

# Boolean algebras for lambda calculus\*

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## Abstract

*In this paper we show that the Stone representation theorem for Boolean algebras can be generalized to combinatory algebras. In every combinatory algebra there is a Boolean algebra of central elements (playing the role of idempotent elements in rings), whose operations are defined by suitable combinators. Central elements are used to represent any combinatory algebra as a Boolean product of directly indecomposable combinatory algebras (i.e., algebras which cannot be decomposed as the Cartesian product of two other nontrivial algebras). Central elements are also used to provide applications of the representation theorem to lambda calculus. We show that the indecomposable semantics (i.e., the semantics of lambda calculus given in terms of models of lambda calculus, which are directly indecomposable as combinatory algebras) includes the continuous, stable and strongly stable semantics, and the term models of all semisensible lambda theories. In one of the main results of the paper we show that the indecomposable semantics is equationally incomplete, and this incompleteness is as wide as possible: for every recursively enumerable lambda theory  $\mathcal{T}$ , there is a continuum of lambda theories including  $\mathcal{T}$  which are omitted by the indecomposable semantics.*

## 1 Introduction

The lambda calculus is not a true equational theory since the variable-binding properties of lambda abstraction prevent variables in lambda calculus from operating as real algebraic variables. Consequently the general methods that have been developed in universal algebra, for defining the semantics of an arbitrary algebraic theory for instance, are not directly applicable. There have been several attempts

to reformulate the lambda calculus as a purely algebraic theory. The earliest, and best known, algebraic models are Curry's combinatory algebras [14]. Combinatory algebras have a simple purely equational characterization and were used to provide an intrinsic first-order, but not equational, characterization of the models of lambda calculus, as a special class of combinatory algebras called  $\lambda$ -models [2, Def. 5.2.7].

Topology is at the center of the known approaches to giving models of the untyped lambda calculus. The first model, found by Scott in 1969 in the category of algebraic lattices, was successfully used to show that all unsolvable  $\lambda$ -terms can be consistently equated. After Scott, a large number of mathematical models for lambda calculus, arising from syntax-free constructions, have been introduced in various categories of domains and were classified into semantics according to the nature of their representable functions, see e.g. [1, 2, 5, 6, 23]. Scott's continuous semantics [27] is given in the category whose objects are complete partial orders and morphisms are Scott continuous functions. The stable semantics (Berry [8]) and the strongly stable semantics (Bucciarelli-Ehrhard [10]) are a strengthening of the continuous semantics, introduced to capture the notion of "sequential" Scott continuous function. All these semantics are structurally and equationally rich [7, 18, 19] in the sense that it is possible to build up  $2^{\aleph_0}$   $\lambda$ -models in each of them inducing, through the kernel congruence relation of the interpretation function, pairwise distinct lambda theories. Nevertheless, the above denotational semantics are equationally *incomplete*: they do not match all possible operational semantics of lambda calculus. The problem of the equational incompleteness was positively solved by Honsell-Ronchi della Rocca [16] for the continuous semantics and by Bastonero-Gouy [4, 15] for the stable semantics. In [25, 26] Salibra has shown in a uniform way that all semantics (including the strongly stable semantics), which involve monotonicity with respect to some partial order and have a bottom element, fail to induce a continuum of lambda theories.

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Salibra [20, 26, 24] has recently launched a research program for exploring lambda calculus and combinatory logic using techniques of universal algebra. In [20] Lusin and Salibra have shown that a lattice identity is satisfied by all congruence lattices of combinatory algebras iff it is trivial (i.e. true in all lattices). As a consequence, it is not possible to apply to combinatory algebras the nice results developed in universal algebra (see [12, 21]) in the last thirty years, which essentially connect lattice identities satisfied by all congruence lattices of algebras in a variety, and Mal'cev conditions (that characterize properties in varieties by the existence of suitable terms involved in certain identities). Thus there is a common belief that lambda calculus and combinatory logic are algebraically pathological.

On the contrary, in this paper we show that combinatory algebras satisfy interesting algebraic properties. One of the milestones of modern algebra is the Stone representation theorem for Boolean algebras, which was generalized by Pierce to commutative rings with unit and next by Comer to the class of algebras with Boolean factor congruences (see [13, 17, 22]). By applying a theorem by Vaggione [30], we show that Comer's generalization of Stone representation theorem holds also for combinatory algebras: any combinatory algebra is isomorphic to a weak Boolean product of directly indecomposable combinatory algebras (i.e., algebras which cannot be decomposed as the Cartesian product of two other nontrivial algebras). Another way to express the representation theorem is in terms of sheaves: any combinatory algebra is isomorphic to the algebra of global sections of a sheaf of indecomposable combinatory algebras over a Boolean space.

The proof of the representation theorem for combinatory algebras is based on the fact that every combinatory algebra has *central elements*, i.e., elements which define a direct decomposition of the algebra as the Cartesian product of two other combinatory algebras, just like idempotent elements in rings or complemented elements in bounded distributive lattices. Central elements in universal algebra were introduced by Vaggione in [29] and were used to investigate the closure of varieties of algebras under Boolean products. In this paper we show that central elements in a combinatory algebra constitute a Boolean algebra, whose Boolean operations can be defined by suitable combinators. This result highlights a connection between propositional classic logic and combinatory logic. What is the real meaning of this flavour of classic logic within combinatory logic remains to be investigated in the future. What we would like to emphasize here is that central elements have been shown fundamental in the application of the representation theorem to lambda calculus, as it will be explained in the next paragraph.

The representation theorem can be roughly summarized as follows: the directly indecomposable combinatory alge-

bras are the 'building blocks' in the variety of combinatory algebras. On the other hand, the result of incompleteness [26], stating that any semantics of lambda calculus given in terms of partial orderings with a bottom element is incomplete, removes the belief that partial orderings are intrinsic to  $\lambda$ -models. It would be interesting to find new Cartesian closed categories, where the partial orderings play no role and the reflexive objects are directly indecomposable as combinatory algebras. In this paper we investigate the class of all models of lambda calculus, which are directly indecomposable as combinatory algebras (*indecomposable semantics*, for short). We show that the indecomposable semantics includes the continuous, stable and strongly stable semantics, and the term models of all semisensible lambda theories (theories which do not equate solvable and unsolvable terms). In one of the main results of the paper we show that the indecomposable semantics is incomplete, and this incompleteness is as wide as possible: for every recursively enumerable lambda theory  $\mathcal{T}$ , there is a continuum of lambda theories including  $\mathcal{T}$  which are omitted by the indecomposable semantics.

It is unknown, in general, whether the set of lambda theories, which are representable in a semantics of lambda calculus, is a lattice with respect to the inclusion ordering. In the last result of the paper we show that the set of lambda theories representable in each of the classic semantics of lambda calculus is not closed under finite intersection, so that it cannot constitute a sublattice of the lattice of all lambda theories.

## 2 Notation and basic definitions

We will generally use the notation of Barendregt's classic work [2] for lambda calculus and combinatory logic and the notation of Burris and Sankappanavar [12] for universal algebra.

### 2.1 The untyped lambda calculus

$\Lambda$  and  $\Lambda^o$  are, respectively, the set of  $\lambda$ -terms and of closed  $\lambda$ -terms.

We will denote  $\alpha\beta$ -conversion by  $\lambda\beta$ . A *lambda theory* is a congruence on  $\Lambda$  (with respect to the operators of abstraction and application) which contains  $\lambda\beta$ ; it can also be seen as a (specific) set of equations between  $\lambda$ -terms. The set of all lambda theories is naturally equipped with a structure of complete lattice, hereafter denoted by  $\lambda\mathcal{T}$ , with meet defined as set theoretical intersection. The join of two lambda theories  $\mathcal{T}$  and  $\mathcal{S}$  is the least equivalence relation including  $\mathcal{T} \cup \mathcal{S}$ . It is clear that  $\lambda\beta$  is the least element of  $\lambda\mathcal{T}$ , while the inconsistent lambda theory  $\Lambda \times \Lambda$  is the top element of  $\lambda\mathcal{T}$ .

The lambda theory generated (or axiomatized) by a set of equations is the least lambda theory containing it. As a matter of notation,  $\mathcal{T} \vdash M = N$  stands for  $M = N \in \mathcal{T}$ ; this is also written as  $M =_{\mathcal{T}} N$ . A lambda theory  $\mathcal{T}$  is consistent if there exists at least an equation  $M = N$  such that  $\mathcal{T} \not\vdash M = N$ .

Solvable  $\lambda$ -terms can be characterized as follows: a  $\lambda$ -term  $M$  is solvable if, and only if, it has a *head normal form*, that is,  $M =_{\lambda\beta} \lambda x_1 \dots x_n . y M_1 \dots M_k$  for some  $n, k \geq 0$  and  $\lambda$ -terms  $M_1, \dots, M_k$ .  $M \in \Lambda$  is *unsolvable* if it is not solvable.

$\mathcal{H}$  is the lambda theory generated by equating all the unsolvable  $\lambda$ -terms, while  $\mathcal{H}^*$  is the unique maximal consistent lambda theory such that  $\mathcal{H} \subseteq \mathcal{H}^*$ . A lambda theory  $\mathcal{T}$  is called *semisensible* [2, Def. 4.1.7(iii)] if  $\mathcal{T} \not\vdash M = N$  whenever  $M$  is solvable and  $N$  is unsolvable.  $\mathcal{T}$  is semisensible iff  $\mathcal{T} \subseteq \mathcal{H}^*$ . A lambda theory  $\mathcal{T}$  is *sensible* if  $\mathcal{H} \subseteq \mathcal{T}$  (see Section 10.2 and Section 16.2 in [2]).

## 2.2 Combinatory algebra

An *applicative structure* is an algebra with a distinguished 2-ary function symbol which we call *application*. We may write it infix as  $s \cdot t$ , or even drop it entirely and write  $st$ . As usual, application associates to the left;  $stu$  means  $(st)u$ .

Curry discovered that a particularly simple applicative structure has tremendous expressive power [14]: a *combinatory algebra*  $\mathbf{C} = (C, \cdot, \mathbf{k}, \mathbf{s})$  is an applicative structure for a signature with two constants  $\mathbf{k}$  and  $\mathbf{s}$ , such that  $\mathbf{k}xy = x$  and  $\mathbf{s}xyz = xz(yz)$  for all  $x, y$ , and  $z$ . See elsewhere [14] for a full treatment.

Call  $\mathbf{k}$  and  $\mathbf{s}$  the *basic combinators*. In the equational language of combinatory algebras the derived combinators  $\mathbf{i}$  and  $\mathbf{1}$  are defined as  $\mathbf{i} \equiv \mathbf{s}\mathbf{k}\mathbf{k}$  and  $\mathbf{1} \equiv \mathbf{s}(\mathbf{k}\mathbf{i})$ . It is not hard to verify that every combinatory algebra satisfies the identities  $\mathbf{i}x = x$  and  $\mathbf{1}xy = xy$ .

We say that  $c \in C$  *represents* a function  $f : C \rightarrow C$  (and that  $f$  is *representable*) if  $cz = f(z)$  for all  $z \in C$ . Call  $c, d \in C$  *extensionally equal* when they represent the same function in  $C$ . For example  $c$  and  $\mathbf{1}c$  are always extensionally equal. (We use  $\mathbf{1}$  below to select a canonical representative inside a class of extensionally equivalent elements.)

For each variable  $x$  we define a transformation  $\lambda x^*$  of the set of combinatory terms as follows:  $\lambda x^* . x = \mathbf{i}$ . Let  $t$  be a combinatory term different from  $x$ . If  $x$  does not occur in  $t$ , define  $\lambda x^* . t = \mathbf{k}t$ . Otherwise,  $t$  must be of the form  $rs$  where  $s$  and  $r$  are combinatory terms, at least one of which contains  $x$ ; in this case define  $\lambda x^* . t = \mathbf{s}(\lambda x^* . r)(\lambda x^* . s)$ . It is well known that  $x$  does not occur in  $\lambda x^* . t$  and that, for every combinatory algebra  $\mathbf{C}$  and combinatory term  $u$ , we

have:

$$\mathbf{C} \models (\lambda x^* . t)u = t[x := u],$$

where the combinatory term  $t[x := u]$  is obtained by substituting  $u$  for  $x$  in  $t$ .

If  $t$  is a combinatory term and  $x_1, x_2, \dots, x_n$  ( $n \geq 2$ ) are variables, then  $\lambda x_1 x_2 \dots x_n^* . t$  is defined by induction as follows:  $\lambda x_1 x_2 \dots x_n^* . t \equiv \lambda x_1^* . (\lambda x_2 \dots \lambda x_n^* . t)$ .

## 2.3 Lambda model

The axioms of an elementary subclass of combinatory algebras, called  $\lambda$ -*models* or models of the lambda calculus, were expressly chosen to make coherent the definition of interpretation of  $\lambda$ -terms (see [2, Def. 5.2.7]). Let  $\mathbf{C}$  be a  $\lambda$ -model and let  $\bar{c}$  be a new symbol for each  $c \in C$ . Extend the language of the lambda calculus by adding  $\bar{c}$  as a new constant symbol for each  $c \in C$ . Let  $\Lambda^o(C)$  be the set of closed  $\lambda$ -terms with constants from  $C$ . The interpretation of terms in  $\Lambda^o(C)$  with elements of  $C$  can be defined by induction as follows (for all  $M, N \in \Lambda^o(C)$  and  $c \in C$ ):

$$|\bar{c}| = c; |(MN)| = |M| |N|; |\lambda x . M| = \mathbf{1}m,$$

where  $m \in C$  is any element representing the following function  $f : C \rightarrow C$ :

$$f(c) = |M[x := \bar{c}]|, \quad \text{for all } c \in C.$$

The *Meyer-Scott axiom* is the most important axiom in the definition of a  $\lambda$ -model. In the first-order language of combinatory algebras it takes the following form

$$\forall x \forall y (\forall z (xz = yz) \Rightarrow \mathbf{1}x = \mathbf{1}y).$$

The combinator  $\mathbf{1}$  becomes an inner choice operator, that makes coherent the interpretation of an abstraction  $\lambda$ -term.

Each  $\lambda$ -model  $\mathbf{C}$  induces a lambda theory, denoted here by  $Th(\mathbf{C})$ , and called *the equational theory of  $\mathbf{C}$* . Thus,  $M = N \in Th(\mathbf{C})$  if, and only if,  $M$  and  $N$  have the same interpretation in  $\mathbf{C}$ .

**Definition 1** Given a lambda theory  $\mathcal{T}$ , a  $\lambda$ -model  $\mathbf{C}$  represents (or induces)  $\mathcal{T}$  if  $\mathcal{T} = Th(\mathbf{C})$ .

The *term model*  $\mathcal{M}_{\mathcal{T}}$  of a lambda theory  $\mathcal{T}$  (see [2, Def. 5.2.11]) consists of the set of the equivalence classes of  $\lambda$ -terms modulo the lambda theory  $\mathcal{T}$  together with the operation of application on the equivalence classes. By [2, Cor. 5.2.13(ii)]  $\mathcal{M}_{\mathcal{T}}$  is a  $\lambda$ -model which represents the lambda theory  $\mathcal{T}$ .

We define various notions of representability of theories in classes of models.

**Definition 2** Given a class  $\mathbb{C}$  of  $\lambda$ -models and a lambda theory  $\mathcal{T}$ ,

1.  $\mathbb{C}$  represents  $\mathcal{T}$  if there is some  $\mathcal{M} \in \mathbb{C}$  representing  $\mathcal{T}$ .
2.  $\mathbb{C}$  omits  $\mathcal{T}$  if there is no  $\mathcal{M} \in \mathbb{C}$  representing  $\mathcal{T}$ .
3.  $\mathbb{C}$  is complete for the set  $S \subseteq \lambda\mathcal{T}$  of lambda theories if  $\mathbb{C}$  represents all elements of  $S$ .
4.  $\mathbb{C}$  is incomplete if it omits a consistent lambda theory.

## 2.4 Algebra

A congruence  $\theta$  on an algebra  $\mathbf{A}$  is an equivalence relation which is compatible with respect to the basic operation of the algebra. Write  $\text{Con}\mathbf{A}$  for the set of congruences of  $\mathbf{A}$ . This has a natural complete lattice structure by inclusion of sets (considering  $\theta$  as a subset of  $A \times A$ , so the meet is just set-intersection).

$\theta$  is *trivial* if it is the top or bottom element in the natural inclusion ordering; write these  $\nabla^{\mathbf{A}}$  (equal to  $A \times A$ ) and  $\Delta^{\mathbf{A}}$  (equal to  $\{(a, a) \mid a \in A\}$ ) respectively. Also, given  $a, b \in A$  write  $\theta(a, b)$  for the least congruence relating  $a$  and  $b$ .

Given two congruences  $\sigma$  and  $\tau$  on the algebra  $\mathbf{A}$ , we can form the *relative product*:

$$\tau \circ \sigma = \{(a, c) \mid a\sigma b\tau c, \text{ for some } b \in A\}.$$

This is a compatible relation on  $\mathbf{A}$ , but not necessarily a congruence.

An algebra  $\mathbf{A}$  is *simple* when its only congruences are  $\Delta^{\mathbf{A}}$  and  $\nabla^{\mathbf{A}}$ .

An algebra  $\mathbf{A}$  is a *subdirect product* of the algebras  $(\mathbf{B}_i : i \in I)$  if there exists an embedding  $f$  of  $\mathbf{A}$  into the direct product  $\prod_{i \in I} \mathbf{B}_i$  such that the projection  $\pi_i \circ f : \mathbf{A} \rightarrow \mathbf{B}_i$  is onto for every  $i \in I$ . We write  $\mathbf{A} \leq \prod_{i \in I} \mathbf{B}_i$  if  $\mathbf{A}$  is a subdirect product of the algebras  $(\mathbf{B}_i : i \in I)$ .

Call a nonempty class  $K$  of algebras of the same similarity type a *variety* if it is closed under subalgebras, homomorphic images and direct products. By Birkhoff's theorem (see [21]) a class of algebras is a variety if, and only if, it is an equational class (that is, it is axiomatized by a set of equations).

## 2.5 Factor congruence

Call  $\theta$  a *factor congruence* when there exists another congruence  $\bar{\theta}$  such that  $\theta \cap \bar{\theta} = \Delta^{\mathbf{A}}$  and  $\nabla^{\mathbf{A}} = \theta \circ \bar{\theta}$ . In this case call  $\theta$  and  $\bar{\theta}$  a *pair of complementary factor congruences*.

It is easy to see that  $\mathbf{A}$  has a pair  $(\theta, \bar{\theta})$  of complementary factor congruences precisely when it is isomorphic to  $\mathbf{B} \times \mathbf{C}$  (with  $\mathbf{B}$  isomorphic to  $\mathbf{A}/\theta$  and  $\mathbf{C}$  isomorphic to  $\mathbf{A}/\bar{\theta}$ ).

So factor congruences are another way of saying 'this algebra is a direct product of simpler algebras'.

The set of factor congruences of  $\mathbf{A}$  is not, in general, a sublattice of  $\text{Con}\mathbf{A}$ .  $\Delta^{\mathbf{A}}$  and  $\nabla^{\mathbf{A}}$  are the *trivial* factor congruences, corresponding to  $\mathbf{A} \cong 1 \times \mathbf{A}$ ; of course, 1 is isomorphic to  $\mathbf{A}/\nabla^{\mathbf{A}}$  and  $\mathbf{A}$  is isomorphic to  $\mathbf{A}/\Delta^{\mathbf{A}}$ .

An algebra  $\mathbf{A}$  is *directly indecomposable* when  $\mathbf{A}$  admits only the two trivial factor congruences ( $\Delta^{\mathbf{A}}$  and  $\nabla^{\mathbf{A}}$ ), while  $\mathbf{A}$  is *directly decomposable* when  $\mathbf{A}$  admits nontrivial factor congruences.

Clearly, a simple algebra is directly indecomposable, though there are algebras which are directly indecomposable but not simple (they just have congruences which do not split the algebra up neatly as a Cartesian product).

It is useful to characterize factor congruences in terms of algebra homomorphisms satisfying certain equalities (the next step will be to express the equalities in the equational language of the algebra itself).

A decomposition operation (see [21, Def. 4.32]) for an algebra  $\mathbf{A}$  is a function  $f : A \times A \rightarrow A$  such that

- $f(x, x) = x$ ;
- $f(f(x, y), z) = f(x, z) = f(x, f(y, z))$ ;
- $f$  is an algebra homomorphism from  $\mathbf{A} \times \mathbf{A}$  into  $\mathbf{A}$ .

By [21, Thm. 4.33] there exists a bijective correspondence between pairs of complementary factor congruences and decomposition operations, and thus between decomposition operations and factorizations  $\mathbf{A} \cong \mathbf{B} \times \mathbf{C}$ .

By this intuition we see that the binary relations  $\theta$  and  $\bar{\theta}$  defined by

$$x \theta y \text{ iff } f(x, y) = y; \quad x \bar{\theta} y \text{ iff } f(x, y) = x,$$

are a pair of complementary factor congruences, and conversely we see that for every pair  $\theta$  and  $\bar{\theta}$  of complementary factor congruences, the map  $f$  defined by

$$f(x, y) = u \text{ iff } x \theta u \bar{\theta} y, \tag{1}$$

is a decomposition operation. Notice that for any  $x$  and  $y$  there is just one element  $u$  such that  $x \theta u \bar{\theta} y$ .

The reader can easily verify these facts, or find proofs elsewhere [21].

An algebra has *Boolean factor congruences* if the factor congruences form a Boolean sublattice of the congruence lattice. Most known examples of varieties in which all algebras have Boolean factor congruences are those with *factorable congruences*, that is, varieties in which every congruence  $\theta$  on  $\mathbf{A} \times \mathbf{B}$  is a product congruence  $\theta_1 \times \theta_2$  of two congruences  $\theta_1 \in \text{Con}\mathbf{A}$  and  $\theta_2 \in \text{Con}\mathbf{B}$ . Recall that  $(b, c) \theta_1 \times \theta_2 (b', c')$  iff  $b \theta_1 b'$  and  $c \theta_2 c'$ .

## 2.6 Boolean product

The Boolean product construction (see [12, Chapter IV]) provides a method for translating numerous fascinating

properties of Boolean algebras into other varieties of algebras. Actually the construction that we call “Boolean product” has been known for several years as “the algebra of global sections of sheaves of algebras over Boolean spaces” (see [13, 17]); however the definition of the latter was unnecessarily involved. We recall that a Boolean space is a compact, Hausdorff and totally disconnected topological space.

A *weak Boolean product* of an indexed family  $(\mathbf{A}_i : i \in I)$  of algebras is a subdirect product  $\mathbf{A} \leq \prod_{i \in I} \mathbf{A}_i$ , where  $I$  can be endowed with a Boolean space topology so that

- (i) the set  $\{i \in I : a_i = b_i\}$  is open for all  $a, b \in A$ , and
- (ii) if  $a, b \in A$  and  $N$  is a clopen subset of  $I$ , then the element  $c$ , defined by  $c_i = a_i$  for every  $i \in N$  and  $c_i = b_i$  for every  $i \in I - N$ , belongs to  $A$ .

A *Boolean product* is just a weak Boolean product such that the set  $\{i \in I : a_i = b_i\}$  is clopen for all  $a, b \in A$ .

### 3 The Stone representation theorem for combinatory algebras

The axioms characterizing the variety of combinatory algebras are suggested by an analysis of recursive processes, not by logic (as for Boolean algebras and Heyting algebras) or by algebra (as for groups and rings). Combinatory algebras are never commutative, associative, finite and recursive, so that there is a common belief that these algebras are algebraically pathological.

On the contrary, in this section we show that combinatory algebras satisfy interesting algebraic properties: the Stone representation theorem for Boolean algebras admits a generalization to combinatory algebras.

#### 3.1 Stone and Pierce

The Stone representation theorem for Boolean rings (the observation that Boolean algebras could be regarded as rings is due to Stone) admits a generalization, due to Pierce, to commutative rings with unit (see [22] and [17, Chapter V]). To make the reader familiar with the argument, we give in this subsection an outline of Pierce construction.

Let  $\mathbf{A} = (A, +, \cdot, 0, 1)$  be a commutative ring with unit, and let  $E(A) = \{a \in A : a \cdot a = a\}$  be the set of idempotent elements of  $A$ . We define a structure of Boolean algebra on  $E(A)$  as follows, for all  $a, b \in E(A)$ :

- $a \wedge b = a \cdot b$ ;
- $a \vee b = a + b - (a \cdot b)$ ;
- $a^- = 1 - a$ .

Then it is possible to show that every idempotent element  $a \neq 0, 1$  defines a pair  $\theta(a, 1), \theta(a, 0)$  of nontrivial complementary factor congruences, where  $\theta(a, 1)$  is the least congruence containing the pair  $(a, 1)$  and similarly for the other congruence  $\theta(a, 0)$ . In other words, the ring  $\mathbf{A}$  can be decomposed in a non trivial way as  $\mathbf{A} = \mathbf{A}/\theta(a, 1) \times \mathbf{A}/\theta(a, 0)$ . If  $E(A) = \{0, 1\}$ , then  $A$  is directly indecomposable. Then the Pierce theorem for commutative rings with unit can be stated as follows: every commutative ring with unit is isomorphic to a Boolean product of directly indecomposable rings. If  $\mathbf{A}$  is a Boolean ring, then we get the Stone representation theorem for Boolean algebras, because the ring of truth values is the unique directly indecomposable Boolean ring.

The remaining part of this section is devoted to the proof of the representation theorem for combinatory algebras.

#### 3.2 The Boolean algebra of central elements

Combinatory logic and lambda calculus internalize many important things (computability theory, for example). ‘To be directly decomposable’ is another internalizable property of these formalisms, as it will be shown in this subsection.

In the equational language of combinatory algebras the combinators  $\mathbf{t}, \mathbf{f}$  representing the booleans are defined by  $\mathbf{t} \equiv \mathbf{k}$  and  $\mathbf{f} \equiv \mathbf{ki}$ . The booleans satisfy the identities  $\mathbf{t}xy = x$  and  $\mathbf{f}xy = y$ .

The combinators  $\mathbf{t}$  and  $\mathbf{f}$  correspond to the constants 0 and 1 in a commutative ring with unit, while, as it will be shown below, the so-called central elements of a combinatory algebra correspond to idempotent elements in a ring. Central elements in universal algebra were introduced by Vaggione in [29].

**Definition 3** Let  $\mathbf{A}$  be a combinatory algebra. We say an element  $e \in A$  is central when it satisfies the following equations, for all  $x, y, z, t \in A$ :

- (i)  $exx = x$ .
- (ii)  $e(exy)z = exz = ex(eyz)$ .
- (iii)  $e(xy)(zt) = exz(eyt)$ .
- (iv)  $e = \mathbf{etf}$ .

The set of central elements of  $\mathbf{A}$  will be denoted by  $E(\mathbf{A})$ .

Every combinatory algebra admits at least two central elements, namely the combinators  $\mathbf{t}$  and  $\mathbf{f}$ . Now we show that central elements, as idempotent elements in a ring, decompose a combinatory algebra  $\mathbf{A}$  as a Cartesian product: if  $e \in E(\mathbf{A})$ , then  $\mathbf{A} = \mathbf{A}/\theta(e, \mathbf{t}) \times \mathbf{A}/\theta(e, \mathbf{f})$ . This will be shown in the next proposition via decomposition operators.

The use of decomposition operators to characterize central elements is new.

Fix some combinatory algebra.

**Proposition 4** *There is a (natural) bijective correspondence between central elements and decomposition operators.*

**Proof.** Given a central element  $e$  we obtain a decomposition operator by taking  $f_e(x, y) = exy$ . It is a simple exercise to show that axioms (i)-(iii) of a central element make  $f_e$  a decomposition operator.

Conversely, given a decomposition operator  $f$ , we have to show that the element  $f(\mathbf{t}, \mathbf{f})$  is central. From Section 2.5 we have that  $f(\mathbf{t}, \mathbf{f})$  is the unique element  $u$  satisfying  $\mathbf{t} \theta u \bar{\theta} \mathbf{f}$ , where  $\theta$  and  $\bar{\theta}$  are the pair of complementary factor congruences associated with the decomposition operator  $f$ . Then, from the property of congruence of  $\theta$  and  $\bar{\theta}$  it follows, for all  $x, y$ :

$$\mathbf{t}xy \theta f(\mathbf{t}, \mathbf{f})xy \bar{\theta} \mathbf{f}xy,$$

that implies  $x \theta f(\mathbf{t}, \mathbf{f})xy \bar{\theta} y$ . Since by definition  $f(x, y)$  is the unique element satisfying  $x \theta f(x, y) \bar{\theta} y$ , then we obtain

$$f(x, y) = f(\mathbf{t}, \mathbf{f})xy \quad (2)$$

Finally, the identities defining  $f$  as decomposition operator make  $f(\mathbf{t}, \mathbf{f})$  a central element.

It is easy to verify that these correspondences form the two sides of a bijection. If  $e$  is central, then the central element  $f_e(\mathbf{t}, \mathbf{f})$  is equal to  $e$ , because  $f_e(\mathbf{t}, \mathbf{f}) = e\mathbf{t}\mathbf{f} = e$  by Def. 3(iv). If  $f$  is a decomposition operator, then by (2) we have that  $f_{f(\mathbf{t}, \mathbf{f})}(x, y) = f(\mathbf{t}, \mathbf{f})xy = f(x, y)$  for all  $x, y$ . ■

For every central element  $e$ , we denote respectively by  $f_e$  and by  $(\theta_e, \bar{\theta}_e)$  the decomposition operator and the pair of complementary factor congruences determined by  $e$ .

**Corollary 5** *If  $e$  is central, then we have:*

1.  $x \theta_e exy \bar{\theta}_e y$ ;
2.  $x \theta_e y$  iff  $exy = y$ ;  $x \bar{\theta}_e y$  iff  $exy = x$ ;
3. The congruence  $\theta_e$  is generated by the pair  $(e, \mathbf{t})$  (i.e.,  $\theta_e = \theta(e, \mathbf{t})$ );
4. The congruence  $\bar{\theta}_e$  is generated by the pair  $(e, \mathbf{f})$  (i.e.,  $\bar{\theta}_e = \theta(e, \mathbf{f})$ ).

We now show that the partial ordering over central elements, defined by

$$d \leq e \text{ iff } \theta_d \subseteq \theta_e \quad (3)$$

is a Boolean ordering. The combinators  $\mathbf{t}$  and  $\mathbf{f}$  are respectively the bottom element and the top element of this ordering, while the combinators  $\lambda xy^*.x\mathbf{t}y$  and  $\lambda x^*.x\mathbf{f}$  (recall

that  $\lambda xy^*$  is defined in Section 2.2) represent respectively the meet operation and the complementation.

**Theorem 6** *Let  $\mathbf{A}$  be a combinatory algebra. Then the algebra  $\mathbf{E}(\mathbf{A}) = (E(\mathbf{A}), \wedge, \bar{\phantom{x}})$  of central elements of  $\mathbf{A}$ , defined by*

$$e \wedge d = etd; \quad e^- = e\mathbf{t}\mathbf{f},$$

*is a Boolean algebra.*

**Proof.** We first show that the factor congruences of  $\mathbf{A}$  form a Boolean sublattice of the congruence lattice  $\text{Con}\mathbf{A}$ . Let  $\mathbf{A} = \mathbf{B} \times \mathbf{C}$  be a decomposition of  $\mathbf{A}$  as the direct product of two combinatory algebras  $\mathbf{B}$  and  $\mathbf{C}$ . It is easy to show, by using the equations defining central elements, that  $E(\mathbf{A}) = E(\mathbf{B}) \times E(\mathbf{C})$ , i.e., every central element of  $\mathbf{A}$  can be decomposed as a pair of central elements of  $\mathbf{B}$  and  $\mathbf{C}$ . In the terminology of universal algebra this means that  $\mathbf{A}$  has no ‘skew factor congruences’. We get the conclusion from [9, Prop. 1.3], where it is shown that an algebra  $\mathbf{A}$  has no skew factor congruences if, and only if, the factor congruences of  $\mathbf{A}$  form a Boolean sublattice of the congruence lattice  $\text{Con}\mathbf{A}$ .

It follows that the partial ordering on central elements, defined in (3), is a Boolean ordering. We have to show now that, for all central elements  $d, e$ , the elements  $e^- = e\mathbf{t}\mathbf{f}$  and  $e \wedge d = etd$  are central and are respectively associated with the pairs  $(\bar{\theta}_e, \theta_e)$  and  $(\theta_e \cap \theta_d, \bar{\theta}_e \vee \bar{\theta}_d)$  of complementary factor congruences (recall that  $e$  is associated with the pair  $(\theta_e, \bar{\theta}_e)$ ).

We check the details for  $e\mathbf{t}\mathbf{f}$ . By Cor. 5(1) we have that  $e\mathbf{t}\mathbf{f}$  is the unique element  $u$  such that  $\mathbf{t} \bar{\theta}_e u \theta_e \mathbf{f}$ . By (1) in Section 2.5 this means that  $e\mathbf{t}\mathbf{f} = g(\mathbf{t}, \mathbf{f})$  for the decomposition operator  $g$  associated with the pair  $(\bar{\theta}_e, \theta_e)$  of complementary factor congruences. We have the conclusion that  $e\mathbf{t}\mathbf{f}$  is central associated with the pair  $(\bar{\theta}_e, \theta_e)$  as in the proof of Prop. 4.

We now consider  $e \wedge d = etd$ . First of all, we show that  $etd = dte$ . By Cor. 5(1) we have that  $\mathbf{t} \theta_e etd \bar{\theta}_e d$ , while  $\mathbf{t} \theta_e dte \bar{\theta}_e d$  can be obtained as follows:  $\mathbf{t} =$  (by Def. 3(i))  $d\mathbf{t}\mathbf{t} \theta_e$  (by  $\mathbf{t}\theta_e e$ )  $dte \bar{\theta}_e$  (by  $e\bar{\theta}_e \mathbf{f}$ )  $d\mathbf{t}\mathbf{f} =$  (by Def. 3(iv))  $d$ . Since there is a unique element  $u$  such that  $\mathbf{t} \theta_e u \bar{\theta}_e d$ , then we have the conclusion  $dte = etd$ . We now show that  $etd$  is the central element associated with the factor congruence  $\theta_e \cap \theta_d$ , i.e.,

$$\mathbf{t} (\theta_e \cap \theta_d) etd (\bar{\theta}_e \vee \bar{\theta}_d) \mathbf{f}.$$

From  $dte = etd$  we easily get that  $\mathbf{t} \theta_e etd$  and  $\mathbf{t} \theta_d etd$ , that is,  $\mathbf{t} (\theta_e \cap \theta_d) etd$ . Finally, by Cor. 5 we have:  $etd \bar{\theta}_e d = d\mathbf{t}\mathbf{f} \bar{\theta}_d \mathbf{f}$ , i.e.,  $etd (\bar{\theta}_e \vee \bar{\theta}_d) \mathbf{f}$ . ■

We now provide the promised representation theorem. If  $I$  is a maximal ideal of the Boolean algebra  $E(\mathbf{A})$ , then  $\cup I$  denotes the congruence on  $\mathbf{A}$  defined by

$$x (\cup I) y \text{ iff } x \theta_e y \text{ for some } e \in I.$$

By a *Pierce variety* (see [30]) we mean a variety of algebras for which there are two constants  $0, 1$  and a term  $u(x, y, z, v)$  such that the following identities hold:  $u(x, y, 0, 1) = x$  and  $u(x, y, 1, 0) = y$ .

**Theorem 7** (Representation Theorem) *Let  $\mathbf{A}$  be a combinatory algebra and  $X$  be the Boolean space of maximal ideals of the Boolean algebra  $\mathbf{E}(\mathbf{A})$  of central elements. Then the map*

$$f : A \rightarrow \prod_{I \in X} (A / \cup I),$$

defined by

$$f(x) = (x / \cup I : I \in X),$$

gives a **weak Boolean product representation** of  $\mathbf{A}$ , where the quotient algebras  $\mathbf{A} / \cup I$  are directly indecomposable.

**Proof.** By Thm. 6 the set of factor congruences of  $\mathbf{A}$  constitutes a Boolean sublattice of  $\text{Con}\mathbf{A}$ . Then by [13]  $f$  gives a weak Boolean product representation of  $\mathbf{A}$ . The quotient algebras  $\mathbf{A} / \cup I$  are directly indecomposable by [30, Thm. 8], because the variety of combinatory algebras is a Pierce variety if we define  $1 \equiv \mathbf{t}$ ,  $0 \equiv \mathbf{f}$  and  $u = \lambda xyzv^*.zyx$ . ■

The map  $f$  of the above theorem does not give in general a Boolean product representation. This follows from two results due to Vaggione [29] and to Plotkin-Simpson [28]. Vaggione has shown that, if a variety has factorable congruences (i.e., every congruence in a product is a product of congruences) and every member of the variety can be represented as a Boolean product of directly indecomposable algebras, then the variety is a discriminator variety (see [12] for the terminology). Discriminator varieties satisfy very strong algebraic properties, in particular they are congruence permutable (i.e., the join of two congruences is just their composition). Plotkin and Simpson have shown that the property of having permutable congruences is inconsistent with combinatory logic.

## 4 The algebraic incompleteness of lambda calculus

The representation theorem of combinatory algebras can be roughly summarized as follows: the directly indecomposable combinatory algebras are the ‘building blocks’ in the variety of combinatory algebras. Then it is natural to investigate the class of models of lambda calculus, which are directly indecomposable as combinatory algebras (*indecomposable semantics*, for short). In this section we show that the indecomposable semantics includes: the continuous, stable and strongly stable semantics, and the term models of all semisensible lambda theories. However, in one of the main results of the paper we give a proof, based on central elements, that the indecomposable semantics is incomplete, and this incompleteness is as wide as possible: for

every recursively enumerable lambda theory  $\mathcal{T}$ , there is a continuum of lambda theories including  $\mathcal{T}$  which are omitted by the indecomposable semantics. In the last result of the paper we show that the set of lambda theories induced by each of the known semantics is not closed under finite intersection, so that it cannot constitute a sublattice of the lattice of lambda theories.

### 4.1 Internalizing ‘indecomposable’

We have shown how to internally represent a factor congruence as a central element. Now we show how to represent the logical assertion that the only factor congruences of a combinatory algebra are trivial.

We recall that an algebra  $\mathbf{A}$  is *directly indecomposable* when it is not trivial and it is not isomorphic to a product of two nontrivial algebras (i.e., there is not a pair of nontrivial complementary factor congruences). A combinatory algebra  $\mathbf{A}$  is directly indecomposable if  $E(\mathbf{A}) = \{\mathbf{t}, \mathbf{f}\}$ .

For two combinatory terms  $t$  and  $u$ , define the pair  $[t, u] \equiv \lambda z^*.ztu$  and, for every sequence  $t_1, \dots, t_n$  ( $n \geq 3$ ), define  $[t_1, \dots, t_n] \equiv [t_1, [t_2, \dots, t_n]]$ .

Define the following combinatory terms:

- $\mathbf{Z} \equiv \lambda e^*.[\lambda x^*.exx, \lambda xyz^*.e(exy)z, \lambda xyz^*.exz, \lambda xyz^*.e(exy)(zu), etf]$ ;
- $\mathbf{U} \equiv \lambda e^*.[\lambda x^*.x, \lambda xyz^*.exz, \lambda xyz^*.ex(eyz), \lambda xyz^*.exz(eyu), e]$ .

**Lemma 8** *The class  $\text{CA}_{DI}$  of the directly indecomposable combinatory algebras is a universal class (i.e., it is an elementary class which can be axiomatized by universal sentences).*

**Proof.** By Def. 3 we have that  $e$  is central if, and only if, the equation  $\mathbf{Z}e = \mathbf{U}e$  holds. Then the class  $\text{CA}_{DI}$  is axiomatized by the following universal formula  $\phi$ :

$$\phi \equiv \forall e((\mathbf{Z}e = \mathbf{U}e \rightarrow e = \mathbf{t} \vee e = \mathbf{f}) \wedge \neg(\mathbf{t} = \mathbf{f})).$$

■

**Corollary 9** *The class  $\text{CA}_{DI}$  of the directly indecomposable combinatory algebras is closed under subalgebras and ultraproducts.*

### 4.2 Algebraic incompleteness

The closure of the class of directly indecomposable combinatory algebras under subalgebras is the key trick in the proof of the algebraic incompleteness theorem.

**Lemma 10** *Given a lambda theory  $\mathcal{T}$  the following conditions are equivalent:*

- (i) The term model  $\mathcal{M}_{\mathcal{T}}$  of  $\mathcal{T}$  is directly decomposable.
- (ii) All  $\lambda$ -models representing  $\mathcal{T}$  are directly decomposable.
- (iii) The indecomposable semantics omits  $\mathcal{T}$ .

**Proof.** (ii  $\Leftrightarrow$  iii): By Def. 2.

(i  $\Rightarrow$  ii): Recall that a  $\lambda$ -model  $\mathbf{C}$  represents a lambda theory  $\mathcal{T}$  (i.e.,  $Th(\mathbf{C}) = \mathcal{T}$ ) if, and only if, the term model  $\mathcal{M}_{\mathcal{T}}$  of  $\mathcal{T}$  is a subalgebra of  $\mathbf{C}$ . Then the conclusion follows from Corollary 9. ■

In every model of lambda calculus the interpretation of the combinators  $\mathbf{t}$  and  $\mathbf{f}$  is equal respectively to that of the  $\lambda$ -terms  $\mathbf{T} \equiv \lambda xy.x$  and  $\mathbf{F} \equiv \lambda xy.y$ . It follows that the  $\lambda$ -terms  $\mathbf{T}$  and  $\mathbf{F}$  can cover the role of trivial central elements in every model of lambda calculus.

**Theorem 11** (The algebraic Incompleteness Theorem) *The indecomposable semantics is incomplete.*

**Proof.** By Lemma 10 it is sufficient to define a lambda theory whose term model is directly decomposable. Let  $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$  be the usual looping term of lambda calculus. Consider two arbitrary consistent lambda theories  $\mathcal{T}$  and  $\mathcal{S}$  satisfying the following conditions:

$$\mathcal{T} \vdash \Omega = \mathbf{T}; \quad \mathcal{S} \vdash \Omega = \mathbf{F}.$$

$\mathcal{T}$  and  $\mathcal{S}$  exist because  $\Omega$  is an easy term (see [2, Prop. 15.3.9]), i.e., it can be equated consistently with any other closed term. It is a simple exercise to verify that the lambda theory  $\mathcal{T} \cap \mathcal{S}$  contains all equations (i)-(iv) of Def. 3 for  $e = \Omega$ , making the equivalence class of  $\Omega$  a nontrivial central element in the term model of  $\mathcal{T} \cap \mathcal{S}$ . ■

We have shown that theories exist with no indecomposable models, so that any class of models which excludes decomposable models cannot be complete.

In the following theorem we show that, although the class of directly indecomposable  $\lambda$ -models is incomplete, it is so wide to include all term models of the semisensible lambda theories.

We recall from Section 2.3 that the term model  $\mathcal{M}_{\mathcal{T}}$  of a lambda theory  $\mathcal{T}$  is a model of lambda calculus and that the elements of  $\mathcal{M}_{\mathcal{T}}$  are equivalence classes of  $\lambda$ -terms modulo  $\mathcal{T}$ .

**Lemma 12** *Let  $e \in \mathcal{M}_{\mathcal{T}}$  be a nontrivial central element in the term model of a lambda theory  $\mathcal{T}$ . Then, every  $\lambda$ -term belonging to the equivalence class  $e$  is unsolvable.*

**Proof.** By Cor. 5 the congruences  $\theta_e = \theta(e, \mathbf{T})$  and  $\bar{\theta}_e = \theta(e, \mathbf{F})$  on the term model of  $\mathcal{T}$  are nontrivial. Then, for every  $\lambda$ -term  $M \in e$ , the lambda theories  $\mathcal{T}_1$  and  $\mathcal{T}_2$ , generated respectively by  $\mathcal{T} \cup \{\mathbf{F} = M\}$  and  $\mathcal{T} \cup \{\mathbf{T} = M\}$ , are consistent. Assume, by the way of contradiction, that  $M$  is

solvable. By [2, Lemma 10.4.1(i)] it is consistent to equate two solvable  $\lambda$ -terms only if they are equivalent according to [2, Def. 10.2.9]. Then the  $\lambda$ -term  $M$  should be equivalent to  $\mathbf{F}$  and  $\mathbf{T}$ . By Remark 10.2.20(ii) in [2] this is possible only if the head variable of  $Mxy$ , where  $x$  and  $y$  are distinct variables, is equal to  $x$  and to  $y$  at the same time. This contradiction provides the conclusion of the lemma. ■

**Theorem 13** *The indecomposable semantics is complete for the set of semisensible lambda theories.*

**Proof.** The conclusion of the theorem follows if we show that the term model of every semisensible lambda theory is directly indecomposable. Assume, by the way of contradiction, that there is a semisensible lambda theory  $\mathcal{T}$  such that the term model  $\mathcal{M}_{\mathcal{T}}$  of  $\mathcal{T}$  admits  $e$  as a nontrivial central element. Since the term model  $\mathcal{M}_{\mathcal{T}}$  satisfies the identity  $exx = x$  (see Def. 3), then  $\mathcal{T} \vdash Mxx = x$  for all  $\lambda$ -terms  $M \in e$ . By Lemma 12  $M \in e$  is unsolvable. Then the unsolvable  $\lambda$ -term  $Mxx$  is provably equal in  $\mathcal{T}$  to the solvable  $\lambda$ -term  $x$ . This contradicts the hypothesis. ■

### 4.3 Continuous, stable and strongly stable semantics

We recall that an algebra is *simple* when it has just two congruences (so that every simple algebra is directly indecomposable).

In the next two theorems we give simple proofs of incompleteness for the classic semantics of lambda calculus.

**Theorem 14** (Honsell-Ronchi della Rocca [16]) *The semantics of lambda calculus given in terms of continuous models is incomplete.*

**Proof.** Let  $\mathbf{C}$  be a continuous model of lambda calculus. The function  $g$ , defined by:  $g(x) = c$  if  $x \not\leq b$  and  $g(x) = \perp$  otherwise, is Scott continuous for every arbitrary element  $c$ . We now show that  $\mathbf{C}$  is simple as a combinatory algebra. Let  $\theta$  be a congruence on  $\mathbf{C}$  such that  $a \theta b$  with  $a \neq b$ . We have  $a \not\leq b$  or  $b \not\leq a$ . Suppose that we are in the first case. Since the continuous function  $g$  is representable in the model, then we have:  $\perp = g(a) \theta g(b) = c$ . By the arbitrariness of  $c$  we get that  $\theta$  is trivial, so that  $\mathbf{C}$  is simple. The conclusion of the theorem follows from the algebraic incompleteness theorem (see Thm. 11), because every simple combinatory algebra is directly indecomposable. ■

The continuous function  $g$  of the above proof is neither stable nor strongly stable (see [5] for a full treatment of stable and strongly stable semantics).

**Theorem 15** (Gouy-Bastonero [15, 4]; Salibra [25, 26]) *The semantics of lambda calculus given in terms of stable or strongly stable models, and whose underlying domain is algebraic, is incomplete.*

**Proof.** Let  $\mathbf{C}$  be a (strongly) stable model of lambda calculus. Take  $a, b \in \mathbf{C}$  such that  $a \neq b$ . We have  $a \not\leq b$  or  $b \not\leq a$ . Suppose that we are in the first case. Then there is a compact element  $d$  of  $\mathbf{C}$  such that  $d \leq a$  and  $d \not\leq b$ . The step function  $f$  defined by  $f(x) = c$  if  $x \geq d$  and  $f(x) = \perp$  otherwise, is stable, and strongly stable for every element  $c$ . This function  $f$  can be used to show that every congruence on  $\mathbf{C}$  is trivial as in the proof of Thm. 14. Then the conclusion is again a consequence of the algebraic incompleteness theorem. ■

We do not know whether the stable and strongly stable models, whose underlying domains are *not* algebraic, are directly indecomposable as combinatory algebras.

Given a class  $\mathbb{C}$  of  $\lambda$ -models, we denote by  $\lambda\mathbb{C}$  the set of lambda theories which are representable in  $\mathbb{C}$  (see Section 2.3). It is unknown, in general, whether  $\lambda\mathbb{C}$  is a lattice with respect to the inclusion ordering of sets and whether  $\lambda\mathbb{C}$  is a sublattice of the lattice  $\lambda\mathcal{T}$  of lambda theories. In the remaining part of this subsection we show that, for each of the classic semantics of lambda calculus, the set  $\lambda\mathbb{C}$  is not closed under finite intersection, so that it is not a sublattice of the lattice  $\lambda\mathcal{T}$  of lambda theories.

**Theorem 16** *Let  $\mathbb{C}$  be a class of directly indecomposable models of lambda calculus. If there are two consistent lambda theories  $\mathcal{T}, \mathcal{S} \in \lambda\mathbb{C}$  such that*

$$\mathcal{T} \vdash \Omega = \mathbf{T}; \quad \mathcal{S} \vdash \Omega = \mathbf{F},$$

*then  $\lambda\mathbb{C}$  is not closed under finite intersection, so it is not a sublattice of  $\lambda\mathcal{T}$ .*

**Proof.** The term model of  $\mathcal{T} \cap \mathcal{S}$  admits a nontrivial central element  $\Omega$ , so that it is directly decomposable. It follows that  $\mathcal{T} \cap \mathcal{S} \notin \lambda\mathbb{C}$ . ■

We recall that the graph  $\lambda$ -models (see, for example, [6, 11]) and the filter  $\lambda$ -models (see, for example, [3]) are classes of models within the continuous semantics.

**Corollary 17** *Let  $\mathbb{C}$  be one of the following semantics: graph semantics, filter semantics, continuous semantics, stable semantics, strongly stable semantics (these last two semantics restricted to models whose underlying domain is algebraic). Then  $\lambda\mathbb{C}$  is not a sublattice of  $\lambda\mathcal{T}$ .*

**Proof.** Semantic proofs that  $\Omega$  is an easy term were given in each of the semantics specified in the statement of the theorem (see [5]). Then the conclusion follows from Thm. 16, because the models in each of these semantics are directly indecomposable as combinatory algebras. ■

#### 4.4 Concerning the number of decomposable models

Now we show that the incompleteness of the indecomposable semantics is as wide as possible.

The proof of following lemma is similar to that of [2, Prop. 17.1.9], where the case  $k = 1$  (due to Visser) is shown, and it is omitted.

**Lemma 18** *Suppose  $\mathcal{T}$  is a recursively enumerable (r.e.) lambda theory and fix arbitrary terms  $M_i, N_i$  for  $1 \leq i \leq k$  which are not provably equal in  $\mathcal{T}$ , that is, such that  $\mathcal{T} \not\vdash M_i = N_i$  for all  $i$ . Then there exists a term  $M$  such that*

$$\mathcal{T} \cup \{M = P\} \not\vdash M_i = N_i, \text{ for all } i \text{ and all closed terms } P.$$

Then the following theorem is a corollary of the algebraic incompleteness theorem.

**Theorem 19** *Let  $\mathcal{T}$  be an r.e. lambda theory. Then, the interval  $[\mathcal{T}] = \{\mathcal{S} : \mathcal{T} \subseteq \mathcal{S}\}$  contains a subinterval  $[\mathcal{S}_1, \mathcal{S}_2] = \{\mathcal{S} : \mathcal{S}_1 \subseteq \mathcal{S} \subseteq \mathcal{S}_2\}$ , constituted by a continuum of lambda theories, satisfying the following conditions:*

- $\mathcal{S}_1$  and  $\mathcal{S}_2$  are r.e. lambda theories;
- Every  $\mathcal{S} \in [\mathcal{S}_1, \mathcal{S}_2]$  is omitted by the indecomposable semantics (in particular,  $\mathcal{S}$  is omitted by the continuous, stable and strongly stable semantics).

**Proof.** The proof is divided into claims.

We first construct  $\mathcal{S}_1$ . We recall that a  $\lambda$ -term  $Q$  is  $\mathcal{T}$ -easy when, for every fixed closed  $\lambda$ -term  $P$ , the lambda theory generated by  $\mathcal{T} \cup \{Q = P\}$  is consistent.

**Claim 20** *There exists a  $\mathcal{T}$ -easy  $\lambda$ -term  $Q$ .*

By Lemma 18.

**Claim 21**  *$\mathcal{T} \not\vdash Q = \mathbf{T}$  and  $\mathcal{T} \not\vdash Q = \mathbf{F}$ .*

Trivial, because  $Q$  is  $\mathcal{T}$ -easy.

Let  $\mathcal{S}_1 = \mathcal{T}_1 \cap \mathcal{T}_2$ , where  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are the consistent lambda theories generated respectively by  $\mathcal{T} \cup \{Q = \mathbf{T}\}$  and  $\mathcal{T} \cup \{Q = \mathbf{F}\}$ .

**Claim 22** *The lambda theory  $\mathcal{S}_1$  is r.e. and contains  $\mathcal{T}$ .*

$\mathcal{S}_1$  is r.e., because it is the intersection of two r.e. lambda theories. The other property follows from  $\mathcal{T} \subseteq \mathcal{T}_1 \cap \mathcal{T}_2 = \mathcal{S}_1$ .

**Claim 23** *The term model of  $\mathcal{S}_1$  has a non trivial central element  $e$ .*

Let  $e = [Q]_{\mathcal{S}_1}$  be the equivalence class of the lambda term  $Q$ . It is easy to show that  $e$  satisfies the equations of Def. 3. Moreover,  $e$  is not trivial because  $\mathcal{S}_1 \not\vdash Q = \mathbf{T}$  and  $\mathcal{S}_1 \not\vdash Q = \mathbf{F}$ .

We now define the lambda theory  $\mathcal{S}_2$ .

**Claim 24** *There exists an r.e. lambda theory  $\mathcal{S}_2$ , which is a proper extension of  $\mathcal{S}_1$ , such that  $\mathcal{S}_2 \not\vdash Q = \mathbf{T}$  and  $\mathcal{S}_2 \not\vdash Q = \mathbf{F}$ .*

We apply Lemma 18 to the lambda theory  $\mathcal{S}_1$  and to the equations  $Q = \mathbf{T}$  and  $Q = \mathbf{F}$ . We get a  $\mathcal{S}_1$ -easy term  $R$  such that  $\mathcal{S}_1 \cup \{R = P\} \not\vdash Q = \mathbf{T}$  and  $\mathcal{S}_1 \cup \{R = P\} \not\vdash Q = \mathbf{F}$ , for all lambda terms  $P$ . Let  $\mathcal{S}_2 = \mathcal{S}_1 \cup \{R = \lambda x.x\}$ .  $\mathcal{S}_2$  is a proper extension of  $\mathcal{S}_1$  because otherwise  $R$  would not be a  $\mathcal{S}_1$ -easy term.

**Claim 25** *The equivalence class of  $Q$  is a non trivial central element of the term model of  $\mathcal{S}_2$ .*

The term model  $\mathcal{M}_{\mathcal{S}_2}$  of  $\mathcal{S}_2$  is a homomorphic image of the term model  $\mathcal{M}_{\mathcal{S}_1}$  of  $\mathcal{S}_1$ . Then, every equation satisfied by  $\mathcal{M}_{\mathcal{S}_1}$  is also satisfied by  $\mathcal{M}_{\mathcal{S}_2}$ . In particular, the equations characterizing  $Q$  as a central element. Finally,  $[Q]_{\mathcal{S}_2}$  is nontrivial as a central element because  $\mathcal{S}_2 \not\vdash Q = \mathbf{T}$  and  $\mathcal{S}_2 \not\vdash Q = \mathbf{F}$ .

**Claim 26** *For every lambda theory  $\mathcal{S}$  such that  $\mathcal{S}_1 \subseteq \mathcal{S} \subseteq \mathcal{S}_2$  the equivalence class of  $Q$  is non trivial central element of the term model of  $\mathcal{S}$ .*

We get the conclusion of the theorem because the interval  $[\mathcal{S}_1, \mathcal{S}_2]$  has a continuum of elements (see [2, Cor. 17.1.11]). ■

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