f_j(y) \in D_j \\ x & \text{otherwise} \end{cases}$$

where $d = \{f_j(y_1), ..., f_j(y_k)\} \bigcap D_j$.

Lemma 3.5 *For all $x \in D$ there exists a unique $j \in J$ such that $f_j(x) \in D_j$.*

Proposition 3.6 *Let M be a closed λ-term and M^i (resp. M^{i_j}) be its interpretation in the canonical product (D, i) (resp. graph model (D_j, i_j)); then we have for all $j \in J$:*

(i) $f_j(x) \in M^i$ for all $x \in M^i$.
(ii) $M^i \cap D_j = M^{i_j}$.

A family of sensible graph models is complete if, for any inequation between closed λ-terms which holds in some sensible graph model, there exists an element of the family in which that inequation holds.

Theorem 3.7 *Let $\mathcal{S} = \{(S_j, i_j)\}_{j\in J}$ be a countable and complete family of sensible graph models, and let (S, i) be the canonical product of $\mathcal{S}$; then the theory of (S, i) is the least sensible graph theory.*

Proof: By the completeness of $\mathcal{S}$ and by Prop. 3.6(ii) we have that $Th(S, i)$ is contained within any sensible graph theory.

In order to prove that (S, i) is sensible, let us suppose that a closed unsolvable term M has a non-empty interpretation in (S, i), i.e., there exists $\alpha \in M^i$. By Lem. 3.5 there exists a unique $j \in J$ such that $f_j(\alpha) \in S_j$. By Prop. 3.6(i) we have that $f_j(\alpha) \in M^i$, and finally, by Prop. refmain(ii), that $f_j(\alpha) \in M^{i_j}$. Since (D_j, i_j) is sensible, this is impossible by Lem. 3.2. Hence $M^i = \emptyset$ for any closed unsolvable M (and actually for any unsolvable in any environment). $\triangle$

4. The minimal graph theory is not $\lambda\beta$

A longstanding open problem is whether there exists a non-syntactic model of lambda calculus whose equational theory is equal to the least λ-theory $\lambda\beta$. In the following theorem we show that this model cannot be found within graph semantics. This result negatively answers Question 1 in [7, Section 6.2] for the restricted class of graph models.

Lemma 4.1 *All the graph models satisfy the inequality $\Omega_3 \leq \lambda y.\Omega_3 y$.*

Proof: Let (D, p) be an arbitrary graph model and $\alpha \in (\Omega_3)^p$. From Lem. 3.1 it follows that there exists a natural number $k \geq 1$ such that $\alpha = b_1 \to_p b_2 \to_p ... \to_p b_k \to_p \alpha$ for suitable finite subsets b_i contained in the interpretation of $\lambda x.xxx$. We have that $\alpha \in (\lambda y.\Omega_3 y)^p$ (that is, $b_1 \to_p b_2 \to_p ... \to_p b_k \to_p \alpha \in (\lambda y.\Omega_3 y)^p$) iff there exists a finite set d such that $d \to_p b_2 \to_p ... \to_p b_k \to_p \alpha \in (\Omega_3)^p$ and $d \subseteq b_1$. This last relation is true by defining $d \equiv b_1$, so that $\alpha \in (\lambda y.\Omega_3 y)^p$. In conclusion, we get $(\Omega_3)^p \subseteq (\lambda y.\Omega_3 y)^p$. $\triangle$

Theorem 4.2 *There exists no graph model whose equational theory is $\lambda\beta$.*

Proof: Assume that there exists a graph model (D, p) whose equational theory is $\lambda\beta$. By Cor. 2.4 in [25] the denotations of two non-$\lambda\beta$-equivalent closed λ-terms must be incomparable in every model of lambda calculus whose equational theory is $\lambda\beta$. Then, for all closed λ-terms M and N

such that $M \neq_{\lambda\beta} N$, we have that neither $M^p \subseteq N^p$ nor $N^p \subseteq M^p$. We get a contradiction because of Lem. 4.1. $\triangle$

In [10] the authors have shown that there exists a minimal graph theory. By Thm. 4.2 we have that $\lambda\beta$ is strictly included within the minimal graph theory. Thus, there exist equations between non-$\lambda\beta$-equivalent terms satisfied by all the graph models. In the following proposition we characterize an equation of this kind.

As a matter of notation, let $f \equiv Y(\lambda zyx.zy(zy(zyx)))$ and $A \equiv \lambda xyzwv.fx(fy(fz(fwv)))$, where Y is the Curry's fixpoint combinator. The λ-term A was defined by Selinger in [25].

Proposition 4.3 *Let $t \equiv x\Omega_3$ and $u \equiv x(\lambda y.\Omega_3 y)$ (for a variable x). Then the equation $Atttu = Attuu$ is satisfied by every graph model, but $Atttu \neq_{\lambda\beta} Attuu$.*

Proof: As a consequence of the proof of [25, Thm. 2.3], the equation $Atttu = Attuu$ holds in every partially ordered model satisfying $\Omega_3 \leq \lambda y.\Omega_3 y$. Then, by Lem. 4.1 it holds in every graph model. Finally, by Lem. 2.2 and Prop. 2.1 in [25] the equation $Atttu = Attuu$ cannot belong to the minimal λ-theory $\lambda\beta$. $\triangle$

5. Omitting equations and theories

A semantics is incomplete if there exists a λ-theory T that is not induced by any model in the semantics. In such a case we say that the semantics *omits* the λ-theory T. More generally, a semantics *omits* (*forces*, respectively) an equation if it fails (holds) in all the models of the semantics. If a semantics omits an equation $M = N$, then it omits all the λ-theories including $M = N$. It is easy to verify that the set of equations 'forced' by a semantics $\mathcal{C}$ constitutes a λ-theory. It is the minimal λ-theory of $\mathcal{C}$ if it is induced by a model of $\mathcal{C}$.

The following two theorems are the main results of the paper. The proof of Thm. 5.1 is postponed to the next section.

Theorem 5.1 *The graph semantics omits all the equations $M = N$ satisfying the following conditions:*

$$M =_{\mathcal{H}^*} N \text{ and } M \neq_{\mathcal{B}} N. \tag{4}$$

In other words, graph semantics omits all the equations $M = N$ between λ-terms which do not have the same Böhm tree, but have the same Böhm tree up to (possibly infinite) η-equivalence (see Section 2.2 and Barendregt [4, Section 10]).

Theorem 5.2 *The λ-theory $\mathcal{B}$ is the unique maximal sensible graph theory.*

Proof: $\mathcal{B}$ is the equational theory of Scott's graph model P_ω (see Section 19.1 in [4]) and of Engeler's graph model $\mathcal{E}_A$ (see [7]). Let T be a sensible graph theory and suppose $M =_T N$. We have that $M =_{\mathcal{H}^*} N$, because $\mathcal{H}^*$ is the unique maximal sensible λ-theory. Since graph semantics does not omit the equation $M = N$, then from $M =_{\mathcal{H}^*} N$ and from Thm. 5.1 it follows that $M =_{\mathcal{B}} N$, so that $T \subseteq \mathcal{B}$. $\triangle$

It is well known that every graph theory is non-extensional (see [7]). We remark that Thm. 5.2 is not trivial, because there exist non-extensional sensible λ-theories that strictly include $\mathcal{B}$ (see [4, Exercize 16.5.5]).

Berline [7] asked whether there is a non-syntactic sensible model of lambda calculus whose theory is strictly included in $\mathcal{B}$. The answer is positive as shown in the following corollary.

Theorem 5.3 *There exists a continuum of different sensible graph theories strictly included in $\mathcal{B}$.*

Proof: Based on a syntactic difficult result (conjectured by Kerth [18] and proved by David [12]), Kerth [18] has shown that there exists a continuum of sensible graph theories. Then the conclusion follows from Thm. 5.2. $\triangle$

It is well known that the λ-term Ω is easy, that is, it can be consistently equated to every other closed λ-term M. We denote by $(\Omega = M)^+$ the λ-theory generated by the equation $\Omega = M$.

Theorem 5.4 *Let M be an arbitrary closed λ-term. Then we have:*

$$P =_{\mathcal{H}^*} Q,\ P \neq_{\mathcal{B}} Q \ \Rightarrow\ (\Omega = M)^+ \not\vdash P = Q.$$

Proof: By [3] the λ-theory $(\Omega = M)^+$ is contained within a graph theory. Then the conclusion follows from Thm. 5.1. $\triangle$

6. The proof of the main theorem

In this section we provide the proof of Thm. 5.1.

We recall that a node of a tree is a sequence of natural numbers and that the level of a node is the length of the sequence. The empty sequence will be denoted by ε.

Let M, N be closed λ-terms such that $M =_{\mathcal{H}^*} N$ and $M \neq_{\mathcal{B}} N$. This last condition expresses the fact that the Böhm tree $BT(M)$ of M is different from the corresponding Böhm tree $BT(N)$ of N.

Let us give an informal overview of the proof. We start by picking a node $u = r_1 \dots r_k$ satisfying the following two conditions: (1) the labels of u in $BT(M)$ and $BT(N)$ are different; (2) the labels of every strict prefix $w = r_1 \dots r_j$ ($j < k$) of u in $BT(M)$ and $BT(N)$ are equal. Then we show that the subterms of M and N, whose Böhm trees are the subtrees of $BT(M)$ and $BT(N)$ at root u, respectively, get different interpretations in all graph models. This is done in Lem. 6.5. In order to get the conclusion, we have to show that in all graph models it is possible to propagate upward, towards the roots of $BT(M)$ and $BT(N)$, the difference "created" at node u. This is done in Lem. 6.6.

Let us introduce now some notations and definitions needed in the proof.

Let $u = r_1 \dots r_k$ be a node at least level, where the labels of $BT(M)$ and $BT(N)$ are different. The sequence ε, $r_1, r_1r_2, r_1r_2r_3$,...,$r_1 \dots r_k$ is the sequence of nodes that are in the path from the root ε to u. These nodes will be denoted by u_0, u_1, u_2,...,u_k. Then, for example, $u_0 = \varepsilon$, $u_2 = r_1r_2$ and $u_k = u$. From the hypothesis of minimality of u it follows that

(i) The label of the node u_j ($0 \leq j < k$) in the Böhm tree of M is equal to the corresponding one in the Böhm tree of N;

(ii) The labels of the node u in $BT(M)$ and $BT(N)$ are different.

From the hypothesis $M =_{\mathcal{H}^*} N$ and $M \neq_{\mathcal{B}} N$ it follows that

(iii) The node u is a starting point for a possibly infinite η-expansion in either $BT(M)$ or $BT(N)$, but not in both. Without loss of generality, we assume to have the η-expansion in $BT(N)$.

We define two sequences M_{u_j} and N_{u_j} ($0 \leq j \leq k$) of λ-terms whose Böhm trees $BT(M_{u_j})$ and $BT(N_{u_j})$ are the subtrees of $BT(M)$ and $BT(N)$ at root u_j, respectively. Let

$$M_{u_0} \equiv M; \quad N_{u_0} \equiv N.$$

If $k = 0$ we have finished. Otherwise, assume by induction hypothesis that we have already defined two λ-terms M_{u_j} and N_{u_j} ($j < k$) and that the Böhm trees of M_{u_j} and N_{u_j} are respectively the subtrees of $BT(M)$ and $BT(N)$ at root u_j. Assume that the principal head normal forms (principal hnfs, for short) of M_{u_j} and N_{u_j} (see [4, Def. 8.3.20]) are respectively

$$M_{u_j} =_{\lambda\beta} \lambda x_1^j \dots x_{n_j}^j.z_j M_1^j \dots M_{s_j}^j; \tag{5}$$

$$N_{u_j} =_{\lambda\beta} \lambda x_1^j \dots x_{n_j}^j.z_j N_1^j \dots N_{s_j}^j.$$

To abbreviate the notation we will write M_{u_j} and N_{u_j} as follows:

$$M_{u_j} =_{\lambda\beta} \lambda\overline{x}_{n_j}^j.z_j M_1^j..M_{s_j}^j;\ N_{u_j} =_{\lambda\beta} \lambda\overline{x}_{n_j}^j.z_j N_1^j..N_{s_j}^j.$$

Then the node u_j in the Böhm trees of M and N has s_j sons. Since $u_{j+1} = u_j r_{j+1}$ is a son of u_j in the Böhm trees of M and N, then we have $r_{j+1} \leq s_j$ and we define

$$M_{u_{j+1}} \equiv M_{r_{j+1}}^j;\ N_{u_{j+1}} \equiv N_{r_{j+1}}^j.$$

Then the Böhm trees of $M_{u_{j+1}}$ and $N_{u_{j+1}}$ are respectively the subtrees of $BT(M)$ and $BT(N)$ at root u_{j+1}. When we calculate the principal hnfs of M_{u_k} and N_{u_k} (recall that $u_k = u$ is the node where the Böhm trees are different), we get

$$M_{u_k} \equiv M_{r_k}^{k-1} =_{\lambda\beta} \lambda\overline{x}_{n_k}^k.z_k M_1^k \dots M_{s_k}^k; \tag{6}$$

$$N_{u_k} \equiv N_{r_k}^{k-1} =_{\lambda\beta} \lambda\overline{x}_{n_k}^k \lambda\overline{y}_r.z_k N_1^k \dots N_{s_k}^k Q_1 \dots Q_r, \tag{7}$$

where $y_i \leq_\eta Q_i$ ($1 \leq i \leq r$) (i.e., Q_i is a possibly infinite η-expansion of the variable y_i), y_i does occur neither free nor bound in N_j^k ($1 \leq j \leq s_k$) and Q_j ($1 \leq j \neq i \leq r$), and it is distinct from each variable $x_1^k, \dots, x_{n_k}^k$, z_k, $y_1, \dots y_{i-1}, y_{i+1}, \dots, y_r$.

Let (D, p) be an arbitrary graph model. First we will show that the terms N_{u_k} and M_{u_k} have different interpretations in (D, p), that is, there exist an element $\alpha_k \in D$ and a D-environment σ_k such that $\alpha_k \in (N_{u_k})_{\sigma_k}^p$, while $\alpha_k \notin (M_{u_k})_{\sigma_k}^p$. Second we will show that this difference at level k can be propagated upward, that is, there exist elements $\alpha_i \in D$ and D-environments σ_i ($i = 1, \dots, k$) such that $\alpha_k \in (N_{u_k})_{\sigma_k}^p$ iff $\alpha_i \in (N_{u_i})_{\sigma_i}^p$ iff $\alpha_0 \in N_{\sigma_0}^p$, and $\alpha_k \in (M_{u_k})_{\sigma_k}^p$ iff $\alpha_i \in (M_{u_i})_{\sigma_i}^p$ iff $\alpha_0 \in M_{\sigma_0}^p$.

To prove these properties of separability, we have to define the elements α_i and the D-environments σ_i. The definition of σ_i is difficult and technical.

We are going to use families of points of the graph models, which are not only pairwise distinct, but also "functionally incompatible", in the sense expressed by the following definition. Then, in the appendix we show that such families actually exist in all graph models.

Definition 6.1 *Let $q > 1$ be a natural number. A sequence $(\beta_n \in D : n \geq 0)$ of distinct elements of D is called a q-*sequence *if the following condition holds:*

$$(\forall i, j)(\forall 0 < t < q)(\forall \overline{a} \in (D^*)^t)\ \beta_j \neq \overline{a}_t \to \beta_i. \tag{8}$$

Recall that, if $\overline{a} \equiv a_1 \dots a_t$, then $\overline{a}_t \to \beta_i$ means $a_1 \to a_2 \to \dots \to a_t \to \beta_i$. Notice that i may be equal to j in the above condition (8).

In the appendix it will be shown the following result.

Lemma 6.2 *q-sequences exist for every $q > 1$.*

Let $(\beta_n : n \geq 0)$ be a q-sequence of elements of D, where

1. $q > (\Sigma_{0 \leq j \leq k} n_j) + (\Sigma_{0 \leq j \leq k} s_j) + r + s$;
2. n_j is the number of external abstractions in the principal hnf of M_{u_j} (see (5) above);
3. s_j is the number of sons of the node u_j in the Böhm tree of M (see (5) above);
4. $r \geq 1$ is the number of η-expansions in N_{u_k} (see (7) above);

5. s is the number of external abstractions in the principal hnf of the subterm Q_r of N_{u_k}:

$$Q_r =_{\lambda\beta} \lambda\overline{w}_s.y_r R_1 \dots R_s \ (s \geq 0). \tag{9}$$

We now define a sequence of environments ρ_j and two sequences of elements $\delta_j, \alpha_j \in D$ ($0 \leq j \leq k$). Next the environments ρ_j will be used to define σ_0 and σ_k. We start by defining ρ_k, δ_k and α_k.

(i) $\delta_k \equiv \overline{\emptyset}_{s_k+r-1} \to \{\overline{\emptyset}_s \to \beta_{k+1}\} \to \beta_k$;

(ii) $\rho_k(z_k) = \{\delta_k\}$, where z_k is the head variable of the principal hnfs of N_{u_k} and M_{u_k};

(iii) $\rho_k(y_r) = \{\overline{\emptyset}_s \to \beta_{k+1}, \beta_k\}$, where y_r is the head variable of the principal hnf of Q_r;

(iv) $\rho_k(x) = \emptyset$ ($x \not\equiv z_k, y_r$);

(v) $\alpha_k \equiv \rho_k(\overline{x}^k_{n_k}) \to \rho_k(\overline{y}_r) \to \beta_k$.

Notice that, if $s = 0$ (i.e., there are no external abstraction in the principal hnf of Q_r), then by definition $\overline{\emptyset}_0 \to \beta_{k+1}$ is just β_{k+1}. Moreover, the notation $\rho_k(\overline{x}^k_{n_k}) \to \rho_k(\overline{y}_r) \to \beta_k$, used in the definition of α_k, means $\rho_k(x^k_1) \to \dots \to \rho_k(x^k_{n_k}) \to \rho_k(y_1) \to \dots \to \rho_k(y_r) \to \beta_k$.

Assume we have defined δ_{j+1}, α_{j+1} and ρ_{j+1} ($j < k$). We define δ_j, α_j and ρ_j as follows.

(i) $\delta_j \equiv \overline{\emptyset}_{r_j-1} \to \{\alpha_{j+1}\} \to \overline{\emptyset}_{s_j-r_j} \to \beta_j$;

(ii) $\rho_j(z_j) = \rho_{j+1}(z_j) \cup \{\delta_j\}$, where z_j is the head variable of the principal hnfs of N_{u_j} and M_{u_j};

(iii) $\rho_j(x) = \rho_{j+1}(x)$ ($x \not\equiv z_j$);

(iv) $\alpha_j \equiv \rho_j(\overline{x}^j_{n_j}) \to \beta_j$.

As a matter of notation, if τ and ρ are environments, we write $\tau \leq \rho$ for $\tau(x) \subseteq \rho(x)$ for all variables x.

Lemma 6.3 *(a) $\rho_j \geq \rho_{j+1}$ ($0 \leq j < k$).*

(b) Let $j < k$ and $\alpha \equiv \overline{c}_t \to \beta_j$ for some sequence $\overline{c}_t$ of length $t < q$. Then, $\alpha \in \rho_0(z_j)$ iff $\alpha \equiv \delta_j$.

Proof: (a) trivially follows from the definition of ρ_j. (b) By definition of ρ_0 we have that $\gamma \in \rho_0(x)$ for some variable x iff γ is one of the following elements of D: $\delta_0, \dots, \delta_k$, β_k, $\overline{\emptyset}_s \to \beta_{k+1}$. To get the conclusion it is sufficient to apply the definition of q-sequence. △

As a matter of notation, for every environment τ, we write

$$\tau[\overline{x}^j_{n_j} := \rho_j(\overline{x}^j_{n_j})] \tag{10}$$

for

$$\tau[x^j_1 := \rho_j(x^j_1)] \dots [x^j_{n_j} := \rho_j(x^j_{n_j})].$$

We now define a sequence $\sigma_0, \dots, \sigma_{k+1}$ of environments as follows:

$$\sigma_0 = \rho_0; \quad \sigma_{j+1} = \sigma_j[\overline{x}^j_{n_j} := \rho_j(\overline{x}^j_{n_j})] \ (0 \leq j \leq k). \tag{11}$$

Lemma 6.4 *(a) $\rho_j \leq \sigma_{j+1} \leq \rho_0$ for every $0 \leq j \leq k$ (in particular, $\sigma_1 = \rho_0$).*

(b) $\delta_j \in \sigma_{j+1}(z_j)$ for all $0 \leq j \leq k$.

Proof: (a) By definition we have $\sigma_1 = \rho_0$. Assume by induction hypothesis that $\rho_{j-1} \leq \sigma_j$. We have to show that $\rho_j \leq \sigma_{j+1}$. By definition $\sigma_{j+1}(x^j_t) = \rho_j(x^j_t)$, for every $1 \leq t \leq n_j$. If z is a variable distinct from x^j_t ($1 \leq t \leq n_j$), then we have $\sigma_{j+1}(z) = \sigma_j(z) \supseteq \rho_{j-1}(z) \supseteq \rho_j(z)$, by induction hypothesis and by $\rho_j \leq \rho_{j-1}$ (see Lem. 6.3).

(b) By definition $\delta_j \in \rho_j(z_j)$. Then the conclusion follows from $\rho_j \leq \sigma_{j+1}$ (see (a)). △

Finally, in the following lemma we show that N_{u_k} and M_{u_k} have different interpretations.

Lemma 6.5 *We have $\alpha_k \in (N_{u_k})^p_{\sigma_k}$ and $\alpha_k \notin (M_{u_k})^p_{\sigma_k}$.*

Proof: Recall that

1. $M_{u_k} \equiv \lambda\overline{x}^k_{n_k}.z_k M^k_1 \dots M^k_{s_k}$;
2. $N_{u_k} \equiv \lambda\overline{x}^k_{n_k}\lambda\overline{y}_r.z_k N^k_1 \dots N^k_{s_k} Q_1 \dots Q_r$;
3. $Q_r \equiv \lambda\overline{w}_s.y_r R_1 \dots R_s$;
4. $\delta_k \equiv \overline{\emptyset}_{s_k+r-1} \to \{\overline{\emptyset}_s \to \beta_{k+1}\} \to \beta_k$;
5. $\alpha_k \equiv \rho_k(\overline{x}^k_{n_k}) \to \rho_k(\overline{y}_r) \to \beta_k$.

As a matter of notation, let

- $\tau \equiv \sigma_k[\overline{x}^k_{n_k} := \rho_k(\overline{x}^k_{n_k})][\overline{y}_r := \rho_k(\overline{y}_r)]$;
- $\overline{Q} \equiv Q_1 \dots Q_r$;
- $\overline{M} \equiv M^k_1 \dots M^k_{s_k}$.
- $\overline{N} \equiv N^k_1 \dots N^k_{s_k}$.
- $\overline{R} \equiv R_1 \dots R_s$.

By the definition of σ_{k+1} we immediately get that $\tau = \sigma_{k+1}[\overline{y}_r := \rho_k(\overline{y}_r)]$. Then we have:

$$\begin{array}{lll}
\alpha_k \in (N_{u_k})^p_{\sigma_k} & \text{iff} & \beta_k \in (z_k)^p_\tau \, \overline{N}^p_\tau \, \overline{Q}^p_\tau \\
& \text{iff} & \beta_k \in (z_k)^p_{\sigma_{k+1}} \overline{N}^p_{\sigma_{k+1}} \overline{Q}^p_\tau, \\
& & \quad \text{by } y_i \neq z_k \text{ not free in } N^k_j \text{ and def. } \tau \\
& \text{iff} & \beta_k \in \{\delta_k\} \overline{N}^p_{\sigma_{k+1}} \overline{Q}^p_\tau, \\
& & \quad \text{by } \sigma_{k+1} \leq \rho_0 \text{ and Lem. 6.3(b)} \\
& \text{iff} & \beta_k \in \{\delta_k\} \overline{\emptyset}_{s_k+r-1} (Q_r)^p_\tau, \\
& & \quad \text{by def. } \delta_k \\
& \text{iff} & \overline{\emptyset}_s \to \beta_{k+1} \in (Q_r)^p_\tau.
\end{array}$$

Finally, we have:

$$\begin{array}{lll}
(Q_r)^p_\tau & = & (\lambda\overline{w}_s.y_r R_1 \dots R_s)^p_\tau, \\
& & \quad \text{by def. } Q_r \text{ (see (9) above)} \\
& = & (\lambda\overline{w}_s.y_r \overline{R})^p_\tau, \\
& & \quad \text{by def. } \overline{R} \\
& = & \{\overline{c}_s \to \sigma : \sigma \in \tau(y_r) \, \overline{R}^p_{\tau[\overline{w}_s := \overline{c}_s]}\}, \\
& & \quad \text{by } y_r \neq w_i \ (i = 1, \dots, s)
\end{array}$$

$$
\begin{array}{ll}
= & \{\overline{c}_s \to \sigma : \sigma \in \rho_k(y_r)\, \overline{R}^p_{\tau[\overline{w}_s:=\overline{c}_s]}\}, \\
 & \quad \text{by } \tau(y_r) = \rho_k(y_r) \\
= & \{\overline{c}_s \to \sigma : \sigma \in \{\overline{\emptyset}_s \to \beta_{k+1}, \beta_k\}\overline{R}^p_{\tau[\overline{w}_s:=\overline{c}_s]}\}, \\
 & \quad \text{by definition of } \rho_k(y_r) \\
\supseteq & \{\overline{c}_s \to \sigma : \sigma \in \{\overline{\emptyset}^s \to \beta_{k+1}\}\overline{R}^p_{\tau[\overline{w}_s:=\overline{c}_s]}\} \\
= & \{\overline{c}_s \to \beta_{k+1} : \overline{c}_s \in D^s\}.
\end{array}
$$

Hence $\alpha_k \in (N_{u_k})^p_{\sigma_k}$, because $\overline{\emptyset}_s \to \beta_{k+1} \in (Q_r)^p_\tau$.

Recall that by (11) $\sigma_{k+1} = \sigma_k[\overline{x}^k := \rho_k(\overline{x}^k)]$.

$$
\begin{array}{lll}
\alpha_k \in (M_{u_k})^p_{\sigma_k} & \text{iff} & \rho_k(\overline{y}_r) \to \beta_k \in (z_k)^p_{\sigma_{k+1}}(\overline{M})^p_{\sigma_{k+1}} \\
 & \text{iff} & \rho_k(\overline{y}_r) \to \beta_k \in \{\delta_k\}(\overline{M})^p_{\sigma_{k+1}}, \\
 & & \quad \text{by } \sigma_{k+1} \le \rho_0 \text{ and Lem. 6.3(b)} \\
 & \text{iff} & \rho_k(\overline{y}_r) \to \beta_k \in \{\delta_k\}\overline{\emptyset}_{s_k}, \\
 & & \quad \text{by def. } \delta_k \\
 & \text{iff} & \rho_k(\overline{y}_r) \to \beta_k = \overline{\emptyset}_{r-1} \to \{\overline{\emptyset}_s \to \beta_{k+1}\} \to \beta_k \\
 & & \quad \text{by def. } \delta_k \\
 & \text{iff} & \rho_k(y_r) = \{\overline{\emptyset}_s \to \beta_{k+1}\}, \\
 & & \quad \text{by def. } \rho_k \\
 & \text{iff} & \{\overline{\emptyset}_s \to \beta_{k+1}, \beta_k\} = \{\overline{\emptyset}_s \to \beta_{k+1}\}.
\end{array}
$$

This last relation is false. Hence $\alpha_k \notin (M_{u_k})^p_{\sigma_k}$. △

The different interpretation of N_{u_k} and M_{u_k} can be propagated upward as shown in the following lemma.

Lemma 6.6 *For every $k > j \ge 0$ we have*

$$\alpha_j \in (N_{u_j})^p_{\sigma_j} \;\Leftrightarrow\; \alpha_{j+1} \in (N_{u_{j+1}})^p_{\sigma_{j+1}}$$

and

$$\alpha_j \in (M_{u_j})^p_{\sigma_j} \;\Leftrightarrow\; \alpha_{j+1} \in (M_{u_{j+1}})^p_{\sigma_{j+1}}.$$

Proof: We prove the result for N_{u_j}. The corresponding proof for M_{u_j} is left to the reader. We recall that $N_{u_j} =_{\lambda\beta} \lambda\overline{x}^j_{n_j}.z_j N^j_1 \dots N^j_{s_j}$, $N_{u_{j+1}} \equiv N^j_{r_j}$ and $\alpha_j \equiv \rho_j(\overline{x}^j_{n_j}) \to \beta_j$. In the following we will write $\overline{N}$ for $N^j_1 \dots N^j_{s_j}$, and $\sigma_j[\overline{x} := \rho_j(\overline{x}^j)]$ for $\sigma_j[\overline{x}^j_{n_j} := \rho_j(\overline{x}^j_{n_j})]$.

$$
\begin{array}{lll}
\alpha_j \in (N_{u_j})^p_{\sigma_j} & \text{iff} & \beta_j \in (z_j)^p_{\sigma_j[\overline{x}:=\rho_j(\overline{x}^j)]}\overline{N}^p_{\sigma_j[\overline{x}:=\rho_j(\overline{x}^j)]} \\
 & & \quad \text{by def. } \alpha_j \\
 & \text{iff} & \beta_j \in (z_j)^p_{\sigma_{j+1}}(\overline{N}^j)^p_{\sigma_{j+1}}, \\
 & & \quad \text{by def. } \sigma_{j+1} \\
 & \text{iff} & \beta_j \in \{\delta_j\}(\overline{N}^j)^p_{\sigma_{j+1}}, \\
 & & \quad \text{by } \sigma_{j+1} \le \rho_0\text{, Lem.6.3(b), 6.4(b)} \\
 & \text{iff} & \beta_j \in \{\delta_j\}\overline{\emptyset}^{r_j-1}(N^j_{r_j})^p_{\sigma_{j+1}}\overline{\emptyset}^{s_j-r_j}, \\
 & & \quad \text{by def. } \delta_j \\
 & \text{iff} & \alpha_{j+1} \in (N_{u_{j+1}})^p_{\sigma_{j+1}}, \\
 & & \quad \text{by } N_{u_{j+1}} \equiv N^j_{r_j} \text{ and def. } \delta_j.
\end{array}
$$

The conclusion of the lemma is now immediate. △

Lemma 6.7 *We have $\alpha_0 \in N^p_{\sigma_0}$, while $\alpha_0 \notin M^p_{\sigma_0}$.*

Proof: Recall that $N \equiv N_{u_0}$ and $M \equiv M_{u_0}$. By applying Lem. 6.6 it is easy to show that that $\alpha_0 \in N^p_{\sigma_0} \Leftrightarrow \alpha_k \in (N_{u_k})^p_{\sigma_k}$, and $\alpha_0 \in M^p_{\sigma_0} \Leftrightarrow \alpha_k \in (M_{u_k})^p_{\sigma_k}$. Then the conclusion is immediate, because by Lem. 6.5 we have that $\alpha_k \in (N_{u_k})^p_{\sigma_k}$ and $\alpha_k \notin (M_{u_k})^p_{\sigma_k}$. △

Appendix

In this appendix we prove Lem. 6.2. Let (D, p) be a graph model and q be an integer greater than 1. We show that there exists a q-sequence in (D, p).

Given $\alpha \in D$, we define the *degree* of α as the smallest natural number $k > 0$ such that there exist finite subsets $b_1, .., b_k$ of D satisfying $\alpha = b_1 \to \dots \to b_k \to \alpha$. If such a natural number does not exist, we say that the degree of α is infinite. The degree of α will be denoted by $deg(\alpha)$.

The proof of Lem. 6.2 is divided into claims.

Claim 6.8 *There exists an element of D whose degree is greater than q.*

Proof: If D has an element whose degree is infinite, we are done. Otherwise, let α_0 be an element of D such that

$$(\forall n > 0)\ \alpha_0 \ne \overline{\emptyset}_n \to \alpha_0. \tag{12}$$

Such an element does exist since otherwise the function $p : D^* \times D \to D$ would not be total.

Let $\alpha_i = \emptyset \to \alpha_{i-1}$ ($i > 0$). In other words, $\alpha_i = \overline{\emptyset}_i \to \alpha_0$. We are going to show that there exists k such that $deg(\alpha_k) > q$.

First remark that, for all j, $deg(\alpha_j) \le deg(\alpha_{j+1})$, since if $\alpha_{j+1} = b_1 \to \dots \to b_k \to \alpha_{j+1}$ then $\alpha_j = b_2 \to \dots \to b_k \to \emptyset \to \alpha_j$.

Hence, either there exist j such that $deg(\alpha_j) > q$, and we are done, or there exist j_0 and n such that $n \le q$ and $deg(\alpha_j) = n$ for all $j \ge j_0$. We are going to show that this latter case is in fact impossible, hence concluding the proof. If j_0 and n are as above, then there exist $c_1, \dots, c_n \subset D$ such that $\alpha_{j_0+n} = c_1 \to \dots \to c_n \to \alpha_{j_0+n}$, i.e. $\overline{\emptyset}_{j_0+n} \to \alpha_0 = c_1 \to \dots \to c_n \to \overline{\emptyset}_{j_0+n} \to \alpha_0$ hence $\alpha_0 = \overline{\emptyset}_n \to \alpha_0$, that contradicts (12). △

Claim 6.9 *There exists a q-sequence.*

Proof: By the above claim there exists an element $\alpha \in D$ whose degree is greater than q. Given a family $\{a_n\}_{n\in\omega}$ of pairwise distinct, finite subsets of D, define $\beta_n = a_n \to \alpha$ ($n \ge 0$). We prove that the sequence ($\beta_n : n \ge 0$) is a q-sequence. By the way of contradiction, assume that $\beta_i = b_1 \to \dots \to b_t \to \beta_j$ ($0 < t < q$) for some i and j, i.e.,

$$a_i \to \alpha = b_1 \to \dots \to b_t \to a_j \to \alpha.$$

It follows that $\alpha = b_2 \to \dots \to b_t \to a_j \to \alpha$. We get a contradiction because the degree of α is greater than q. △

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