A Note on a Capacity Result for the Hopfield Neural Network

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Abstract

In this note we prove, by using straightforward arguments, a stronger version of a well-known capacity result proved by Abu-Mostafa and St. Jacques. Our results states that no Hopfield memory can have a "capacity" (in Abu-Mostafa and St. Jacques's sense) exceeding 1, thereby showing the inadequacy of their capacity notion.

Key words: Hopfield networks, Associative memory, Capacity.

In a well-known paper, Abu-Mostafa and St. Jacques [1] proved the following theorem, which provides an upper bound for the "capacity" of the (discrete) Hopfield neural network.

Theorem 1 *Let $p \leq 2^{n-1}$ be an integer satisfying the following condition. For any p-set of binary vectors $\xi^{(1)}, \cdots, \xi^{(p)} \in \{-1, +1\}^n$ there exists a zero-diagonal $n \times n$ matrix $W = (w_{ij})$ and an n-vector $\mathbf{t}$ such that*

$$\xi_i^{(\alpha)} = sgn\left(\sum_{j=1}^{n} w_{ij}\xi_j^{(\alpha)} - t_i\right) \tag{1}$$

for all $\alpha = 1 \ldots p$ and $i = 1 \ldots n$. Then $p \leq n$.

Condition (1) amounts to saying that there exists a Hopfield network with n binary neurons that can store the p patterns $\xi^{(1)}, \cdots, \xi^{(p)}$.

Despite the fact that several authors have long pointed out its intrinsic limitations [2, 5], the above theorem is considered as one of the most general results concerning the capacity of the Hopfield model [4, 3]. In fact, in contrast to other results, no assumption is made on the way in which the matrix W is constructed, and no restriction is imposed as to its structure, apart from being zero-diagonal. Moreover, the result holds no matter whether the network is operating in the synchronous or asynchronous mode.

In this note it is shown, using straightforward arguments, that the bound derived by Abu-Mostafa and St. Jacques turns out to be excessively optimistic for the true "capaicty" (in the strong sense specified above) of the Hopfield memory cannot in fact exceed 1. This follows immediately from the next proposition which, in turn, is an obvious consequence of the (essential) requirement that the weight matrix W be zero-diagonal.

Proposition 2 *No Hopfield associative memory can store two binary patterns $\xi^{(1)}$ and $\xi^{(2)}$ at Hamming distance 1.*

Proof: Let the two n-vectors $\xi^{(1)}$ and $\xi^{(2)}$ differ (only) in their k-th component, that is:

$$\xi_j^{(2)} = \begin{cases} -\xi_j^{(1)}, & \text{if } j = k \\ \xi_j^{(1)}, & \text{if } j \neq k \end{cases}$$

To store the two patterns we have to find a zero-diagonal weight matrix W and a threshold vector $\mathbf{t}$ such that condition (1) holds for all $\alpha = 1, 2$ and $i = 1 \ldots n$. Since $\xi^{(1)}$ and $\xi^{(2)}$ do not differ in the components $j \neq k$, it follows that, for $i = k$,

$$\sum_{j=1}^{n} w_{kj} \xi_j^{(1)} - t_k = \sum_{j=1}^{n} w_{kj} \xi_j^{(2)} - t_k$$

and this quantity is required to be positive and negative at once. Clearly, no such matrix and vector exist, and this proves the proposition. $\square$

On of the most immediate implications of the above result is that as soon as $p \geq 2$ we are *always* able to find a set of p data vectors that cannot be stored together in a Hopfield memory, and this immediately proves the following assertion, which is a stronger version of Theorem 1.

Theorem 3 *Let p be an integer satisfying the condition of Theorem 1. Then $p = 1$.*

Indeed, we mention that Baldi [2] did anticipate the preceding fact but in a less explicit fashion, and apparently without recognition of its generality. As he put it: "...the result [of Abu-Mostafa and St. Jacques] as stated is problematic for it is easy to find small configurations of vectors...that cannot be stored. However, these counterexamples are artificial and often use vectors at Hamming distance 1" (p. 528). Thus, Baldi developed his arguments simply on the grounds of specific examples, not in terms of a general statement as we did.

References

[1] Y. S. Abu-Mostafa and J. M. St. Jacques, "Information capacity of the Hopfield model," *IEEE Trans. Inform. Theory*, vol. 31, no. 4, pp. 461–464, 1985.

[2] P. Baldi, "Neural networks, orientations of the hypercube, and algebraic threshold functions," *IEEE Trans. Inform. Theory*, vol. 34, no. 3, pp. 523–530, 1988.

[3] L. Fausett, *Fundamentals of Neural Networks.* Prentice-Hall, Upper Saddle River, N.J., 1994.

[4] Y. Kamp and M. Hasler, *Recursive Neural Networks for Associative Memory.* Wiley, Chichester, UK, 1990.

[5] R. J. McEliece, E. C. Posner, E. R. Rodemich, and S. S. Venkatesh, "The capacity of the Hopfiel associative memory," *IEEE Trans. Inform. Theory*, vol. 33, no. 4, pp. 461–482, 1987.