The Lattice of Lambda Theories*

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Abstract

Lambda theories are equational extensions of the untyped lambda calculus that are closed under derivation. The set of lambda theories is naturally equipped with a structure of complete lattice, where the meet of a family of lambda theories is their intersection, and the join is the least lambda theory containing their union. In this paper we study the structure of the lattice of lambda theories by universal algebraic methods. We show that nontrivial quasi-identities in the language of lattices hold in the lattice of lambda theories, while every nontrivial lattice identity fails in the lattice of lambda theories if the language of lambda calculus is enriched by a suitable finite number of constants. We also show that there exists a sublattice of the lattice of lambda theories which satisfies: (i) a restricted form of distributivity, called meet semidistributivity; and (ii) a nontrivial identity in the language of lattices enriched by the relative product of binary relations.

Keywords: lambda calculus, lattice of lambda theories, lattice identities, commutator, lambda abstraction algebras

1 Introduction

Lambda theories are equational extensions of the untyped λ-calculus that are closed under derivation. They arise by syntactical and semantic considerations: a lambda theory may correspond to an operational semantics of the lambda calculus, as well as it may be induced by a model of lambda calculus through the kernel congruence relation of the interpretation function (see e.g. [4] [6]). The set of lambda theories is naturally equipped with a structure of complete lattice (see [4, Chapter 4]), where the meet of a family of lambda theories is their intersection, and the join is the least lambda theory containing their union. The bottom element of this lattice is the minimal lambda theory $\lambda\beta$, while the top element is the inconsistent lambda theory. The lattice of lambda theories, hereafter denoted by λT, has a very rich and complex structure. For example, every countable partially ordered set embeds into λT by an order-preserving map, and every interval of λT, whose bounds are recursively enumerable lambda theories, has a continuum of elements (see Visser [33]).

The lattice of lambda theories is isomorphic to the congruence lattice of the term algebra of the minimal lambda theory $\lambda\beta$. This remark is the starting point for studying the structure

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of λT by universal algebraic methods. In [28] Salibra has shown that the variety (i.e., equational class) generated by the term algebra of $\lambda\beta$ is axiomatized by the finite schema of identities characterizing *lambda abstraction algebras* (LAA's). The equational theory of lambda abstraction algebras, introduced by Pigozzi and Salibra in [23] and [24], is intended as an alternative to combinatory logic in this regard since it is a first-order algebraic description of lambda calculus, which keeps the lambda notation and hence all the functional intuitions. Lambda abstraction algebras are axiomatized by the equations that hold between contexts of the lambda calculus (i.e., λ-terms with 'holes' [4, Def. 14.4.1]), as opposed to lambda terms with free variables. The essential feature of a context is that a free variable in a λ-term may become bound when we substitute it for a 'hole' within the context. Thus, 'holes' play the role of algebraic variables, and the contexts are the algebraic terms in the similarity type of lambda abstraction algebras. Hence the explicit finite equational axiomatization for the variety of lambda abstraction algebras provides also an explicit axiomatization of the identities between contexts satisfied by the term algebra of the minimal lambda theory $\lambda\beta$. Moreover, any lattice identity or quasi-identity that holds in the congruence lattices of all LAA's must necessarily hold in the congruence lattice of the term algebra of $\lambda\beta$ (because this algebra is an LAA) and then it holds in the lattice of lambda theories.

In order to fully appreciate the results of the paper it is important that the reader understands the rather peculiar relation between the lattice λT of lambda theories and the variety LAA. In [28] Salibra has shown that, for every variety of LAA's, there exists exactly one lambda theory whose term algebra generates the variety. Thus, the properties of an arbitrary lambda theory can be studied by means of the variety of LAA's generated by its term algebra. This result implies that the lattice λT of lambda theories is isomorphic to the lattice of the equational theories of LAA's, and dual isomorphic to the lattice of varieties of LAA's.

It is well known that the structure of an algebra is affected by the shape of its congruence lattice. For example, it is easy to show that if a group G has three normal subgroups (recall that normal subgroups of a group correspond to congruences), any two of which intersect in the trivial subgroup and generate the whole G, then G must be Abelian. Put in another way, if the finite lattice $\mathbf{M}_3$ (consisting of three atoms, a zero and one, and no other elements) is a 0-1-sublattice of the lattice of congruences of G, then G is Abelian. Alternatively, the concept of Abelian group, and other important concepts, can be defined in terms of the commutator operation $[M,N] = Sg(\{a^{-1}b^{-1}ab : a \in M, b \in N\})$ on normal subgroups: G is Abelian if, and only if, $[G,G]$ is the trivial subgroup. The extension of the commutator to algebras other than groups is due to the pioneering papers of Smith [31] and Hagemann-Hermann [11]. General commutator theory has to do with a binary operation, the commutator, that can be defined on the set of congruences of any algebra. The operation is very well behaved in congruence modular varieties (see Freese-McKenzie [9] and Gumm [10]). In [27] Salibra has recently shown that the variety of lambda abstraction algebras is not congruence modular, as a consequence it is not possible to apply to it the nice theory of commutator developed for congruence modular varieties. However, Lipparini [19] [20] and Kearnes-Szendrei [13] have recently shown that under very weak hypotheses the commutator proves also useful in studying algebras without congruence modularity. Their deep results essentially connect: (a) identities or quasi-identities in the language of lattices satisfied by congruence lattices; (b) properties of the commutator; and (c) Mal'cev conditions, that characterize properties in varieties by the existence of certain terms involved in certain identities. We believe that this new theory can be fruitfully applied to lambda calculus.

We use the isomorphism between the lattice of lambda theories and the congruence lattice of the term algebra of $\lambda\beta$ to apply commutator theory in the present context. We show that the commutator of any lambda theory with the inconsistent lambda theory is the given lambda theory. This is a nontrivial relation, which implies a lattice quasi-identity on the lattice of lambda theories, that, among other things, has the ET and the Zipper conditions as consequences. These latter conditions are known to hold in any lattice of equational theories, so that they are also satisfied by λT as a consequence of the isomorphism, described in [28], between λT and the lattice of the equational theories of LAA's.

An identity in the language of lattices (a lattice identity, for short) is trivial if it holds in every lattice and nontrivial otherwise. We conjecture that the lattice λT does not satisfy any nontrivial lattice identity. However, we show that, for every nontrivial lattice identity e, there exists a natural number n such that e fails in the lattice of lambda theories in a language of λ-terms with n constants. In a more general result, we show that the congruence lattices of all LAA's satisfy a lattice identity e if, and only if, e is trivial. In other words, for every nontrivial lattice identity e, there exists a lambda abstraction algebra (not necessarily equal to the term algebra of $\lambda\beta$) whose congruence lattice does not satisfy e.

In the last section of the paper we analyze a sublattice of λT satisfying nice lattice properties. We show that there exists a lambda theory $\mathcal{J}$, whose term algebra admits a semilattice term operation. The consistency of $\mathcal{J}$ is obtained by model-theoretic means by defining an intersection type system (see [5], [7], [2]) for synthesizing a model of the lambda calculus. The lattice of all lambda theories extending $\mathcal{J}$ is shown to satisfy a restricted form of distributivity, called meet semidistributivity, and a nontrivial congruence identity, i.e., an identity in the language of lattices enriched by the relative product of binary relations.

The consistency of the theory $\mathcal{J}$ has been proved by M. Dezani-Ciancaglini with the help of the first author. The authors are grateful to M. Dezani-Ciancaglini for her contribution to this paper.

2 Basic notions and notations

Unless otherwise stated we shall use the terminology of Barendregt [4] for lambda calculus and that of Salibra [28] for lambda abstraction algebras. For the general theory of universal algebras the reader may consult McKenzie, McNulty and Taylor [21]. For commutator theory the reader may consult Freese and McKenzie [9], and Gumm [10].

2.1 Lambda calculus

The set $\Lambda_I(C)$ of λ-terms of lambda calculus over an infinite set I of variables and a set C of constants is constructed as usual: every variable $x \in I$ and every constant $c \in C$ is a λ-term; if t and s are λ-terms, then so are (st) and $\lambda x.t$ for each variable $x \in I$. We will write Λ_I for $\Lambda_I(\emptyset)$, the set of λ-terms without constants.

The symbol $\equiv$ denotes syntactic equality. The following are some well-known λ-terms:

$$\mathbf{i} \equiv \lambda x.x; \quad \mathbf{s} \equiv \lambda xyz.xz(yz); \quad \mathbf{k} \equiv \mathbf{T} \equiv \lambda xy.x; \quad \mathbf{F} \equiv \lambda xy.y; \quad \Omega \equiv (\lambda x.xx)(\lambda x.xx).$$

A *compatible λ-relation* $\mathcal{T}$ over $\Lambda_I(C)$ is any set of equations between λ-terms that is closed under the following two rules:

(i) If $t_1 = u_1 \in \mathcal{T}$ and $t_2 = u_2 \in \mathcal{T}$, then $t_1t_2 = u_1u_2 \in \mathcal{T}$;

(ii) If $t = u \in \mathcal{T}$ then $\lambda x.t = \lambda x.u \in \mathcal{T}$ for every variable $x \in I$.

We will write either $\mathcal{T} \vdash t = s$ or $t =_{\mathcal{T}} s$ for $t = s \in \mathcal{T}$.

If $\bar{t}$ and $\bar{u}$ are two sequences of terms, i.e. $\bar{t} = t_1 \dots t_n$ and $\bar{u} = u_1 \dots u_n$, we write $\mathcal{T} \vdash \bar{t} = \bar{u}$ for $\mathcal{T} \vdash t_i = u_i$ for every $i = 1, \dots, n$.

As a matter of notation, $\mathcal{T} \circ \mathcal{S}$ denotes the compatible λ-relation defined as follows: $\mathcal{T} \circ \mathcal{S} \vdash t = s$ if, and only if, there exists a λ-term u such that $\mathcal{T} \vdash t = u$ and $\mathcal{S} \vdash u = s$.

A *lambda theory* $\mathcal{T}$ (over $\Lambda_I(C)$) is any compatible λ-relation which is an equivalence relation and includes (α)- and (β)-conversion. The set of all lambda theories over $\Lambda_I(C)$ is naturally equipped with a lattice structure, with meet defined as set theoretical intersection. The join of two lambda theories $\mathcal{T}$ and $\mathcal{S}$ is the least equivalence relation including $\mathcal{T} \cup \mathcal{S}$.

The lattice of lambda theories over Λ_I will be denoted by λT. $\lambda\beta$ denotes the minimal lambda theory over Λ_I, while $1_{\lambda T}$ is the inconsistent lambda theory. The lambda theory $\mathcal{H}$, generated by equating all the unsolvable λ-terms, is consistent by [4, Thm. 16.1.3] and admits a unique maximal consistent extension $\mathcal{H}^*$ [4, Thm. 16.2.6]. A lambda theory $\mathcal{T}$ is called *sensible* [4, Def. 4.1.7(ii)] if $\mathcal{H} \subseteq \mathcal{T}$, while it is *semisensible* [4, Def. 4.1.7(iii)] if $\mathcal{T} \not\vdash t = u$ whenever t is solvable and u is unsolvable. By [4, Lemma 17.1.1] $\mathcal{T}$ is semisensible iff $\mathcal{T} \subseteq \mathcal{H}^*$.

2.2 Algebras

In this section we recall some algebraic notions that will be used in the following.

A *lattice order* is a partial order on a nonempty set L with respect to which every subset of L with exactly two elements has both a least upper bound (join) and a greatest lower bound (meet). The join and meet of two elements a and b are denoted respectively by $a + b$ and $a \cdot b$ (as usual, we will write ab for $a \cdot b$ if there is no danger of confusion). If $a \in L$, then the interval $[a) = \{b : a \leq b\}$ is a sublattice of L. A lattice L is *complete* if every subset of L admits a least upper bound and a greatest lower bound.

Let $\mathbb{N}$ be the set of natural numbers. A *similarity type* of algebras is constituted by an $\mathbb{N}$-indexed family $\mathcal{F}$ of function symbols. If $f \in \mathcal{F}_n$ we say that f is an n-ary function symbol. An *algebra* $\mathbf{A}$ of type $\mathcal{F}$ is determined by a nonempty set A together with a finitary operation $f^{\mathbf{A}} : A^n \to A$ for every $f \in \mathcal{F}_n$. The set A is called the *universe* (or *underlying set*) of $\mathbf{A}$.

An algebra is *trivial* if its universe is a singleton set.

Given an algebra $\mathbf{A}$ of type $\mathcal{F}$, a binary relation $\vartheta \subseteq A \times A$ is *compatible* if for all $f \in \mathcal{F}_n$ and all $a_i, b_i \in A$, we have

$$a_1 \vartheta b_1, ..., a_n \vartheta b_n \Rightarrow \; f^{\mathbf{A}}(a_1, \ldots, a_n) \vartheta f^{\mathbf{A}}(b_1, \ldots, b_n).$$

A *congruence* ϑ of $\mathbf{A}$ is just a compatible equivalence relation, that is, the kernel of some homomorphism. Con$\mathbf{A}$, the set of all congruences of the algebra $\mathbf{A}$, is naturally equipped with a lattice structure, with meet defined as set theoretical intersection. Con$\mathbf{A}$ is a complete lattice: we shall use $\sqcap$ and $\sqcup$ for infinite meets and joins. A congruence ϑ is *trivial* if ϑ is either the largest congruence $1_{\mathbf{A}}$ of $\mathbf{A}$ or the least congruence $0_{\mathbf{A}}$ of $\mathbf{A}$, where $1_{\mathbf{A}}$ and $0_{\mathbf{A}}$ denote, respectively, $A \times A$ and $\{(a, a) \mid a \in A\}$. If $\overline{a} = a_1 \ldots a_n$, $\overline{b} = b_1 \ldots b_n$ and $\sigma \in$ **Con** $\mathbf{A}$ we will write: $\overline{a}\sigma\overline{b} \iff a_i \sigma b_i \quad$ for every $i = 1, \ldots, n$.

Given two congruences σ and τ on the algebra $\mathbf{A}$, we can form the relative product:

$$\sigma \circ \tau = \{(a, c) \mid a\sigma b\tau c, \text{ for some } b \in A\},$$

which is a compatible relation on $\mathbf{A}$, but not necessarily a congruence.

A nonempty class K of algebras of type $\mathcal{F}$ is called a *variety* if it is closed under subalgebras, homomorphic images and direct products. By Birkhoff's theorem (see [21]) a class of algebras is a variety if, and only if, it is an equational class (that is, it is axiomatized by a set of equations). A variety K of algebras is generated by an algebra $\mathbf{A} \in K$ if every equation satisfied by $\mathbf{A}$ is also satisfied by every algebra in K.

We denote by $Eq(K)$ the set of equations satisfied by all the algebras in a variety K.

A set E of equations of type $\mathcal{F}$ is called an *equational theory* if there exists a variety K of algebras of type $\mathcal{F}$ such that $E = Eq(K)$.

Let K be a variety. The set $Th(K)$ of all equational theories including $Eq(K)$ is naturally equipped with a complete lattice structure, with meet defined as set theoretical intersection. $Eq(K)$ is the least equational theory of this lattice, while the set constituted by all the equations of type $\mathcal{F}$ is the largest equational theory of this lattice.

The class $Sv(K)$ of all subvarieties of K (i.e., varieties H such that $H \subseteq K$) is also naturally equipped with a complete lattice structure, with meet defined as set theoretical intersection. The variety K is the largest element of this lattice, while the singleton set constituted by the trivial algebra is the least element of this lattice. It is a well known result that the lattice $Sv(K)$ of the subvarieties of a variety K is dually isomorphic to the lattice $Th(K)$ of the equational theories including $Eq(K)$.

Let K be a class of algebras of type $\mathcal{F}$, $\mathbf{U}$ be an algebra of the same type and X be a subset of $\mathbf{U}$. We say that $\mathbf{U}$ has the *universal mapping property for K over X* iff, for every $\mathbf{A} \in K$ and for every mapping $g : X \to A$, there is a homomorphism $f : \mathbf{U} \to \mathbf{A}$ that extends g (i.e. $f(x) = g(x)$ for every $x \in X$). We say that $\mathbf{U}$ is *free for K over X* iff $\mathbf{U} \in K$, $\mathbf{U}$ is generated by X and $\mathbf{U}$ has the universal mapping property for K over X. If $\mathbf{U}$ is free in K over X, then X is called a *free generating set for U*, and $\mathbf{U}$ is said to be *freely generated by X*.

An identity in the binary symbols $\{\cdot, +, \circ\}$ is called a *congruence identity*, while an identity in the language $\{\cdot, +\}$ of lattices is called a *lattice identity.* We interpret the variables of a congruence (lattice) identity as congruence relations, and for arbitrary binary relations γ and δ we interpret $\gamma + \delta$ as the congruence relation generated by the union of the two relations, $\gamma \cdot \delta$ as the intersection and $\gamma \circ \delta$ as the composition of the two relations (as usual, we will write $\gamma\delta$ for $\gamma \cdot \delta$). We say that a variety $\mathcal{V}$ *satisfies a congruence (lattice) identity* if it holds in all congruence lattices of members of $\mathcal{V}$. A congruence identity is *trivial* if it holds in the congruence lattice of any algebra, while a lattice identity is *trivial* if it holds in every lattice. Any congruence identity $u = v$ is equivalent to the pair of inclusions $u \leq v$ and $v \leq u$. Conversely, the congruence inclusion $u \leq v$ is equivalent to the congruence identity $u = uv$. Varieties are often characterized in terms of congruence identities. For example, the variety of groups is congruence 2-permutable, i.e., it satisfies the following congruence identity

$$\gamma \circ \delta = \delta \circ \gamma,$$

while it is was shown in [27] that the variety of lambda abstraction algebras is not congruence modular, i.e., it does not satisfy the following lattice identity (modular law):

$$\gamma(\delta + \gamma\rho) = \gamma\delta + \gamma\rho.$$

A *quasi-identity* is an implication with an equational conclusion and a finite number of equational premises. For example, the quasi-identity

$$\gamma\rho = \gamma\delta \;\Rightarrow\; \gamma\rho = \gamma(\rho + \delta),$$

called meet semidistributivity, holds in the congruence lattice of an algebra if the conclusion $\gamma\rho = \gamma(\rho+\delta)$ is satisfied by all the congruences γ, ρ and δ which satisfy the premise $\gamma\rho = \gamma\delta$.

2.3 Lambda abstraction algebras

Let I be an infinite set. The similarity type of *lambda abstraction algebras of dimension I* is constituted by a binary operation symbol "$\cdot$" formalizing application, an unary operation symbol "λx" for every $x \in I$ formalizing functional abstraction, and a constant symbol (i.e., nullary operation symbol) "x" for every $x \in I$. The elements of I are the variables of lambda calculus although in their algebraic transformation they no longer play the role of variables in the usual sense. In the remaining part of the paper we will refer to them as *λ-variables.* The actual variables of the lambda abstraction theory will be referred to as *context variables* and denoted by the Greek letters ξ, ν, and μ, possibly with subscripts.

The terms of the language of lambda abstraction theory are called *contexts.* They are constructed in the usual way: every λ-variable $x \in I$ and context variable ξ is a context; if t and s are contexts, then so are $t \cdot s$ and $\lambda x(t)$.

Because of their similarity to the terms of the lambda calculus we use the standard notational conventions of the latter. The application operation symbol "$\cdot$" is normally omitted, and the application of t and s is written as juxtaposition ts. When parentheses are omitted, association to the left is assumed. The left parenthesis delimiting the scope of a lambda abstraction is replaced with a period and the right parenthesis is omitted. For example, $\lambda x(ts)$ is written $\lambda x.ts$. Successive λ-abstractions $\lambda x \lambda y \lambda z \dots$ are written $\lambda xyz \dots$. Note that contexts without any occurrence of context variables coincide with ordinary terms of the lambda calculus.

Our notion of a context coincides with the notion of *context* defined in Barendregt [4, Def. 14.4.1]; our context variables correspond to Barendregt's notion of a 'hole'. The main difference between Barendregt's notation and our's is that 'holes' are denoted here by Greek letters $\xi, \mu, \dots$, while in Barendregt's book by $[\,], [\,]_1, \dots$. The essential feature of a context is that a free λ-variable in a λ-term may become bound when we substitute it for a 'hole' within the context. For example, if $C(\xi) \equiv \lambda x.x(\lambda y.\xi)$ is a context, in Barendregt's notation: $C([\,]) \equiv \lambda x.x(\lambda y.[\,])$, and $t \equiv xy$ is a λ-term, then $C(t) \equiv \lambda x.x(\lambda y.xy)$.

A lambda theory over the language $\Lambda_I(C)$ has a natural algebraic interpretation. It is a congruence on the algebra

$$\mathbf{\Lambda}_I(C) := (\Lambda_I(C),\ \cdot^{\mathbf{\Lambda}_I(C)},\ \lambda x^{\mathbf{\Lambda}_I(C)},\ x^{\mathbf{\Lambda}_I(C)})_{x \in I}, \tag{1}$$

where $\Lambda_I(C)$ is the set of the λ-terms over the infinite set I of λ-variables and the set C of constants, and, for all $s, t \in \Lambda_I(C)$,

$$s \cdot^{\mathbf{\Lambda}_I(C)} t = (st); \quad \lambda x^{\mathbf{\Lambda}_I(C)}(t) = \lambda x.t; \quad x^{\mathbf{\Lambda}_I(C)} = x.$$

We denote by $\mathbf{\Lambda}^{\mathcal{T}}_{I,C}$ the quotient of $\mathbf{\Lambda}_I(C)$ by $\mathcal{T}$ and call it the *term algebra* of the lambda theory $\mathcal{T}$. As a matter of notation, we write $\mathbf{\Lambda}_I$ ($\mathbf{\Lambda}^{\mathcal{T}}_I$, respectively) for $\mathbf{\Lambda}_I(\emptyset)$ ($\mathbf{\Lambda}^{\mathcal{T}}_{I,\emptyset}$).

We say that $\mathcal{T}$ satisfies an identity between contexts $t(\xi_1, \dots, \xi_n) = u(\xi_1, \dots, \xi_n)$ if the term algebra of $\mathcal{T}$ satisfies it; i.e., if all the instances of the above identity, obtained by substituting λ-terms for context variables in it, fall within the lambda theory: $\mathcal{T} \vdash t(s_1, \cdots, s_n) = u(s_1, \cdots, s_n)$, for all λ-terms $s_1, \cdots, s_n$. For example, every lambda theory satisfies the identity $(\lambda x.x)\xi = \xi$ because $\lambda\beta \vdash (\lambda x.x)s = s$ for every λ-term s.

Lambda abstraction algebras are meant to axiomatize those identities between contexts that are valid for the lambda calculus and were studied by Goldblatt, Pigozzi and Salibra in a series of papers [23, 24, 25, 27, 28, 29]. We now give the formal definition of a lambda abstraction algebra.

Definition 2.1 *Let I be an infinite set. By a* **lambda abstraction algebra of dimension** *I we mean an algebraic structure of the form*

$$\mathbf{A} := (A, \cdot^{\mathbf{A}}, \lambda x^{\mathbf{A}}, x^{\mathbf{A}})_{x \in I}$$

satisfying the following identities between contexts, for all $x, y, z \in I$:

(β_1) $(\lambda x.x)\xi = \xi$;

(β_2) $(\lambda x.y)\xi = y, \quad x \neq y$;

(β_3) $(\lambda x.\xi)x = \xi$;

(β_4) $(\lambda xx.\xi)\mu = \lambda x.\xi$;

(β_5) $(\lambda x.\xi\mu)\nu = (\lambda x.\xi)\nu((\lambda x.\mu)\nu)$;

(β_6) $(\lambda xy.\mu)((\lambda y.\xi)z) = \lambda y.(\lambda x.\mu)((\lambda y.\xi)z), \quad x \neq y, z \neq y$;

(α) $\lambda x.(\lambda y.\xi)z = \lambda y.(\lambda x.(\lambda y.\xi)z)y, \quad z \neq y$.

I is called the **dimension set** *of* $\mathbf{A}$. $\cdot^{\mathbf{A}}$ *is called* **application** *and* $\lambda x^{\mathbf{A}}$ *is called* **λ-abstraction** *with respect to* x.

The class of lambda abstraction algebras of dimension I is denoted by LAA_I and the class of all lambda abstraction algebras of any dimension by LAA. We also use LAA_I as shorthand for the phrase "lambda abstraction algebra of dimension I", and similar for LAA.

LAA_I is a variety (i.e., equational class) for every dimension set I, and therefore it is closed under the formation of subalgebras, homomorphic (in particular isomorphic) images, and Cartesian products.

In [25, 28, 29] it was shown that the term algebra of every lambda theory is a lambda abstraction algebra and that every variety of lambda abstraction algebras is generated by the term algebra of a suitable lambda theory over the language Λ_I. In particular, the term algebra of the least lambda theory $\lambda\beta$ generates the variety LAA_I. Hence the explicit finite equational axiomatization for the variety of lambda abstraction algebras provides also an explicit axiomatization of the identities between contexts satisfied by the term algebra of the least lambda theory $\lambda\beta$. The variety of LAA_I's generated by the term algebra $\mathbf{\Lambda}_I^{\mathcal{T}}$ will be denoted by $\mathsf{LAA}_I^{\mathcal{T}}$.

We would like to explicitly mention at this point that the equational theory axiomatized by $(\beta_1) - (\beta_6)$ and (α) is a conservative extension of lambda beta-calculus: for any two λ-terms $t, u \in \Lambda_I$, the identity $t = u$ between λ-terms is a logical consequence of $(\beta_1) - (\beta_6)$ and (α) (in symbols, $\mathsf{LAA}_I \models t = u$) if, and only if, $t = u$ is derivable in the lambda beta-calculus. This can be immediately inferred from the fact, stated previously, that LAA_I is generated as a variety by the term algebra of the lambda theory $\lambda\beta$.

The following theorem will be repeatedly used in the remaining part of the paper. We recall that the algebra $\mathbf{\Lambda}_I$ is defined in this section (see (1) above), while the lattices of the equational theories and of the subvarieties of a given variety are defined in Section 2.2.

Theorem 2.2 ([28]) *The following lattices are isomorphic.*

1. *The lattice* λT *of lambda theories over* Λ_I*;*
2. *The interval sublattice* $[\lambda\beta) = \{\mathcal{T} : \lambda\beta \subseteq \mathcal{T}\}$ *of the congruence lattice of* $\mathbf{\Lambda}_I$*;*
3. *The congruence lattice of the term algebra* $\mathbf{\Lambda}_I^{\lambda\beta}$*;*
4. *The lattice* $Th(\mathsf{LAA}_I)$ *of the equational theories including* $Eq(\mathsf{LAA}_I)$*;*
5. *The dual lattice of the lattice* $Sv(\mathsf{LAA}_I)$ *of the subvarieties of* LAA_I*.*

The isomorphism of (1) and (5) is one of the main results in [28], while the isomorphisms of (1)-(2)-(3) and of (4)-(5) are trivially verified. From Thm. 2.2 it follows that any lattice identity or quasi-identity that holds in the congruence lattices of all LAA_I's must necessarily hold in the lattice of lambda theories.

2.3.1 Locally finite LAA's

An element a of an LAA_I $\mathbf{A}$ is said to be *algebraically dependent* on $x \in I$ if $(\lambda x.a)z \neq a$ for some $z \in I$; otherwise a is *algebraically independent* of x. The set of all $x \in I$ such that a is algebraically dependent on x is called the *dimension set* of a and is denoted by Δa. An element a is *finite* (*infinite*) *dimensional* if Δa is finite (infinite). An element a is called *zero-dimensional* if $\Delta a = \emptyset$.

If t is a λ-term and $\mathbf{A}$ is an LAA_I, then $t^{\mathbf{A}}$ will denote the value of t in $\mathbf{A}$ when each λ-variable x occurring in t is interpreted as $x^{\mathbf{A}}$. The dimension set of $t^{\mathbf{A}}$ is a subset of the set of free λ-variables of t (see [24, Prop. 1.7]).

There is a strong connection between the lambda theories and the subclass of LAA's whose elements are finite dimensional according to the following definition.

Definition 2.3 *A lambda abstraction algebra* $\mathbf{A}$ *is* **locally finite** *if every $a \in A$ is of finite dimension.*

The class of locally finite LAA_I's is denoted by LFA_I, which is also used as shorthand for the phrase "locally finite lambda abstraction algebra of dimension I".

The term algebra $\mathbf{\Lambda}^{\mathcal{T}}_{I,C}$ of a lambda theory $\mathcal{T}$ over the language $\Lambda_I(C)$ is locally finite. This is a direct consequence of the trivial fact that every λ-term is a finite string of symbols and hence contains only finitely many λ-variables.

The following two results characterize the locally finite LAA_Is and those congruences on $\mathbf{\Lambda}_I(C)$ that are lambda theories.

Proposition 2.4 ([28]) *Let I be an infinite set. A congruence θ on $\mathbf{\Lambda}_I(C)$ is a lambda theory over the language $\Lambda_I(C)$ if, and only if, the following two conditions are satisfied:*

(i) The quotient algebra $\mathbf{\Lambda}_I(C)/\theta$ is an LAA_I;

(ii) $(\lambda x.c)y \;\theta\; c$ for all $c \in C$ and all $x, y \in I$, i.e., the equivalence class c/θ of every element $c \in C$ is a zero-dimensional element of $\mathbf{\Lambda}_I(C)/\theta$.

Proposition 2.5 [25, Prop. 2.4] *Let I be an infinite set. An algebra $\mathbf{A}$ in the similarity type of lambda abstraction algebras of dimension I is (isomorphic to) the term algebra of a lambda theory if, and only if, it is an LFA_I.*

2.3.2 Lambda algebras and LFA's

There have been several attempts to reformulate the lambda calculus as a purely algebraic theory. The earliest, and best known, algebraic models are the combinatory algebras of Curry [8] and Schönfinkel [30]. They discovered that a formal system of combinators having the same expressive power of lambda calculus can be based on only two primitive combinators. Curry also specified (by a considerably less natural set of axioms) a purely equational subclass of combinatory algebras, the *λ-algebras*, that he viewed as algebraic models of the lambda calculus (see [8], [4, Def. 5.2.5], [28, Section 4]).

By the *zero-dimensional subreduct* of an LAA_I $\mathbf{A}$ we mean the algebra

$$\mathbf{Zd\,A} = (\mathrm{Zd}\,\mathbf{A}, \cdot^A, \mathbf{k}^A, \mathbf{s}^A),$$

where $\mathrm{Zd}\,\mathbf{A} = \{\, a \in A : \Delta a = \emptyset \,\}$ is the set of zero-dimensional elements of $\mathbf{A}$, $\mathbf{k}^A = (\lambda xy.x)^{\mathbf{A}}$ and $\mathbf{s}^A = (\lambda xyz.xz(yz))^{\mathbf{A}}$.

Theorem 2.6 ([25]) *The zero-dimensional subreduct* $\mathbf{Zd\,A}$ *of an LAA_I is a λ-algebra.*

The opposite is also true: every λ-algebra is isomorphic to a zero-dimensional subreduct of an LFA_I. This leads to a categorical equivalence between the category of λ-algebras and the category of LFA_I's described in Thm. 2.7 below.

Let $\mathbf{C}$ be a λ-algebra. We denote by $\mathbf{C}[I]$ the *free extension of* $\mathbf{C}$ *by* I in the variety of combinatory algebras. $\mathbf{C}[I]$ is an expansion of $\mathbf{C}$ defined up to isomorphism by the following universal mapping properties: ($C[I]$ is the universe of $\mathbf{C}[I]$.) (1) $I \subseteq C[I]$; (2) $\mathbf{C}[I]$ is a combinatory algebra; (3) for every homomorphism $h : \mathbf{C} \to \mathbf{A}$ from $\mathbf{C}$ into a combinatory algebra $\mathbf{A}$ and every mapping $g : I \to A$ there exists a unique homomorphism $f : \mathbf{C}[I] \to \mathbf{A}$ extending both h and g. A concrete description of $\mathbf{C}[I]$ as a quotient of the algebra of the combinatory polynomials over C (with λ-variables from I) may be found on page 109 of [22]. In [25] it was shown that the free extension $\mathbf{C}[I]$ of a λ-algebra $\mathbf{C}$ (by a set I) in the variety of combinatory algebras can be turned into a LFA_I whose zero-dimensional subreduct is isomorphic to $\mathbf{C}$.

Theorem 2.7 [25, Thm. 3.2] *Let $\mathcal{F}$ be the functor mapping every LFA_I $\mathbf{A}$ into its zero-dimensional subreduct* $\mathbf{Zd}\,\mathbf{A}$ *and every homomorphism $f : \mathbf{A} \to \mathbf{B}$ of LFA_I's into its restriction $f_{|\mathrm{Zd}\mathbf{A}} : \mathbf{Zd}\,\mathbf{A} \to \mathbf{Zd}\,\mathbf{B}$. Let $\mathcal{G}$ be the functor mapping every λ-algebra $\mathbf{C}$ into its free extension $\mathbf{C}[I] \in \mathsf{LFA}_I$ and every homomorphism $g : \mathbf{C} \to \mathbf{D}$ of λ-algebras into the unique homomorphism $h : \mathbf{C}[I] \to \mathbf{D}[I]$ extending both g and the identity map on I (if g is one-one and/or onto, so is its extension h). Then the functors $\mathcal{F}$ and $\mathcal{G}$ make the category of λ-algebras and the category of LFA_I's equivalent.*

It follows that, for every LFA_I $\mathbf{A}$, the congruence lattice of $\mathbf{A}$ is isomorphic to the congruence lattice of $\mathbf{Zd}\,\mathbf{A}$: for every $a, b \in A$ such that $\Delta(a) \cup \Delta(b) = \{x_1, \dots, x_n\}$, and for every congruence $\vartheta \in \mathrm{Con}\mathbf{A}$, we have that

$$a \;\vartheta\; b \;\Leftrightarrow\; \lambda x_1 \dots x_n.a \;\vartheta\; \lambda x_1 \dots x_n.b.$$

3 Lattice quasi-identities

In this section we apply techniques of universal algebra to study the properties of the lattice λT of lambda theories. We use the isomorphism between the lattice λT and the congruence lattice of the term algebra $\mathbf{\Lambda}_I^{\lambda\beta}$ to define the binary commutator on the set of lambda theories. We show that it has a good behavior if one of its arguments is the inconsistent lambda theory (see Thm. 3.2 below). As a consequence, we get that the lattice λT satisfies a condition (in the form of quasi-identity) that, among other things, implies the ET and Zipper conditions. However, these latter conditions are known to hold in any lattice of equational theories, so that their validity in λT is also a consequence of the isomorphism between λT and the lattice of equational theories expressed in Thm. 2.2 above.

If $\mathcal{T}$ and $\mathcal{S}$ are lambda theories, then $M(\mathcal{T}, \mathcal{S})$ is the set of all 2×2 matrices M of the form:

$$M = \begin{pmatrix} t(\overline{s}_1, \overline{u}_1) & t(\overline{s}_1, \overline{u}_2) \\ t(\overline{s}_2, \overline{u}_1) & t(\overline{s}_2, \overline{u}_2) \end{pmatrix}$$

where $\overline{s}_1, \overline{s}_2 \in (\Lambda_I)^n$, $\overline{u}_1, \overline{u}_2 \in (\Lambda_I)^m$, for some $n, m \geq 0$, t is any $m + n$-ary context, and $\mathcal{T} \vdash \overline{s}_1 = \overline{s}_2$, $\mathcal{S} \vdash \overline{u}_1 = \overline{u}_2$. That is, if in matrix M we shift along a line then we shift modulus $\mathcal{S}$, if we shift along a column we shift modulus $\mathcal{T}$.
If $M = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix}$ and $M' = \begin{pmatrix} t'_1 & t'_2 \\ t'_3 & t'_4 \end{pmatrix}$ are matrices then $M =_{\lambda\beta} M'$ means that $\lambda\beta \vdash t_i = t'_i$ for $i = 1, \dots, 4$.
If $\mathcal{T}$, $\mathcal{S}$ and $\mathcal{G}$ are lambda theories, we say that $\mathcal{T}$ **centralizes** $\mathcal{S}$ **modulo** $\mathcal{G}$ (see e.g. [9]), in symbols $C(\mathcal{T}, \mathcal{S}; \mathcal{G})$, if and only if, for every matrix M such that:

$$M = \begin{pmatrix} t & u \\ s & w \end{pmatrix} \in M(\mathcal{T}, \mathcal{S})$$

we have:

$$\mathcal{G} \vdash t = u \;\Rightarrow\; \mathcal{G} \vdash s = w.$$

The set of all lambda theories $\mathcal{G}$ such that $C(\mathcal{T}, \mathcal{S}; \mathcal{G})$ is not empty and closed under arbitrary intersection (see [9]). The **commutator** $[\mathcal{T}, \mathcal{S}]$ of the lambda theories $\mathcal{T}$ and $\mathcal{S}$ is defined as the least lambda theory $\mathcal{G}$ satisfying $C(\mathcal{T}, \mathcal{S}; \mathcal{G})$. Notice that $\mathcal{T}$ always centralizes $\mathcal{S}$ modulo $\mathcal{T}\mathcal{S}$ (recall that $\mathcal{T}\mathcal{S}$ is a notation for $\mathcal{T} \cap \mathcal{S}$), so that it is always $[\mathcal{T}, \mathcal{S}] \leq \mathcal{T}\mathcal{S}$.

Example 3.1 *Consider the lambda theory $\lambda\eta$ (i.e. $\lambda\beta$ extended with η-rule: $\lambda x.tx = t$ provided that x is not free in t) and the lambda theory $\mathcal{S}$ axiomatized by $\Omega = \mathbf{i}$. Let $\Delta_2^\eta \equiv \lambda xy.xxy$ and $\Delta_2 \equiv \lambda x.xx$. If $u(\xi, \mu) \equiv \mu\xi$ is a binary context, then we have the following matrix*

$$M \equiv \begin{pmatrix} u(\mathbf{i}, \Delta_2^\eta) & u(\mathbf{i}, \Delta_2) \\ u(\Omega, \Delta_2^\eta) & u(\Omega, \Delta_2) \end{pmatrix} \equiv \begin{pmatrix} \Delta_2^\eta \mathbf{i} & \Delta_2 \mathbf{i} \\ \Delta_2^\eta \Omega & \Delta_2 \Omega \end{pmatrix} =_\beta \begin{pmatrix} \mathbf{i} & \mathbf{i} \\ \lambda y.\Omega\Omega y & \Omega\Omega \end{pmatrix} \in M(\mathcal{S}, \lambda\eta).$$

It follows that $(\lambda y.\Omega\Omega y, \Omega\Omega) \in [\mathcal{S}, \lambda\eta]$.

In this first result we show that the behavior of the commutator is good when one of the involved lambda theories is the inconsistent lambda theory $1_{\lambda T}$.

Theorem 3.2 *Let* $\mathcal{T}$ *be a lambda theory. Then*

$$[1_{\lambda T}, \mathcal{T}] = [\mathcal{T}, 1_{\lambda T}] = \mathcal{T}.$$

Proof: Let $t(\xi, \mu, \nu) \equiv \xi\mu\nu$ be a context and s, u be λ-terms such that $\mathcal{T} \vdash s = u$. Recall that $\mathbf{T} \equiv \lambda xy.x$ and $\mathbf{F} \equiv \lambda xy.y$. Define:

$$M \equiv \begin{pmatrix} t(\mathbf{F}, s, \mathbf{F}) & t(\mathbf{F}, u, \mathbf{F}) \\ t(\mathbf{T}, s, \mathbf{F}) & t(\mathbf{T}, u, \mathbf{F}) \end{pmatrix} \equiv \begin{pmatrix} \mathbf{F}s\mathbf{F} & \mathbf{F}u\mathbf{F} \\ \mathbf{T}s\mathbf{F} & \mathbf{T}u\mathbf{F} \end{pmatrix} =_{\lambda\beta} \begin{pmatrix} \mathbf{F} & \mathbf{F} \\ s & u \end{pmatrix} \in M(1_{\lambda T}, \mathcal{T})$$

From $[1_{\lambda T}, \mathcal{T}] \vdash \mathbf{F} = \mathbf{F}$ it follows that $[1_{\lambda T}, \mathcal{T}] \vdash s = u$. By the arbitrariness of s and u such that $\mathcal{T} \vdash s = u$ we obtain that $\mathcal{T} \leq [1_{\lambda T}, \mathcal{T}]$. And since it always holds $[1_{\lambda T}, \mathcal{T}] \leq \mathcal{T}$ we obtain the conclusion.

We now show that $[\mathcal{T}, 1_{\lambda T}] = \mathcal{T}$. Define:

$$M = \begin{pmatrix} t(\mathbf{T}, s, s) & t(\mathbf{F}, s, s) \\ t(\mathbf{T}, u, s) & t(\mathbf{F}, u, s) \end{pmatrix} \equiv \begin{pmatrix} \mathbf{T}ss & \mathbf{F}ss \\ \mathbf{T}us & \mathbf{F}us \end{pmatrix} =_{\lambda\beta} \begin{pmatrix} s & s \\ u & s \end{pmatrix} \in M(\mathcal{T}, 1_{\lambda T})$$

From $[\mathcal{T}, 1_{\lambda T}] \vdash s = s$ it follows that $[\mathcal{T}, 1_{\lambda T}] \vdash u = s$. By the arbitrariness of s and u such that $\mathcal{T} \vdash s = u$ we obtain that $\mathcal{T} \leq [\mathcal{T}, 1_{\lambda T}]$. And since it always holds $[\mathcal{T}, 1_{\lambda T}] \leq \mathcal{T}$ we obtain the conclusion. □

The language of bounded lattices is constituted by two binary operators, "·" (meet) and "+" (join), and two constants "0" (bottom) and "1" (top). If q is an identity or a quasi-identity in the language of bounded lattices and λT is the lattice of lambda theories, we write $\lambda T \models q$ if the (quasi-)identity holds in the lattice λT.

In the following result we show that the lattice λT satisfies a nontrivial quasi-identity. It is easy to verify that the quasi-identity (2) below does not hold in the congruence lattice of the unary algebra $(\{a, b, c\}, f)$, where $f(x) = x$ for all $x \in \{a, b, c\}$. Notice that the above congruence lattice is isomorphic to the finite lattice $\mathbf{M}_3$, having 3 atoms, a zero and one, and no other elements.

Theorem 3.3 *Let* $\mathcal{T}$, $\mathcal{G}$ *and* $\mathcal{S}_k$ $(k \in K)$ *be lambda theories. Then*

$$\lambda T \models \sqcup_{k\in K}\mathcal{S}_k = 1,\quad \mathcal{T} \geq \mathcal{G}(\mathcal{S}_k + \mathcal{T}\mathcal{G})\ (k \in K) \ \Rightarrow\ \mathcal{G} \leq \mathcal{T}. \qquad (2)$$

Proof: Assume $\sqcup_{k\in K}\mathcal{S}_k = 1_{\lambda T}$ and $\mathcal{T} \geq \mathcal{G}(\mathcal{S}_k + \mathcal{T}\mathcal{G})$ for all $k \in K$. The proof is divided into claims.

Claim 3.4 $C(\mathcal{S}_k, \mathcal{G}; \mathcal{T})$, *i.e.,* $\mathcal{S}_k$ *centralizes* $\mathcal{G}$ *modulo* $\mathcal{T}$.

We have to prove that, if $M = \begin{pmatrix} t & u \\ s & w \end{pmatrix} \in M(\mathcal{S}_k, \mathcal{G})$ and $t =_\mathcal{T} u$ then $s =_\mathcal{T} w$. Since

$$s =_{\mathcal{S}_k} t =_{\mathcal{G}\mathcal{T}} u =_{\mathcal{S}_k} w \ \text{ and } \quad s =_\mathcal{G} w,$$

then

$$s =_{\mathcal{G}(\mathcal{S}_k \circ (\mathcal{G}\mathcal{T}) \circ \mathcal{S}_k)} w.$$

The compatible λ-relation $\mathcal{G}(\mathcal{S}_k \circ (\mathcal{G}\mathcal{T}) \circ \mathcal{S}_k)$ is contained within the lambda theory $\mathcal{G}(\mathcal{S}_k + \mathcal{G}\mathcal{T})$ so that we have:

$$s =_{\mathcal{G}(\mathcal{S}_k + \mathcal{G}\mathcal{T})} w.$$

Finally we obtain the conclusion $s =_\mathcal{T} w$ from the hypothesis $\mathcal{T} \geq \mathcal{G}(\mathcal{S}_k + \mathcal{G}\mathcal{T})$.

Claim 3.5 *$C(1_{\lambda T}, \mathcal{G}; \mathcal{T})$, i.e., $1_{\lambda T}$ centralizes $\mathcal{G}$ modulo $\mathcal{T}$.*

Let $M = \begin{pmatrix} t & u \\ s & w \end{pmatrix} \in M(1_{\lambda T}, \mathcal{G})$ and $\mathcal{T} \vdash t = u$. We have to show that $\mathcal{T} \vdash s = w$. By hypothesis there exist a context $a(\overline{\xi}, \overline{\mu})$ and elements $\overline{b}_1, \overline{c}_1, \overline{b}_2, \overline{c}_2$ for which $\overline{c}_1 =_{\mathcal{G}} \overline{c}_2$ and

$$\begin{pmatrix} t & u \\ s & w \end{pmatrix} \equiv \begin{pmatrix} a(\overline{b}_1, \overline{c}_1) & a(\overline{b}_1, \overline{c}_2) \\ a(\overline{b}_2, \overline{c}_1) & a(\overline{b}_2, \overline{c}_2) \end{pmatrix}$$

Since $1_{\lambda T} = \sqcup_{k \in k} \mathcal{S}_k$ there exist $i_1, \ldots, i_{s-1} \in K$ and sequences of elements $\overline{r}_1, \ldots, \overline{r}_s$ such that

$$\overline{b}_1 \equiv \overline{r}_1 =_{\mathcal{S}_{i_1}} \overline{r}_2 =_{\mathcal{S}_{i_2}} \ldots =_{\mathcal{S}_{i_{s-1}}} \overline{r}_s \equiv \overline{b}_2.$$

Then for every $j = 1, \ldots, s-1$ we have:

$$\begin{pmatrix} a(\overline{r}_j, \overline{c}_1) & a(\overline{r}_j, \overline{c}_2) \\ a(\overline{r}_{j+1}, \overline{c}_1) & a(\overline{r}_{j+1}, \overline{c}_2) \end{pmatrix} \in M(\mathcal{S}_{i_j}, \mathcal{G})$$

Since $a(\overline{r}_1, \overline{c}_1) \equiv a(\overline{b}_1, \overline{c}_1) \equiv t =_{\mathcal{T}} u \equiv a(\overline{b}_1, \overline{c}_2) \equiv a(\overline{r}_1, \overline{c}_2)$, from Claim 3.4 it follows that $a(\overline{r}_j, \overline{c}_1) =_{\mathcal{T}} a(\overline{r}_j, \overline{c}_2)$ for every $j = 1, \ldots, s$. In particular:

$$s \equiv a(\overline{r}_k, \overline{c}_1) =_{\mathcal{T}} a(\overline{r}_k, \overline{c}_2) \equiv w$$

so that $C(1_{\lambda T}, \mathcal{G}; \mathcal{T})$. This concludes the proof of Claim 3.5.

The conclusion of the theorem is obtained as follows:

$$\mathcal{G} = [1_{\lambda T}, \mathcal{G}] \leq \mathcal{T},$$

because $C(1_{\lambda T}, \mathcal{G}; \mathcal{T})$ and the commutator $[1_{\lambda T}, \mathcal{G}]$ is the least lambda theory ϑ such that $1_{\lambda T}$ centralizes $\mathcal{G}$ modulo ϑ. □

The ET condition and the Zipper condition defined below hold in any lattice of equational theories (see [18]).

Theorem 3.6 (ET Condition) *Let $(\mathcal{S}_k : k \in K)$ be a family of lambda theories. If*

$$\sqcup_{k \in K} \mathcal{S}_k = 1_{\lambda T}$$

then there is a finite sequence $\mathcal{S}_0, \ldots, \mathcal{S}_n$ with $\mathcal{S}_j \in \{\mathcal{S}_k : k \in K\}$ for $j \leq n$ such that:

$$(((\ldots(((\mathcal{S}_0\mathcal{G}) + \mathcal{S}_1)\mathcal{G}) + \mathcal{S}_2)\mathcal{G}) + \ldots + \mathcal{S}_n)\mathcal{G} = \mathcal{G} \tag{3}$$

for every lambda theory $\mathcal{G}$.

Proof: By Thm. 2.2 the lattice λT of lambda theories is isomorphic to the lattice of the equational theories including $Eq(\mathsf{LAA}_I)$. Ernè and Tardos (see [18]) independently pointed out that the lattices of equational theories always satisfy the ET condition. □

The following Corollary can be obtained as a consequence either of the ET condition or of Thm. 3.3.

Corollary 3.7 (Zipper Condition) *Let $\mathcal{T}$, $\mathcal{G}$ and $(\mathcal{S}_k : k \in K)$ be lambda theories. Then,*

$$\lambda T \models \sqcup_{k \in K} \mathcal{S}_k = 1, \quad \mathcal{S}_k\mathcal{G} = \mathcal{T} \ (k \in K) \ \Rightarrow \ \mathcal{G} = \mathcal{T}.$$

Proof: We can obtain the conclusion from Thm. 3.3. Since $\mathcal{G}(\mathcal{S}_k+\mathcal{T}\mathcal{G}) = \mathcal{G}(\mathcal{S}_k+\mathcal{S}_k\mathcal{G}\mathcal{G}) = \mathcal{G}\mathcal{S}_k = \mathcal{T}$, then Thm. 3.3 implies $\mathcal{G} \le \mathcal{T}$. The other inequality $\mathcal{T} \le \mathcal{G}$ follows from $\mathcal{S}_k\mathcal{G} = \mathcal{T}$. □

Corollary 3.8 *Let $\mathcal{S}$ and $\mathcal{T}$ be two lambda theories. If $\mathcal{S} + \mathcal{T} = 1_{\lambda T}$ and $\mathcal{S}\mathcal{G} = \mathcal{T}\mathcal{G}$ then $\mathcal{G} \le \mathcal{S}$ and $\mathcal{G} \le \mathcal{T}$.*

Corollary 3.9 *Let $\mathcal{S}$ and $\mathcal{G}$ be two nonsemisensible lambda theories and let $\mathcal{T}$ be a sensible lambda theory. Then $\mathcal{S}\mathcal{G} \neq \mathcal{T}\mathcal{G}$.*

A magic triple [4, Sect. 17.1] is a set of three λ-terms $\{t, u, w\}$ such that Cons$(t = u)$ (i.e., $\lambda\beta \cup \{t = u\}$ is consistent), Cons$(u = w)$, Cons$(t = w)$ but not Cons$(t = u = w)$. For example $\{\mathbf{i}, \Omega, \Omega\mathbf{s}\}$ is a magic triple.

Corollary 3.10 *Let $\{t, u, w\}$ be a magic triple and let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ be the lambda theories generated respectively by the equations $t = u$, $u = w$ and $w = t$. Then, for every permutation i, j, k of $1, 2, 3$ we have that $\mathcal{T}_i\mathcal{T}_j \neq \mathcal{T}_i\mathcal{T}_k$.*

Proof: If we assume $\mathcal{T}_i\mathcal{T}_j = \mathcal{T}_i\mathcal{T}_k$, then from $\mathcal{T}_k + \mathcal{T}_j = 1_{\lambda T}$ and from Cor. 3.8 it follows $\mathcal{T}_i \le \mathcal{T}_k$ and $\mathcal{T}_i \le \mathcal{T}_j$. This result contradicts the hypothesis $\mathcal{T}_k+\mathcal{T}_i = 1_{\lambda T}$ and the consistency of $\mathcal{T}_k$. □

We conclude this section by showing that some well-known lattices cannot be sublattices of λT.

Corollary 3.11 *Let $\mathbf{M}_k$ be the lattice having k atoms, a zero and one, and no other elements. $\mathbf{M}_k$ ($k \ge 3$) cannot be a sublattice of λT, provided the top element is the inconsistent lambda theory $1_{\lambda T}$.*

Proof: The conclusion follows from Cor. 3.8. □

In every lattice L the modularity law is equivalent to the requirement that L has no sublattice isomorphic to the "pentagon" $\mathbf{N}_5$ (see [21, Thm. 2.25]). The pentagon $\mathbf{N}_5$ is constituted by five distinct elements $0, a, b, c, 1$ such that $a \le c$, $1 = c + b = a + b$ and $0 = ab = cb$. In [27] it was shown the following result.

Theorem 3.12 (Salibra [27]) *Let $\mathcal{T}$ be the lambda theory generated by the equation $\Omega = \mathbf{i}$. The lattice λT is not modular because the pentagon $\mathbf{N}_5$, defined by*

$$0 \equiv \mathcal{T}\mathcal{H}^*; \quad 1 \equiv 1_{\lambda T}; \quad b \equiv \mathcal{T}; \quad a \equiv \mathcal{H} + (\mathcal{T}\mathcal{H}^*); \quad c \equiv \mathcal{H}^*,$$

is a sublattice of λT.

4 Congruence and Lattice Identities

In this section we turn our attention from the lattice of lambda theories, equivalently the congruence lattice of the term algebra $\mathbf{\Lambda}_I^{\lambda\beta} \in \mathsf{LAA}_I$, to the congruence lattices of all LAA_I's. We show that a lattice identity holds in the congruence lattices of all LAA's if, and only if, it is trivial. This result has no immediate consequence for the lattice of lambda theories. In fact, for every nontrivial lattice identity e, we only know that there exists an LAA_I $\mathbf{A}$ whose congruence lattice does not satisfy e. However, later on we show that $\mathbf{A}$ can be chosen to be a term algebra if we modify the language of lambda calculus with a finite number of constants.

A context $t(\xi_1, \dots, \xi_n)$ is *idempotent* in a variety $\mathcal{V}$ of lambda abstraction algebras if $\mathcal{V} \models t(\xi, \xi, \dots, \xi) = \xi$.

The following theorem is a consequence of the sequentiality theorem of lambda calculus (see [4, Thm. 14.4.8]).

Theorem 4.1 *Let $\mathcal{T}$ be a semisensible lambda theory. Then the variety $\mathsf{LAA}_I^{\mathcal{T}}$ generated by the term algebra of $\mathcal{T}$ satisfies a congruence identity e if, and only if, e is trivial.*

Proof: Let $\mathcal{H}$ be the lambda theory generated by equating all unsolvable λ-terms and $\mathcal{H}^*$ be the unique maximal consistent extension of $\mathcal{H}$. $\mathsf{LAA}_I^{\mathcal{H}^*}$ is a subvariety of $\mathsf{LAA}_I^{\mathcal{T}}$ because the term algebra of $\mathcal{H}^*$ is a homomorphic image of the term algebra of $\mathcal{T}$ (recall that $\mathcal{T} \subseteq \mathcal{H}^*$). Then it is sufficient to prove the conclusion of the theorem for $\mathsf{LAA}_I^{\mathcal{H}^*}$.

Assume, by the way of contradiction, that $\mathsf{LAA}_I^{\mathcal{H}^*}$ satisfies a nontrivial congruence identity. Then by Lemma 4.3 and Lemma 4.6 in [13] there exists a natural number $n > 1$, an idempotent n-ary context $t(\xi_1, \xi_2, \dots, \xi_n)$ and n linear identities satisfied in $\mathsf{LAA}_I^{\mathcal{H}^*}$:

$$t(\xi_{11}, \xi_{12} \dots, \xi_{1n}) = t(\mu_{11}, \mu_{12} \dots, \mu_{1n})$$

$$t(\xi_{21}, \xi_{22}, \dots, \xi_{2n}) = t(\mu_{21}, \mu_{22}, \dots, \mu_{2n})$$

$$.$$

$$t(\xi_{n1}, \xi_{n2}, \dots, \xi_{nn}) = t(\mu_{n1}, \mu_{n2}, \dots, \mu_{nn}),$$

where

$\xi_{ii} \equiv \xi$; $\mu_{ii} \equiv \mu$ for all $1 \le i \le n$;

ξ and μ are two distinct context variables;

ξ_{ij} and μ_{ij} $(1 \le i \ne j \le n)$ are not necessarily distinct context variables.

Since $\mathsf{LAA}_I^{\mathcal{H}^*}$ is generated by the term algebra $\mathbf{\Lambda}_I^{\mathcal{H}^*}$, an identity (between contexts) holds throughout $\mathsf{LAA}_I^{\mathcal{H}^*}$ iff it holds in $\mathbf{\Lambda}_I^{\mathcal{H}^*}$. From the idempotence of t it follows that

$$\mathbf{\Lambda}_I^{\mathcal{H}^*} \models t(\xi, \dots, \xi) = \xi. \tag{4}$$

Let $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$ be an unsolvable λ-term, and let $\mathbf{i} \equiv \lambda x.x$ be solvable. From (4) we get:

$$\mathcal{H}^* \vdash t(\Omega, \dots, \Omega) = \Omega; \quad \mathcal{H}^* \vdash t(\mathbf{i}, \dots, \mathbf{i}) = \mathbf{i}. \tag{5}$$

By definition of $\mathcal{H}^*$ (5) implies that $t(\Omega, \dots, \Omega)$ is unsolvable and $t(\mathbf{i}, \dots, \mathbf{i})$ is solvable.

We now show that there exists a natural number k $(1 \le k \le n)$ satisfying the following property for all λ-terms $u_1, \dots, u_n$:

$$t(u_1, \dots, u_n) \text{ is unsolvable if, and only if, } u_k \text{ is unsolvable.} \tag{6}$$

Let ε be the root node in the Böhm tree of a lambda term. From (5) it is not the case that $t(u_1, \dots, u_n)$ is unsolvable for all λ-terms $u_1, \dots, u_n$. Then the unsolvable $t(\Omega, \dots, \Omega)$ is not ε-constant according to the Def. 14.4.2(i) in [4]. The conclusion (6) is obtained by the sequentiality theorem of lambda calculus (see [4]; Thm. 14.4.8 and Def. 14.4.2(ii)).

Consider now the identity

$$t(\xi_{k1}, \dots, \xi_{k(k-1)}, \xi, \xi_{k(k+1)}, \dots, \xi_{kn}) = t(\mu_{k1}, \dots, \mu_{k(k-1)}, \mu, \mu_{k(k+1)}, \dots, \mu_{kn}).$$

If we interpret ξ as Ω and μ as $\mathbf{i}$ and the other context variables in an arbitrary way, then we get that the following identity should hold:

$$\mathcal{H}^* \vdash t(u_{k1}, \dots, u_{k(k-1)}, \Omega, u_{k(k+1)}, \dots, u_{kn}) = t(r_{k1}, \dots, \mu_{k(k-1)}, \mathbf{i}, r_{k(k+1)}, \dots, r_{kn}).$$

By (6)

$$t(u_{k1},\dots,u_{k(k-1)},\Omega,u_{k(k+1)},\dots,u_{kn})$$

is unsolvable, while

$$t(r_{k1},\dots,\mu_{k(k-1)},\mathbf{i},r_{k(k+1)},\dots,r_{kn})$$

is solvable. This contradicts the property of semisensibility of $\mathcal{H}^*$: $\mathcal{H}^*$ does not equate a solvable and an unsolvable. Contradiction. □

As a consequence of the above theorem, for every nontrivial lattice identity e, there exists an LAA_I $\mathbf{A}$ whose congruence lattice does not satisfy e. In the following theorem we show that $\mathbf{A}$ can be chosen to be a term algebra if we modify the language of lambda calculus with a finite number of constants.

Let $C_n = \{c_1,\dots,c_n\}$ be a finite set of constants. Recall from Section 2.1 that $\Lambda_I(C_n)$ is the set of lambda terms constructed from the set I of λ-variables and the set C_n of constants. The minimal lambda theory over $\Lambda_I(C_n)$ will be denoted by $\lambda\beta_n$, while $\mathbf{\Lambda}_I(C_n)$ is the algebra defined in (1) above (see Section 2.3).

Lemma 4.2 *The lattice of lambda theories over $\Lambda_I(C_n)$ is isomorphic to the congruence lattice of the free algebra over n generators in the variety of λ-algebras.*

Proof: The term algebra $\mathbf{\Lambda}^{\lambda\beta_n}_{I,C_n}$ of the lambda theory $\lambda\beta_n$ is an LFA_I by Prop. 2.5. It is a simple matter to show that the lattice of lambda theories over $\Lambda_I(C_n)$ is isomorphic to the congruence lattice of $\mathbf{\Lambda}^{\lambda\beta_n}_{I,C_n}$, which is isomorphic to the congruence lattice of its zero-dimensional subreduct $\mathbf{Zd}\,\mathbf{\Lambda}^{\lambda\beta_n}_{I,C_n}$ by the remark after Thm. 2.7. Since $\mathbf{Zd}\,\mathbf{\Lambda}^{\lambda\beta_n}_{I,C_n}$ is a λ-algebra by Thm. 2.6, then we obtain the conclusion of the lemma if we can show that this zero-dimensional subreduct is isomorphic to the free algebra over n generators in the variety of λ-algebras (see Section 2.2).

Let $\mathbf{B}$ be a λ-algebra and $g : C_n \to B$ be an arbitrary map. We have to show that g can be extended to a homomorphism from $\mathbf{Zd}\,\mathbf{\Lambda}^{\lambda\beta_n}_{I,C_n}$ into $\mathbf{B}$. Recall from Section 2.3 that the free extension $\mathbf{B}[I]$ of $\mathbf{B}$ by I in the variety of combinatory algebras can be turned into a locally finite LAA_I, whose zero-dimensional elements are all the elements of B. We define a function $f : \Lambda_I(C_n) \to B[I]$ as follows, for all λ-terms $t, u \in \Lambda_I(C_n)$:

$$f(x) = x^{\mathbf{B}[I]};\quad f(c_i) = g(c_i);\quad f(tu) = f(t)\cdot^{\mathbf{B}[I]} f(u);\quad f(\lambda x.t) = \lambda x^{\mathbf{B}[I]}.f(t).$$

Function f is a homomorphism from the algebra $\mathbf{\Lambda}_I(C_n)$ into the LFA_I $\mathbf{B}[I]$. The kernel ϑ of the function f is a congruence satisfying the two conditions of Prop. 2.4, so that ϑ is a lambda theory over $\Lambda_I(C_n)$. Since $\lambda\beta_n$ is the least lambda theory over $\Lambda_I(C_n)$, then $\lambda\beta_n \subseteq \vartheta$. It follows that the term algebra $\mathbf{\Lambda}^{\vartheta}_{I,C_n}$ is a homomorphic image of the term algebra $\mathbf{\Lambda}^{\lambda\beta_n}_{I,C_n}$. In conclusion, there exists a homomorphism $h : \mathbf{\Lambda}^{\lambda\beta_n}_{I,C_n} \to \mathbf{\Lambda}^{\vartheta}_{I,C_n}$ and an embedding $i : \mathbf{\Lambda}^{\vartheta}_{I,C_n} \to \mathbf{B}[I]$. By Thm. 2.7 the restriction of $i \circ h$ to $\mathbf{Zd}\,\mathbf{\Lambda}^{\lambda\beta_n}_{I,C_n}$ is a homomorphism from $\mathbf{Zd}\,\mathbf{\Lambda}^{\lambda\beta_n}_{I,C_n}$ into $\mathbf{Zd}\,\mathbf{B}[I]$ extending g. We get the conclusion of the lemma by recalling that $\mathbf{Zd}\,\mathbf{B}[I]$ is isomorphic to $\mathbf{B}$. □

Theorem 4.3 *Let e be a nontrivial lattice identity. Then there exists a natural number n such that the identity e fails in the lattice of the lambda theories over the language $\Lambda_I(C_n)$.*

Proof: By Lemma 4.2 we obtain the conclusion of the theorem if there exists a natural number n such that the identity e fails in the congruence lattice of the free algebra over n generators in the variety of λ-algebras.

Assume $e \equiv (p = q)$ is a nontrivial lattice identity, where $p = p(\vartheta_1, ..., \vartheta_k)$ and $q = q(\vartheta_1, ..., \vartheta_k)$ are terms in the language $+, \cdot$ of lattices and $\vartheta_1, ..., \vartheta_k$ is the sequence of all the variables occurring in p and q. Then by Thm. 4.1 e fails in the lattice of congruences of some lambda abstraction algebra. This means that either $p \leq q$ or $q \leq p$ fails [1] in LAA_I (note that the inclusion $p \leq q$ is equivalent to the identity $p = pq$). Without loss of generality, we assume that $p \leq q$ fails in LAA_I. Recall that the join $\sigma + \gamma$ of two congruences σ and γ is the least equivalence relation containing σ and γ. If $\circ$ is the relative product of binary relations, and $\circ_n$ is defined by induction as follows: $\sigma \circ_2 \gamma = \sigma \circ \gamma$; $\sigma \circ_{n+1} \gamma = \sigma \circ (\gamma \circ_n \sigma)$, then we get:

$$\sigma + \gamma = \bigcup_{n \geq 2} \sigma \circ_n \gamma. \tag{7}$$

For $p = p(\vartheta_1, ..., \vartheta_k)$, in which both $+$, $\cdot$ may occur, and for each integer $r \geq 2$, let p^r be the join free term obtained from p by replacing each occurrence of $+$ in p by the r-fold relation product $\circ_r$ of the operands (see [26]). For example, if:

$$p = ((\vartheta_1 + \vartheta_2)\vartheta_3) + \vartheta_4,$$

then:

$$p^3 = ((\vartheta_1 \circ_3 \vartheta_2)\vartheta_3) \circ_3 \vartheta_4.$$

By expanding $\circ_3$ and by recalling the associativity of the relative product we get:

$$p^3 = ((\vartheta_1 \circ \vartheta_2 \circ \vartheta_1)\vartheta_3) \circ \vartheta_4 \circ ((\vartheta_1 \circ \vartheta_2 \circ \vartheta_1)\vartheta_3).$$

Since $p \leq q$ fails in LAA_I, then there exists a natural number $r \geq 2$ such that $p^r \leq q$ fails to hold in LAA_I.

We now show that $p^r \leq q$ fails in the variety LA of λ-algebras. By Thm. 2.2 in [26] we have that $p^r \leq q$ holds in (the lattices of the congruences of all algebras in) a variety $\mathcal{V}$ of type $\mathcal{F}$ iff $\mathcal{V}$ satisfies the following Mal'cev condition: there exists a $k \geq 2$ and algebraic terms $t_i(x_1, \ldots, x_n)$ $(i = 1, \ldots, k)$ in the similarity type $\mathcal{F}$ such that $\mathcal{V}$ satisfies a suitable set U of identities (depending on $p^r \leq q$) having the following structure:

$$t_i(x_{s1}, \ldots, x_{sn}) = t_j(x_{w1}, \ldots, x_{wn}), \tag{8}$$

where s and w are two suitable functions from $\{1, \ldots, n\}$ into $\{1, \ldots, n\}$ depending on the identity $t_i = t_j \in U$. Assume, by the way of contradiction, that the variety LA of λ-algebras satisfies $p^r \leq q$. Then there exist suitable combinatory terms $t_i(x_1, \ldots, x_n)$ $(i = 1, \ldots, k)$ such that LA satisfies the above Mal'cev conditions (8). Consider the usual translation $(\ )_\lambda$ of the combinatory terms $t = t(x_1, ..., x_n)$ into lambda terms: $(x_i)_\lambda = x_i$, $\mathbf{k}_\lambda = \lambda x_1 x_2.x_1$, $\mathbf{s}_\lambda = \lambda x_1 x_2 x_3.x_1 x_3(x_2 x_3)$, $(tu)_\lambda = t_\lambda u_\lambda$. By [4, Thm. 7.3.10] we have that if $\mathsf{LA} \models t_i(x_{s1}, \ldots, x_{sn}) = t_j(x_{w1}, \ldots, x_{wn})$ then

$$\lambda\beta \vdash (t_i)_\lambda(x_{s1}, \ldots, x_{sn}) = (t_j)_\lambda(x_{w1}, \ldots, x_{wn}), \tag{9}$$

for all identities $t_i = t_j$ in U. It is a simple matter to show by induction over the complexity of the combinatory term $t = t(x_1, ..., x_n)$ that every free occurrence of a variable x_i $(1 \leq i \leq n)$ in the λ-term $t_\lambda = t_\lambda(x_1, ..., x_n)$ is not in the scope of any lambda abstraction λx_j. Then from (9) it follows that

$$\lambda\beta \vdash (t_i)_\lambda(u_{s1}, \ldots, u_{sn}) = (t_j)_\lambda(u_{w1}, \ldots, u_{wn}), \tag{10}$$

for all identities $t_i = t_j$ in U and all λ-terms $u_1, \ldots, u_n$, where the substitution in $(t_i)_\lambda$ and $(t_j)_\lambda$ is made without α-conversion. By (10) we get that the term algebra $\mathbf{\Lambda}_I^{\lambda\beta}$ of the minimal lambda theory $\lambda\beta$ (without constants) satisfies the following identities between contexts:

$$(t_i)_\lambda(\xi_{s1}, \ldots, \xi_{sn}) = (t_j)_\lambda(\xi_{w1}, \ldots, \xi_{wn})$$

[1] By "$p \leq q$ is true in $\mathbf{Con}\,\mathbf{A}$" we mean that the binary relation, obtained by substituting elements of $\mathbf{Con}\,\mathbf{A}$ for the variables of p, is included in the corresponding binary relation obtained from q.

for all $t_i = t_j$ in U. Since $\mathbf{\Lambda}_I^{\lambda\beta}$ generates LAA_I (see [28, Thm. 14]), then we have that the above identities hold in LAA_I. By applying now the Mal'cev condition for $p^r \leq q$ to the variety LAA_I, we obtain that LAA_I satisfies $p^r \leq q$, and this is a contradiction.

Then the variety LA does not satisfy the lattice identity $p^r \leq q$. Since the Mal'cev condition for $p^r \leq q$ is expressed in terms of n-ary combinatory terms, by applying again Thm. 2.2 in [26], we get that $p^r \leq q$ fails to hold into the congruence lattice of the LA-free algebra $\mathbf{B}$ freely generated by a set of cardinality n. By Lemma 4.2 we obtain the conclusion of the theorem. □

5 The lambda theory $\mathcal{J}$

In this section we show that there exists a sublattice of λT satisfying good lattice properties. First we introduce a lambda theory $\mathcal{J}$, whose consistency is obtained by using intersection types for defining a filter model for it (see [5], [7], [2]). Then the equations defining $\mathcal{J}$ are used to define a semilattice term operation on the term algebra $\mathbf{\Lambda}_I^{\mathcal{J}}$. It follows from this result that the lattice of all lambda theories including $\mathcal{J}$ has the lattice properties described in Thm. 5.3 below.

The complete proof of the following theorem can be found in Appendix.

Theorem 5.1 *The lambda theory $\mathcal{J}$, axiomatized by*

$$\Omega xx = x; \quad \Omega xy = \Omega yx; \quad \Omega x(\Omega yz) = \Omega(\Omega xy)z, \tag{11}$$

is consistent.

An algebra $\mathbf{A}$ is *neutral* if $[\sigma, \tau] = \sigma\tau$ for all congruences $\sigma, \tau \in \mathbf{Con}\,\mathbf{A}$ (see, for example, [13]). A variety $\mathcal{V}$ is neutral if every algebra $\mathbf{A} \in \mathcal{V}$ is neutral.

Theorem 5.2 *The variety $\mathsf{LAA}_I^{\mathcal{J}}$ generated by the term algebra of the lambda theory $\mathcal{J}$ satisfies a nontrivial congruence identity.*

Proof: From the identities (11) defining $\mathcal{J}$ it follows that the λ-term Ω defines a semilattice operation on every algebra in the variety $\mathsf{LAA}_I^{\mathcal{J}}$. From this and from [21, 4.186.5] it follows the neutrality of every algebra in $\mathsf{LAA}_I^{\mathcal{J}}$. The conclusion of the theorem now follows from Cor. 4.7 in [13]. □

Theorem 5.3 *The interval sublattice $[\mathcal{J}) = \{\mathcal{T} : \mathcal{J} \subseteq \mathcal{T}\}$ of the lattice of lambda theories satisfies the following properties:*

(i) $[\mathcal{T}, \mathcal{S}] = \mathcal{T}\mathcal{S}$ *for all lambda theories* $\mathcal{S}, \mathcal{T} \in [\mathcal{J})$.

(ii) The finite lattice $\mathbf{M}_3$ *is not a sublattice of* $[\mathcal{J})$.

(iii) $[\mathcal{J})$ *satisfies a nontrivial congruence identity.*

(iv) $[\mathcal{J})$ *is congruence meet semidistributive, i.e. the following implication holds for all lambda theories* $\mathcal{S}, \mathcal{T}, \mathcal{G} \in [\mathcal{J})$.

$$\mathcal{S}\mathcal{T} = \mathcal{S}\mathcal{G} \;\Rightarrow\; \mathcal{S}\mathcal{T} = \mathcal{S}(\mathcal{T} + \mathcal{G}).$$

Proof: We recall that the interval sublattice $[\mathcal{J})$ is isomorphic to the congruence lattice of the term algebra $\mathbf{\Lambda}_I^{\mathcal{J}}$.

(i) By (11) the context $\Omega\xi\mu$ defines a semilattice operation on the term algebra $\mathbf{\Lambda}_I^{\mathcal{J}}$. From this and from [21, 4.186.5] it follows the neutrality of $\mathbf{\Lambda}_I^{\mathcal{J}}$.

(ii) From the characterization of neutrality in [20].
(iii) From Thm. 5.2.
(iv) From (i) and from Cor. 4.7 in [13]. □

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Appendix

In this appendix we prove the consistency of the lambda theory $\mathcal{J}$ defined in Section 5. We use intersection types [5] [7] as a tool for synthesizing a model of lambda calculus whose equational theory contains $\mathcal{J}$. The present construction is a particular case of the metodology developed in [2].

We need infinitely many sets of constants to build the set of intersection types.

Definition 5.4 *Let $\langle -, - \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a computable bijective map. The set $\mathcal{C}$ of type constants is defined by*

$$\mathcal{C} = \bigcup_{n \in \mathbb{N}} \mathcal{C}^n$$

where

$$\mathcal{C}^1 = \{\omega, \chi\} \qquad \mathcal{C}^{n+1} = \mathcal{C}^n \cup \{\varsigma_{\langle n,m \rangle} \mid m \in \mathbb{N}\};$$

The set $\mathbb{T}$ of types is defined by

$$\mathbb{T} = \bigcup_{n \in \mathbb{N}} \mathbb{T}^n$$

where

$$\mathbb{T}^n = \mathcal{C}^n \mid \mathbb{T}^n \to \mathbb{T}^n \mid \mathbb{T}^n \cap \mathbb{T}^n.$$

Notation. Upper case Roman letters $(A, B, \ldots)$ will denote arbitrary types; greek letters will denote constants in $\mathcal{C}$. When we write intersection types we shall use the following convention: the constructor $\cap$ takes precedence over the constructor $\to$; when parentheses are omitted, the constructor $\to$ associates to the right. For example, $A \to B \to C$ stands for $A \to (B \to C)$.

Much of the expressive power of intersection types comes from the fact that types can be endowed with a *preorder relation* $\leq$, which induces the structure of a meet semi-lattice with respect to $\cap$, the top element being ω.

Definition 5.5 *The preorder relation $\leq$ on types is defined by the following set of axioms and rules (we write $A \sim B$ for $A \leq B$ & $B \leq A$):*

(refl)	$A \leq A$	*(idem)*	$A \leq A \cap A$
(incl$_L$)	$A \cap B \leq A$	*(incl$_R$)*	$A \cap B \leq B$
(mon)	$\dfrac{A \leq A' \quad B \leq B'}{A \cap B \leq A' \cap B'}$	*(trans)*	$\dfrac{A \leq B \quad B \leq C}{A \leq C}$
(ω)	$A \leq \omega$	*(ω-η)*	$\omega \leq \omega \to \omega$
($\to$-$\cap$)	$(A \to B) \cap (A \to C) \leq A \to B \cap C$	*(η)*	$\dfrac{A' \leq A \quad B \leq B'}{A \to B \leq A' \to B'}$
(χ)	$\chi \sim \omega \to \chi$	*($\varsigma_{\langle n,m\rangle}$)*	$\varsigma_{\langle n,m\rangle} \sim \varsigma_{\langle n,m\rangle} \to W_{\langle n,m\rangle}$

where, for every n, $\langle W_{\langle n,m\rangle} : m \in \mathbb{N}\rangle$ is an enumeration of the set $\{A \to B \to A \cap B \mid A, B \in \mathbb{T}^n\}$.

From these axioms we can derive other properties; for example $\omega \sim \omega \to \omega$ by axioms (ω) and $(\omega$-$\eta)$.

Associativity and commutativity of $\cap$ (modulo $\sim$) follow easily from the axioms and rules of $\leq$ introduced in Definition 5.5. The operator $\cap$ is commutative and associative; then we can write $\bigcap_{i \leq n} A_i$ for $A_1 \cap \ldots \cap A_n$. Similarly, we write $\bigcap_{i \in I} A_i$, where I always denotes a finite non-empty set.

We associate with each type the maximum number of nested arrows in the leftmost path.

Definition 5.6 *The mapping $\# : \mathbb{T} \to \mathbb{N}$ is defined inductively on types as follows:*

$$\begin{array}{lcll} \#(A) & = & 0 & \textit{if } A \in \mathcal{C}; \\ \#(A \to B) & = & \#(A) + 1; & \\ \#(A \cap B) & = & \max\{\#(A), \#(B)\}. & \end{array}$$

It is immediate to verify that $\leq$ induces an *easy intersection type theory* according to [2]. This implies:

Theorem 5.7 *For all $A \in \mathbb{T}$ with $\#(A) \geq 1$ there is $B \in \mathbb{T}$ such that $A \sim B$, $B \equiv \bigcap_{i \in I}(C_i \to D_i)$, and $\#(B) = \#(A)$.*

Theorem 5.8 *For all I, and $A_i, B_i, C, D \in \mathbb{T}$,*

$$\bigcap_{i \in I}(A_i \to B_i) \leq C \to D \;\&\; D \not\sim \omega \Rightarrow \exists J \subseteq I.\; C \leq \bigcap_{i \in J} A_i \;\&\; \bigcap_{i \in J} B_i \leq D.$$

Before introducing the notion of an *intersection-type assignment system* we define bases.

Definition 5.9 *A* basis *is a (possibly infinite) set of statements of the shape $x : B$, where $B \in \mathbb{T}$, with all variables distinct. $\Gamma, x : A$ stands for $\Gamma \cup \{x : A\}$ when $x \notin \Gamma$.*

Definition 5.10 *The* intersection type assignment system *is a formal system for deriving judgements of the form $\Gamma \vdash t : A$, where the* subject *t is an untyped λ-term, the* predicate *A is in $\mathbb{T}$, and Γ is a basis. Its axioms and rules are the following:*

$$(Ax)\ \frac{(x:A) \in \Gamma}{\Gamma \vdash x:A} \qquad (Ax\text{-}\omega)\ \ \Gamma \vdash t : \omega$$

$$(\to \mathrm{I})\ \frac{\Gamma, x:A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B} \qquad (\to \mathrm{E})\ \frac{\Gamma \vdash t : A \to B \quad \Gamma \vdash u : A}{\Gamma \vdash tu : B}$$

$$(\cap \mathrm{I})\ \frac{\Gamma \vdash t : A \quad \Gamma \vdash t : B}{\Gamma \vdash t : A \cap B} \qquad (\leq)\ \frac{\Gamma \vdash t : A \quad A \leq B}{\Gamma \vdash t : B}.$$

Notice that the intersection elimination rules

$$(\cap E)\ \frac{\Gamma \vdash t : A \cap B}{\Gamma \vdash t : A} \qquad \frac{\Gamma \vdash t : A \cap B}{\Gamma \vdash t : B}$$

can be immediately proved by the rules introduced in Definition 5.10.

As expected the above type assignment system satisfies a Generation Theorem. The proof is done in [2] for all type assignment systems induced by easy intersection type theories.

Theorem 5.11 (Generation Theorem)

1. *Assume $A \not\sim \omega$. $\Gamma \vdash x : A$ iff there exists $B \in \mathbb{T}$ such that $(x:B) \in \Gamma$ and $B \leq A$.*
2. *$\Gamma \vdash tu : A$ iff there exists $B \in \mathbb{T}$ such that $\Gamma \vdash t : B \to A$ and $\Gamma \vdash u : B$.*
3. *$\Gamma \vdash \lambda x.t : A$ iff there exist I and $B_i, C_i \in \mathbb{T}$ such that $\Gamma, x : B_i \vdash t : C_i$ and $\bigcap_{i \in I}(B_i \to C_i) \leq A$.*
4. *$\Gamma \vdash \lambda x.t : B \to C$ iff $\Gamma, x:B \vdash t : C$.*

Now we discuss how to build a model of lambda calculus out of the type assignment system introduced in Definition 5.10. The rules ($\cap$I) and ($\leq$) imply that the set $\{A : \Gamma \vdash t : A\}$ of types of a λ-term t is an upper set closed under finite intersection. This remark can be formalized in the definition of a *filter* of types. Then we show how to turn the space of filters into an applicative structure, where λ-terms can be properly interpreted. The *filter model* thus obtained is a model of lambda calculus satisfying the following axiom of extensionality:

$$\forall xy((\forall z.xz = yz) \to x = y).$$

Filter models arise naturally in the context of the generalizations of Stone duality that are used in discussing domain theory in logical form (see [1], [7], [32]). This approach provides a conceptually independent semantics to intersection types, the *lattice semantics*. Types are viewed as *compact elements* of domains. The type ω denotes the least element, intersection denotes join of compact elements, and arrow types internalize the space of continuous endomorphisms. Following the paradigm of Stone duality, intersection type assignment systems give rise to filter models, where the interpretation of a λ-term can be given through a finitary logical description.

Definition 5.12

1. *A filter (over $\mathbb{T}$) is a set $X \subseteq \mathbb{T}$ such that:*
   - *$\omega \in X$;*
   - *if $A \leq B$ and $A \in X$, then $B \in X$;*

- *if $A, B \in X$, then $A \cap B \in X$;*

2. *$\mathcal{F}$ denotes the set of filters over $\mathbb{T}$;*
3. *if $X \subseteq \mathbb{T}$, $\uparrow X$ denotes the filter generated by X;*
4. *a filter is* principal *if it is of the shape $\uparrow \{A\}$, for some type A (we shall denote $\uparrow \{A\}$ simply by $\uparrow A$).*

It is easy to check that the set $\mathcal{F}$ of filter constitutes a complete lattice, where the meet of a family of filters is their intersection, and the join is the least filter containing their union. The bottom element of this lattice is the the principal filter $\uparrow \omega$, while the top element is the whole set $\mathbb{T}$ of types. The lattice $\mathcal{F}$ is ω-algebraic: the compact (or finite) elements of $\mathcal{F}$ are the principal filters $\uparrow A$.

Next we endow the space of filters with a binary operation of application which makes possible the interpretation of the λ-terms. Let $\mathsf{Env}_{\mathcal{F}}$ be the set of all environments, i.e., mappings from the set of variables into $\mathcal{F}$.

Definition 5.13

1. *The application $\cdot : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ is defined as*

$$X \cdot Y = \{B \mid \exists A \in Y. A \to B \in X\}.$$

2. *The interpretation function: $[\![\ ]\!] : \Lambda \times \mathsf{Env}_{\mathcal{F}} \to \mathcal{F}$ is defined by*

$$[\![t]\!]_\rho = \{A \in \mathbb{T} \mid \exists \Gamma \models \rho.\ \Gamma \vdash t : A\},$$

where ρ ranges over $\mathsf{Env}_{\mathcal{F}}$ and $\Gamma \models \rho$ if and only $(x : B) \in \Gamma$ implies $B \in \rho(x)$;

3. *The triple $\langle \mathcal{F}, \cdot, [\![\ ]\!] \rangle$ is called a* filter model.

The previous definition is sound because $X \cdot Y$ is a filter. Easy intersection type theories induce extensional models of lambda calculus (see [2]) in the sense of Hindley-Longo [12]. Then we get the following theorem:

Theorem 5.14 *The filter model $\langle \mathcal{F}, \cdot, [\![\ ]\!] \rangle$ is an extensional model of lambda calculus.*

We provide now a lemma (proved in [2]) which characterizes the types derivable for $\Delta \equiv \lambda x.xx$ and $\Omega \equiv \Delta\Delta$.

Lemma 5.15

1. *$\vdash \Delta : A \to B$ iff $A \leq A \to B$;*
2. *$\vdash \Omega : B$ iff $A \leq A \to B$ for some $A \in \mathbb{T}$ such that $\vdash \Delta : A$.*
3. *If $\vdash \Omega : B$ then there exists $A \in \mathbb{T}$ such that $\#(A) = 0$, $A \leq A \to B$ and $\vdash \Delta : A$.*

We show that there are some types which cannot be deduced for Δ.

Lemma 5.16 *$\not\vdash \Delta : \chi$ and $\not\vdash \Delta : (\chi \to \chi) \to \chi \to \chi$.*

Proof: Recall the definition of the types W_i from Definition 5.5. We define $\mathcal{E}_\omega$ as the minimal set such that:

$$\begin{array}{ll} \omega \in \mathcal{E}_\omega; & A \in \mathbb{T}, B \in \mathcal{E}_\omega \ \Rightarrow\ A \to B \in \mathcal{E}_\omega; \\ A, B \in \mathcal{E}_\omega \ \Rightarrow\ A \cap B \in \mathcal{E}_\omega; & W_i \in \mathcal{E}_\omega \ \Rightarrow\ \varsigma_i \in \mathcal{E}_\omega. \end{array}$$

It is easy to check by induction on $\leq$ that for all $A \in \mathbb{T}$:[2]

$$B \leq A \,\&\, B \in \mathcal{E}_\omega \;\Rightarrow\; A \in \mathcal{E}_\omega.$$

This implies

$$\omega \not\leq \chi, \tag{12}$$

since $\omega \in \mathcal{E}_\omega$ and $\chi \notin \mathcal{E}_\omega$.

By the way of contradiction suppose that $\vdash \Delta : \chi$. Then $\vdash \Delta : \omega \to \chi$, and by Lemma 5.15(1) we get $\omega \leq \omega \to \chi \sim \chi$. This contradicts (12).

Similarly, from $\vdash \Delta : (\chi \to \chi) \to \chi \to \chi$ and from Lemma 5.15(1) it follows that $\chi \to \chi \leq (\chi \to \chi) \to (\chi \to \chi) \to \chi \to \chi$, which implies $\chi \to \chi \leq \chi$ by Theorem 5.8. Applying again Theorem 5.8 we get $\omega \leq \chi$. This again contradicts (12). □

The following lemma characterizes the types deducible for Ω.

Lemma 5.17 *$\vdash \Omega : A \to B \to C$ iff $A \cap B \leq C$.*

Proof: Let W_j be an arbitrary type of the shape $A \to B \to A \cap B$. By an easy computation, we may give to Δ the type ς_j for any integer j. In fact, since $\varsigma_j \sim \varsigma_j \to W_j$, it follows $x : \varsigma_j \vdash xx : W_j$, hence $\vdash \lambda x.xx : \varsigma_j \to W_j \sim \varsigma_j$. From this last fact, by applying ($\to$E), we obtain $\vdash \Omega : W_j$ for all j. So we conclude, using rule ($\leq$), that $A \cap B \leq C$ implies $\vdash \Omega : A \to B \to C$.

On the other side, let $\vdash \Omega : A \to B \to C$. Then by applying Lemma 5.15(3), it follows that there exists D such that $\#(D) = 0$, $\vdash \Delta : D$ and $D \leq D \to A \to B \to C$. Then $D \equiv \bigcap_{i \in I} \psi_i$ for some type constants ψ_i. By Lemma 5.16 we get that, for all $i \in I$, ψ_i is either ω or ς_j for some integer j. Since $A \leq \omega$ for all A, this implies that either $D \sim \omega \to \omega$ or $D \sim \bigcap_{j \in J}(\varsigma_j \to W_j)$ for some subset J of I. From $D \leq D \to A \to B \to C$ and from Theorem 5.8 it follows that either $\omega \leq A \to B \to C$ or $\exists L \subseteq J$ such that $\bigcap_{j \in L} W_j \leq A \to B \to C$. Let $W_j \equiv A_j \to B_j \to A_j \cap B_j$. By applying again Theorem 5.8 we get $A \leq \bigcap_{j \in I} A_j$, $B \leq \bigcap_{j \in I} B_j$, and $\bigcap_{j \in I}(A_j \cap B_j) \leq C$ for some $I \subseteq J$, so that $A \cap B \leq C$. □

We now prove the main theorem of this section.

Theorem 5.18 *The filter model $\langle \mathcal{F}, \cdot, [\![\;]\!] \rangle$ is not trivial and validates the theory $\mathcal{J}$.*

Proof: The model is not trivial since it is easy to derive $\vdash \mathbf{i} : (\chi \to \chi) \to \chi \to \chi$ while $\not\vdash \Delta : (\chi \to \chi) \to \chi \to \chi$ by Lemma 5.16.

We show that $[\![\Omega xy]\!]_\rho = [\![\Omega yx]\!]_\rho$ and $[\![\Omega x(\Omega yz)]\!]_\rho = [\![\Omega(\Omega xy)z]\!]_\rho$ for all ρ. The proof of $[\![\Omega xx]\!]_\rho = [\![x]\!]_\rho$ for all ρ is similar.

The condition $[\![t]\!]_\rho = [\![u]\!]_\rho$ for all ρ is equivalent to $\Gamma \vdash t : A$ iff $\Gamma \vdash u : A$ for all Γ, A by Definition 5.13.

By Theorem 5.11(2) $\{x : A, y : B\} \vdash \Omega xy : C$ iff $\vdash \Omega : D \to E \to C$ for some D, E such that $\{x : A\} \vdash x : D$ and $\{y : B\} \vdash y : E$. Theorem 5.11(1) implies $A \leq D$ and $B \leq E$, i.e., by rule ($\leq$) we have that $\vdash \Omega : A \to B \to C$. By Lemma 5.17 this is true iff $A \cap B \leq C$. Similarly $\{x : A, y : B\} \vdash \Omega yx : C$ iff $A \cap B \leq C$, so we conclude that $[\![\Omega xy]\!]_\rho = [\![\Omega yx]\!]_\rho$ for all ρ.

From Theorem 5.11(2) $\{x : A, y : B, z : C\} \vdash \Omega x(\Omega yz) : D$ iff $\vdash \Omega : E \to F \to D$ for some E, F such that $\{x : A\} \vdash x : E$ and $\{y : B, z : C\} \vdash \Omega yz : F$. Reasoning as in previous case we get $A \cap F \leq D$ and $B \cap C \leq F$, i.e. $A \cap B \cap C \leq D$. Similarly $\{x : A, y : B, z : C\} \vdash$

[2] It is also easy to check by induction on the definition of $\mathcal{E}_\omega$ that $A \in \mathcal{E}_\omega \Rightarrow A \sim \omega$, so that $\mathcal{E}_\omega$ is the set of types equivalent to ω.

$\Omega(\Omega xy)z : D$ iff $A \cap B \cap C \leq D$, so we conclude that $[\![\Omega x(\Omega yz)]\!]_\rho = [\![\Omega(\Omega xy)z]\!]_\rho$ for all ρ. $\square$

Notice that in the lattice $\mathcal{F}$ the filter $\uparrow (A \cap B)$ is the join of the filters $\uparrow A$ and $\uparrow B$. Moreover, $[\![\Omega]\!] \cdot \uparrow A \cdot \uparrow B = \uparrow (A \cap B)$ by Lemma 5.17. Hence, Ω is interpreted as the join. More precisely, if X, Y are arbitrary filters, then $[\![\Omega]\!] \cdot X \cdot Y$ is the join of X and Y. So it is natural to ask if Ω could be interpreted as meet. We conjecture that this is possible if we add the union type constructor $\cup$ (as defined in [3]), since the meet of the filters $\uparrow A$ and $\uparrow B$ is the filter $\uparrow (A \cup B)$.