
Lambda Calculus: Visser topology

by

Antonino Salibra

Università Ca'Foscari di Venezia

www.dsi.unive.it/~salibra

Visser and Ershov Topologies

- Scott topology stands to partial ordering theory as Visser and Ershov topologies stand to computability theory.

Let Λ be the set of lambda terms. We consider the following family RE of subsets O of Λ :

- O is β -closed (i.e., $M \in O, M =_{\beta} N \implies N \in O$);
- O is a recursively enumerable (r.e., for short) set.

RE is a distributive sublattice of the Boolean algebra of all subsets of Λ (in other words, RE is closed under finite intersection and under finite union).

The **Ershov topology** on Λ is the topology generated by RE as a base of open sets, while the **Visser topology** is the topology generated by RE as a base of closed sets.

- If T is a lambda theory, the Ershov and Visser topologies on Λ/T are the quotient topologies.
- **Fact:** If T is a r.e. lambda theory, then the Ershov topology is the discrete topology, because the T -equivalence class of a lambda term is r.e.
- **Fact:** The application operator on Λ is Visser and Ershov continuous in the first coordinate and in the second coordinate. For example, the inverse image of a β -closed and r.e. set w.r.t. the map

$$P \mapsto MP$$

is a β -closed and r.e. set.

Visser Topology

Theorem (Visser). The Visser topology is hyperconnected on Λ (i.e., the intersection of two Visser opens is never empty).

Proof. Let $|\cdot| : \Lambda \rightarrow \mathbb{N}$ be any bijective and computable function from Λ onto the set \mathbb{N} of natural numbers. We denote by $*$ the inverse map of $|\cdot|$: $|M|^* = M$. Recall from Chapter 1 the definition of the numerals in lambda calculus: $\underline{0} = I$; $\underline{n+1} = [F, \underline{n}]$.

The proof is divided into claims.

Claim 1. There exists a lambda term E such that $E|M| \rightarrow_{\beta} M$, for every term M (in other word, $E\underline{k} \rightarrow_{\beta} k^*$).

See Section 8.1 in Barendregt's book.

Claim 2. For every partial computable function $f: \mathbb{N} \rightarrow_p \mathbb{N}$ there exists a total computable function g such that

$$f \downarrow n \implies f(n)^* =_{\beta} g(n)^*.$$

Let F be a lambda term representing the function f in lambda calculus. We define

$$g(\underline{n}) = |E(F\underline{n})|.$$

Assume $f(\underline{n}) = k$. Then we get the conclusion as follows:

$$g(\underline{n})^* = |E(F\underline{n})|^* = E(F\underline{n}) =_{\beta} (\text{by } f(\underline{n})=k) E\underline{k} =_{\beta} (\text{by Claim 1}) k^* = f(\underline{n})^*.$$

Visser Topology

- **Claim 3.** If $O \cup U = \Lambda$, where O and U are r.e. and β -closed sets, then either O is Λ or U is Λ .

By contraposition assume that neither O nor U is Λ . Let $a \in O - U$ and $b \in U - O$. Let A and B be two recursively inseparable sets of natural numbers (i.e., $A \not\sim B = \hat{}$ and there exists no recursive set C such that $A \subseteq C$ and $B \subseteq \bar{C}$). Define the partial map

$$f(x) = \begin{cases} |a| & \text{if } x \in A \\ |b| & \text{if } x \in B \end{cases}$$

Then there exists a total computable map g satisfying the conclusion of Claim 2.

Define $P = \{x : g(x)^* \in O\}$ and $Q = \{x : g(x)^* \in U\}$. Then P and Q are r.e. such that $P \cup Q = \mathbb{N}$. We show that $A \subseteq P - Q$ and $B \subseteq Q - P$. By definition $f(x) = |a|$ for every $x \in A$, that implies $g(x)^* =_{\beta} a$ for every $x \in A$ (by Claim 2). Since O and U are β -closed and $a \in O - U$, then $g(x)^* \in O - U$ for every $x \in A$. Then $A \subseteq P - Q$. Similarly, we can prove the other inclusion $B \subseteq Q - P$. Starting from P and Q we can construct a recursive set C such that $A \subseteq C$ and $B \subseteq \bar{C}$. Then we get a contradiction.

- **Corollary.** Every β -closed set of lambda terms (different from the empty set and Λ) is not recursive.

By hyperconnection.

Visser Topology

- **Corollary.** If T is a r.e. lambda theory, then the Visser topology on Λ/T satisfies the separation axiom T_1 , but it is not sober.

Proof. The T -equivalence class of every lambda term is β -closed and r.e., so that it is an element of the base of closed sets. A T_1 space is sober if the unique irreducible closed sets are the singleton sets, while by hyperconnection Λ/T is irreducible.

- **Range Theorem.** Let T be an r.e. lambda theory. Every unary function representable on the term algebra of T is either a constant function or has an infinite range.

Proof. Assume that the range of a representable function is finite with cardinality ≥ 2 . Since the Visser topology is T_1 by hypothesis, then the topology induced on the finite range is discrete. Then the inverse image of an arbitrary proper subset of the range is clopen (i.e., contemporaneously open and closed). This contradicts the hyperconnectedness of the Visser topology.

- **Corollary.** The application operator is not Visser continuous.