1. Consider the universe of teachers. Formalise the following sentences in the language:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>The teacher John (constant)</td>
</tr>
<tr>
<td>$b$</td>
<td>The teacher Mary (constant)</td>
</tr>
<tr>
<td>$W$</td>
<td>to be a teacher of math (unary predicate)</td>
</tr>
<tr>
<td>$E$</td>
<td>to be a good teacher (unary predicate)</td>
</tr>
<tr>
<td>$L$</td>
<td>older than (binary predicate)</td>
</tr>
<tr>
<td>$H$</td>
<td>hates (binary predicate)</td>
</tr>
</tbody>
</table>

- Every good teacher is older than some teacher of math

$$\forall x (E(x) \rightarrow \exists y (W(y) \land L(x, y)))$$

For every teacher $x$, if $x$ is good, then there exists a teacher $y$ of math such that $x$ is older than $y$

- Some teacher older than John and Mary hates every good teacher

$$\exists x (L(x, a) \land L(x, b) \land \forall y (E(y) \rightarrow H(x, y)))$$

There exists a teacher $x$, who is older than John and older than Mary, and who hates every good teacher

2. Find proofs in natural deduction of the following formulas (when this is possible). (Remember that the assumptions are on the left part of the symbol $\vdash$ and the conclusion on the right part).

(a) $A \land B \vdash (A \lor B) \land B$

$$
\frac{A \land B [\land e]}{A [\lor i]} \frac{A \land B [\land e]}{B [\land e]} \frac{A \lor B [\lor i]}{(A \lor B) \land B [\land i]}
$$
To prove $A \land B$ assuming $A \lor B$ is equivalent to prove the formula $(A \lor B) \rightarrow (A \land B)$. But this formula is not a tautology. It becomes false when $A$ is true and $B$ is false.

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(c) $A \rightarrow B, C \rightarrow D \vdash (A \land C) \rightarrow (B \land D)$

\[
\frac{A \land C}{A} \quad \frac{A \land C}{C} \\
\hline
\frac{A \rightarrow B}{B} \quad \frac{C \rightarrow D}{D} \\
\hline
\frac{B \land D}{\bot} \quad \frac{A \land C}{\bot} \\
\hline
\frac{B \land D}{\bot} \\
\]

(d) $A \lor B, \neg B \vdash A$

\[
\frac{[B]^* \neg B}{\bot} \quad \frac{\bot}{[\neg] A} \\
\hline
\frac{A \lor B}{A} \\
\]

3. (a) Provide a proof of $\neg \neg A \rightarrow A$ in natural deduction.

\[
\frac{[\neg A]^* \land [\neg \neg A]^*}{\neg \neg A \rightarrow A} \\
\]

(b) Show that the deduction rule (RRA) can be proven from the other ones if we assume the formula $\neg \neg A \rightarrow A$.

\[
\frac{[\neg A]^*}{\bot} \quad \frac{\bot}{[\neg] A} \\
\hline
\frac{\neg \neg A}{[\neg \neg] A} \quad \frac{\neg \neg A \rightarrow A}{[\rightarrow] A} \\
\]

4. Explain the completeness theorem of first-order logic.

Notation: $T \vdash \phi$ means “there exists a formal proof of $\phi$ whose assumptions are in $T$. If $M$ is a model, then $M \models \phi$ means that the sentence $\phi$ is true in $M$. If $T$ is a set of sentences, $M \models T$ means that every sentence $\phi \in T$ is true in $M$. 


The following is the completeness theorem of first-order logic:

\[ T \vdash \phi \text{ if and only if } \forall M (M \models T \Rightarrow M \models \phi) \]

To prove that there exists a formal proof of \( \phi \) from \( T \), it is sufficient to prove that \( \phi \) is true in every model satisfying the set \( T \) of axioms.

5. Let \( \phi \equiv \forall x \forall y (C(a) \land C(x) \rightarrow \exists z B(y, z)) \) be a sentence, where \( a \) is a constant.

(a) Define a model of \( \phi \).

Model \( \mathcal{M} \) with universe: \( M = \{ \text{dog, cat, book} \} \)
\( a \) is a constant symbol: \( a^\mathcal{M} = \text{dog} \).

\( C \) is a unary relation symbol. The interpretation is a function \( C^\mathcal{M} : M \rightarrow \{ \text{true, false} \} \):
\( C^\mathcal{M}(\text{dog}) = \text{false}; C^\mathcal{M}(\text{cat}) = \text{false}; C^\mathcal{M}(\text{book}) = \text{false}. \)

\( B \) is a binary relation symbol. We interpret \( B^\mathcal{M} : M \times M \rightarrow \{ \text{true, false} \} \) as follows: for all \( x, y \in M \), \( B^\mathcal{M}(x, y) = \text{true} \) iff \( x = y \).

We show that \( M \models \phi \).

Consider the formula \( C(a) \land C(x) \rightarrow \exists z B(y, z) \), where \( x, y \) ranges over \( M \). Since \( C^\mathcal{M}(a) \land C^\mathcal{M}(\text{dog}), C^\mathcal{M}(a) \land C^\mathcal{M}(\text{cat}), \) and \( C^\mathcal{M}(a) \land C^\mathcal{M}(\text{book}) \) are all false, then the implication is true for every \( x \) and \( y \).

(b) Define a model of \( \neg \phi \).

Model \( \mathcal{M} \) with universe: \( M = \{ \text{dog} \} \)
\( a^\mathcal{M} = \text{dog} \).

\( C^\mathcal{M}(\text{dog}) = \text{true}. \)
\( B^\mathcal{M}(\text{dog}, \text{dog}) = \text{false}. \)

Then \( \mathcal{M} \models \neg \phi \).