Hierarchical Matching via Kronecker Products of Transitive Closures *

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Abstract

In this technical report we propose an alternative to the concept of association graphs for matching attributed rooted trees T_1 and T_2. Instead of constructing the sum of two Kronecker products (the association graph), we deal with the Kronecker product $K(\overline{T_1}, \overline{T_1})$ of the transitive closures of the two trees. Thus, order preserving mappings (of maximum similarity) between T_1 and T_2 may be characterized in terms of subgraphs of $K(\overline{T_1}, \overline{T_1})$. The problem of finding these subgraphs, in turn, is approximated by the solution of a weighted matroid intersection problem.

1 Introduction

Representing images by attributed graphs and using graph matching techniques to compare images is an appealing concept because of the invariance of graphs with respect to geometric transformations. However, one has to cope with the following problems.

- Finding common subgraphs of maximum similarity is an NP-hard problem, in general.
- Due to diverse resolutions or viewpoints the same object may be represented by a fine or a coarse graph and the contractions to compensate for these differences are not known a-priori.
- "Similar objects" may be represented by graphs with small (but many) common minors. A single common subgraph of maximum similarity is not necessarily a good measure of image similarity.

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The idea behind this technical report is to relax the conditions for matching in order to solve the above problems simulataneously. We restrict ourselves to matching rooted trees with unordered branches such that the hierarchy is preserved. However, " a mapping between two rooted trees T_1 and T_2 that preserves the hierarchy" can be understood in many ways.

(a) An isomorphism between a minor of T_1 and a minor of T_2.

(b) A one-to-one correspondence β between edges from T_1 and edges from T_2 such that for any two edges e_1, e_1' in T_1 with $e_1 \prec e_1'$ it holds that $\beta(e_1) \prec \beta(e_1')$.

(c) A one-to-one correspondence β between edges from T_1 and edges from T_2 such that for any two edges e_1, e_1' in T_1 with $e_1 \prec e_1'$ it never holds that $\beta(e_1) \succ \beta(e_1')$.

In contrast to [2] a mapping β is not characterized by a (maximal) clique in the association graph of T_1 and T_2 but by a (maximal) sub-forest in the Kronecker product of the transitive closures of T_1 and T_2. Via the weighted matroid intersection problem, we will thus arrive at an approximation of the matching problem in polynomial time.

The plan of this technical report is as follows. In Section 2 we provide basic defintions and characterize the minors of a rooted tree T in terms of subgraphs of transitive closures of T. In section 3 we define relaxed conditions on graph matching, i.e. *order preserving path bijections* and characterize the latter in terms of Kronecer products of transitive closures. In Section 3, analogous to weighted association graphs, we intoduce weights (here on the edges of the Kronecker product of the transitive closures) and thus arrive at a characterization of order preserving path bijections of maximum similarity. Finally, in Section 5, the problem of finding order preserving path bijections of maximum similarity is approximated via a weighted matroid intersection problem.

2 Contracting rooted trees

We restrict ourselves to simple and directed graphs (short: graphs).

Definition 2.1 (Graph $G = (V, E)$, $C(V)$, $source(e)$, $target(e)$, leaf) *A graph $G = (V, E)$ is given by a finite set V and a subset E of*

$$C(V) := \{(v, w) \ : \ v, w \in V, \ v \neq w\}. \tag{1}$$

The elements of V and E are called vertices and (directed) edges, respectively. Let $e = (v, w) \in E$. Then v and w are called the source and the target of e, respectively. We write $v = source(e)$ and $w = target(e)$. A vertex $l \in V$ is called leaf of G if there exists no $e \in E$ with $v = target(e)$.

Definition 2.2 (Subgraph, vertex and edge induced subgraphs)
Let $G = (V, E)$ be a graph. A subgraph of G is a graph $S = (V_S, E_S)$ with $V_S \subseteq V$ and $V_E \subseteq E$. For $V_S \subseteq V$ the graph $S = (V_S, E_S)$ defined by

$$E_S := \{(v, w) \in E \mid v, w \in V_S\}$$

is called subgraph of G induced by V_S. Conversely, for $E_S \subseteq E$ the graph $S = (V_S, E_S)$ defined by

$$V_S := \{v \in V \mid \exists w \in V \text{ with } (v, w) \in E \ \vee \ (w, v) \in E\}$$

is called subgraph of G induced by E_S.

Definition 2.3 (Rooted tree)
A graph $T = (V, E)$ is called rooted tree if there exists unique $r \in V$ (the root) such that r is not a source of any $e \in E$ and such that each $w \in V$, $w \neq r$ is the source of exactly one $e \in E$.

Definition 2.4 (Path Π from u_1 to u_k on G)
Let $G = (V, E)$ be a graph and let $\Pi = (u_1, \ldots, u_k)$, $k \geq 2$ be a sequence of vertices from V such that $(u_i, u_{i+1}) \in E$ for all i, $1 \leq i < k$. Then Π is called path from u_1 to u_k on G.

Definition 2.5 (Predecessor, successor relations $\prec$, $\succ$, $\preceq$, $\succeq$)
Let $G = (V, E)$ be a graph and let $v, w \in V$. Vertex v is called predecessor of vertex w, write $v \prec w$, if there exists a path from v to w on G. In this case w is called a successor of v and we write $w \succ v$. Moreover, we write $v \preceq w$ and $w \succeq v$, if $v = w$ or $v \prec w$. Relations $\prec$, $\succ$ extend to binary relations on E by

- $e \prec e' :\Longleftrightarrow target(e) \preceq source(e')$,
- $e \preceq e' :\Longleftrightarrow e \prec e'$ *or* $e = e'$, *and*
- $e \succ e' :\Longleftrightarrow e' \prec e$, *and*
- $e \succeq e' :\Longleftrightarrow e' \preceq e$.

Definition 2.6 ($\overline{E}$, transitive closure $\overline{G}$)
Let $G = (V, E)$ be a graph. Setting $\overline{E} := \{(v, w) \mid v \prec w\}$, the transitive closure $\overline{G}$ of G is the graph $\overline{G} := (V, \overline{E})$.

Definition 2.7 ($Pred_G(v)$, contraction, coarsening of T, minor)
Let $G = (V, E)$ be a graph, let $(v, w) \in E$, and let $Pred_G(v)$ be the (possibly empty) set of vertices $t \in V$ such there exists $(t, v) \in E$. The contraction of v into w on G transforms G into $G_{v,w} = (V_{v,w}, E_{v,w})$ defined by $V_{v,w} := V \setminus \{v\}$ and

$$E_{v,w} := \{e = (x, y) \in E \mid x, y \neq v\} \ \cup \ \{(u, w) \mid u \in Pred_G(v)\}. \tag{2}$$

A graph obtained from G by a (possibly empty) sequence of contractions is called a coarsening of T. A minor of G is a subgraph of a coarsening of T.

Definition 2.8 (Closed subgraph)
Let G be a graph with edge set E and let $S = (V_S, E_S)$ be a subgraph of G. Then S is called a closed subgraph of G, if for each $e = (v, w) \in E$ with $v, w \in V_S$ there exists a path from v to w on S.

Proposition 2.9 (Coarsening of $T \leftrightarrow$ closed rooted subtree of $\overline{T}$)
Let T be rooted tree with root r. Then each coarsening of T is a closed subtree of $\overline{T}$ with root r and vice versa.

Proof: "$\longrightarrow$"
Let C be a coarsening of T. Clearly, C is a subtree of T with root r. It remains to show that C is a closed subgraph of $\overline{T}$. The proof is by induction on the number n of contractions needed to generate C from T. The case $n = 0$ is clear. Now assume that we have already performed $n - 1$ contractions resulting in a coarsening $C^{n-1} = (V^{n-1}, E^{n-1})$ of $\overline{T}$. Furthermore, let $(v, w) \in E^{n-1}$, and let $C = (V_C, E_C)$ be the coarsening obtained by contracting v into w on C^{n-1}. From the assumption of the induction and Definition 2.7 it follows that $V_C = V^{n-1} \setminus \{v\}$. To see that C is a closed subgraph of $\overline{T}$ it suffices to consider an edge (u, w) of $\overline{T}$ with $u \in Pred_{C^{n-1}}(v)$ and to show that there exists a path from u to w on C. By the assumption of the induction there exists a path $(u, \ldots, v, w)$ on C^{n-1} and, since C is obtained from C^{n-1} by contracting v into w on C^{n-1}, we get that there also exists a path from u to w on C.
"$\longleftarrow$"
Conversely, let $T = (V, E)$ and let $C = (V_C, E_C)$ be a closed subtree of $\overline{T} = (V, \overline{E})$ with root r. We show that C is the coarsening of T obtained from T by contracting all edges with source in $V \setminus V_C$. We perform an induction on the cardinality n of $V \setminus V_C$. The case $n = n_0$ is clear. Now assume that the cardinality of $V \setminus V_C$ is $n > 0$. Then there exists an edge $(v, w) \in E$ such that $v \in V \setminus V_C$ and $w \in V_C$ (take $v \in V \setminus V_C$ closest to the root of T, for example). Furthermore, let

$$Pred_{C,v}(w) := \{z \in Pred_C(w) \mid \text{the path from } z \text{ to } w \text{ on } T \text{ contains } v\}.$$

Define the graph $C^{n-1} = (V^{n-1}, E^{n-1})$ by

$$\begin{aligned} V^{n-1} &:= V_C \cup \{v\} \text{ , and} \\ E^{n-1} &:= E_C \cup \{(v, w)\} \cup \{(z, v) \mid z \in Pred_{C,v}(w)\} \setminus \\ &\quad\ \{(z, w) \mid z \in Pred_{C,v}(w)\}. \end{aligned}$$

From C being a closed subtree of $\overline{T}$ with root r it follows that also C^{n-1} is a closed subtree of $\overline{T}$ with root r. The assumption of the induction now implies that C^{n-1} is the coarsening of T obtained from T by contracting all edges with source in $V \setminus (V_C \cup \{v\})$. Contracting v into w on C^{n-1} yields C. Hence, C is the coarsening of T obtained from T by contracting all edges with source in $V \setminus V_C$. $\square$

3 Order preserving path bijections

Definition 3.1 (Projections p_1 and p_2) *Let $G = (V_1 \times V_2, E)$ be a graph. The projections p_i of V onto V_i $(i = 1, 2)$ are defined by*

$$p_i(u_1, u_2) := u_i \quad \forall (u_1, u_2) \in V. \tag{3}$$

The projections p_i extend to mappings from E to $V_i \times V_i$ by

$$p_i((u_1, u_2), (v_1, v_2)) := (u_i, v_i) \quad \forall ((u_1, u_2), (v_1, v_2)) \in E. \tag{4}$$

Definition 3.2 (Kronecker product $K = K(G_1, G_2)$, $p_i(S)$)
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. The Kronecker product of G_1 and G_2 is the graph $K = K(G_1, G_2) = (V_1 \times V_2, E_K)$, where E_K consists of all $e \in C(V_1 \times V_2)$ (see Definition 2.1) with

$$p_1(e) \in E_1 \wedge p_2(e) \in E_2. \tag{5}$$

Let $S = (V_S, E_S)$ be a subgraph of K. Then the graphs $p_i(S) = (V_i^S, E_i^S)$ are defined by $V_i^S := p_i(V_S)$, $E_i^S := p_i(E_S)$, $i = 1, 2$.

Proposition 3.3 (Closed subgraphs of Kronecker products)
Let S be a closed subgraph of $K(G_1, G_2)$. Then $p_i(S)$ is an closed subgraph of G_i, $i = 1, 2$.

Proof: Let $G_i = (V_i^G, E_i^G)$, $p_i(S) = (V_i^S, E_i^S)$, $i = 1, 2$. Furthermore, let S be a closed subgraph of $K(G_1, G_2)$ and assume that, without lack of generality, $p_1(S)$ is not a closed subgraph of G_1. Then there exists an edge $e_1 = (v_1, w_1) \in E_1^G$ with $v_1, w_1 \in V_1^S$ but there exists no path from v_1 to w_1 on $p_1(S)$. Let $e = (v, w)$ be an edge of $K(G_1, G_2)$ with $p_1(e) = e_1$ and $p_2(e) \in E_2^S$. Clearly, v, w are vertices of S, but there exists no path from v to w on S - a contradiction to S being a closed subgraph of $K(G_1, G_2)$. □

Notation 3.1 ($T_i = (V_i, E_i)$, r_i)
Throughout the paper $T_i = (V_i, E_i)$ denotes a tree with root r_i, $i = 1, 2$.

Definition 3.4 (Order preserving [OP] path bijection β, $\mathcal{B}(T_1, T_2)$)
Let M_i be a minor of T_i with edge set E_i^M, $i = 1, 2$. Moreover, let β: $E_1^M \mapsto E_2^M$ be a bijection such that there exists no pair $(v_i, w_i), (v_{3-i}, w_{3-i}) \in V_\beta$ with $v_i \prec w_i$ in T_i and $w_{3-i} \prec v_{3-i}$ in T_{3-i} for $i = 1$ or $i = 2$. Then β is called an order preserving [OP] path bijection. The collection of all OP path bijections between T_1 and T_2 is denoted by $\mathcal{B}(T_1, T_2)$.

Definition 3.5 (Graph $G_\beta = (V_\beta, E_\beta)$ of β)
Let M_i be a minor of T_i with edge set E_i^M $(i = 1, 2)$ and let β: $E_1^M \mapsto E_2^M$ be a bijection. Then G_β is the subgraph of $K(\overline{T_1}, \overline{T_2})$ induced by

$$E_\beta := \{(v_1, v_2), (w_1, w_2) \mid (v_1, w_1) \in E_1^M \;\wedge\; (v_2, w_2) \in E_2^M\}.$$

The vertex set of G_β is denoted by V_β.

The proof of the following proposition is straight forward.

Proposition 3.6 (Characterization of OP path bijections in $K(\overline{T_1}, \overline{T_2})$) *Let $S = (V_S, E_S)$ be a subgraph of $K(\overline{T_1}, \overline{T_2})$. If*

1. *$p_i(e) \neq p_i(e')$ for all $e \neq e' \in E_S$ $(i = 1, 2)$,*
2. *$p_i(S)$ is a minor of T_i, $i = 1, 2$, and*
3. *there exists no pair $(v_i, v_{3-i}), (w_i, w_{3-i}) \in V_S$ with $v_i \prec w_i$ in T_i and $w_{3-i} \prec v_{3-i}$ in T_{3-i} for $i = 1$ or $i = 2$,*

then G_β is the graph of the OP path bijection β: $p_1(E_S) \mapsto p_2(E_S)$ defined by

$$\beta(e_1) := p_2(e) \text{ , where } e \in E_S \text{ is such that } p_1(e) = e_1.$$

Conversely, let $S = G_\beta$ for some OP path bijection β. Then S and the edge set E_S of S fulfill the three items above.

4 OP path bijections of maximum similarity

Notation 4.1 (Similarity measure σ) *From now on let $\sigma : \overline{E_1} \times \overline{E_2} \mapsto \mathbb{R}_{>0}$, where $\mathbb{R}_{>0}$ denotes the set of positive real numbers. The mapping σ extends to a mapping from subgraphs $S = (V_S, E_S)$ of $K(\overline{T_1}, \overline{T_2})$ to $\mathbb{R}_{>0}$ by*

$$\sigma(S) := \sum_{e \in E_S} \sigma(e). \tag{6}$$

Definition 4.1 (σ-maximum OP path bijection) *Let $\beta \in \mathcal{B}(T_1, T_2)$. Then β is said to be σ-maximum, if $\sigma(G_\beta) \geq \sigma(G_{\beta'})$ for all $\beta' \in \mathcal{B}(T_1, T_2)$.*

Notation 4.2 (Edge set $\overline{E}_K$ of $K(\overline{T_1}, \overline{T_2})$)
For the rest of the paper the edge set of $K(\overline{T_1}, \overline{T_2})$ is denoted by $\overline{E}_K$.

Proposition 4.2 (Leafs of G_β with σ-maximum β)
Let β be a σ-maximum OP path bijection. Then each leaf of G_β is also a leaf of $K(\overline{T_1}, \overline{T_2})$.

Proof: Let $l \in V_\beta$ be a leaf of G_β. We assume that l is not a leaf of $K(\overline{T_1}, \overline{T_2})$. It suffices to show that then there exists $\beta' \in \mathcal{B}(T_1, T_2)$ with $\sigma(G_{\beta'}) > \sigma(G_{\beta'})$. Indeed, from the assumption it follows that there exist $(u_1, u_2) \in V_1 \times V_2$ and $e \in \overline{E}_K$ such that $p_i(e) = (u_i, p_i(l))$ in $\overline{E_i}$, $i = 1, 2$. We define $G' = (V', E')$ by

- $V' := V_\beta \cup \{(u_1, u_2)\}$, and
- $E' := E_\beta \cup \{((u_1, u_2) \text{ , } (p_1(l), p_2(l)))\}$.

It remains to show that G' fulfills items 1., 2., and 3. of Proposition 3.6. Items 1. and 3. follow directly from the construction of G'.
To see that item 2. of Proposition 3.6 holds for G' instead of G, let $C_i = (V_i^C, E_i^C)$ denote a coarsening of T_i such that $p_i(E_\beta) \subseteq E_i^C$. Furthermore, set

$$\begin{aligned} V_i' &:= V_i^C \cup \{u_i\} \ , \\ E_i' &:= E_i^C \cup \{(u_i, p_i(l))\} \ , \end{aligned}$$

and let $C_i' = (V_i', E_i')$. Clearly, $p_i(E') \subseteq E_i'$. Now it remains to show that C_i' is a coarsening of T_i. Using Proposition 2.9 we may as well show that C_i' is a closed subtree of $\overline{T_i}$ with root r_i. By construction, C_i' is a subtree of $\overline{T_i}$ with root r_i. To see that C_i' is a closed subgraph of $\overline{T_i}$ it suffices to consider an edge $(u_i, v_i) \in \overline{E_i}$ with $v_i \in p_i(V_\beta)$ and to make sure that there exists a path from u_i to v_i on C_i'. But this is clear since one can go from u_i to $p_i(l)$ first, and then, due to C_i being a closed subgraph of $\overline{T_i}$ (see Proposition 2.9 again), one can go on to v_i. □

Definition 4.3 (Concatenation of edges, $Con(E)$, $\lhd$)
Let $G = (V, E)$ be a graph and let $e, e^a, e^b \in E$. Then e is called concatenation of e^a and e^b, if

$$source(e^a) = source(e) \ \wedge \ target(e^b) = target(e) \ \wedge \ target(e^a) = source(e^b).$$

In this case we write $e^a \lhd e$, $e^b \lhd e$. The set of all edges from E that are concatenations of two edges from E is denoted by $Con(E)$.

Assumption 4.1 *From now on we assume that for any $e^a \neq e^b$, $f^a \neq f^b$ in $\overline{E}_K$ and $e, f \in Con(\overline{E}_K)$ with $e^a, e^b \lhd e$, $f^a, f^b \lhd f$, it holds that $\sigma(e, f) \leq \sigma(e^a, f^a) + \sigma(e^b, f^b)$.*

Definition 4.4 (Basic edge in graph G)
Let G be a graph. An edge e in G such that e is not a concatenation of two other edges in G is called basic in G.

Proposition 4.5 ($\exists$ σ-maximum β : E_β contains basic edges only)
There always exist a σ-maximum $\beta \in \mathcal{B}(T_1, T_2)$ such that E_β contains only edges that are basic in $\overline{E}_K$.

Proof: Let $e \in E_\beta$ be non-basic in $\overline{E}_K$. Then there exist $e^a, e^b \in \overline{E}_K$ such that e is the concatenation of e^a and e^b. Furthermore, set

- $V' := V_\beta \cup \{target(e^a)\}$,
- $E' := E_\beta \cup \{e^a, e^b\} \setminus \{e\}$,

and let $G' = (V', E')$. Due to Assumption 4.1 it suffices to show that $G' \in \mathcal{B}(T_1, T_2)$, i.e. that items 1., 2. and 3 of Proposition 3.6 hold not only for G_β but also for G'. Items 1. and 3. follow directly from the construction of G'.

To see that item 2. of Proposition 3.6 holds also for G', let $C_i = (V_i^C, E_i^C)$ denote a coarsening of T_i such that $p_i(E_\beta) \subseteq E_i^C$. Furthermore, set

$$\begin{array}{rcl} V_i' & := & V_i^C \cup \{p_i(target(e^a))\} \ , \\ E_i' & := & E_i^C \cup \{p_i(e^a), p_i(e^b)\} \setminus \{p_i(e)\} \ , \end{array}$$

and let $C_i' = (V_i', E_i')$. Clearly, $p_i(E') \subseteq E_i'$. Now it remains to show that C_i' is a coarsening of T_i. Using Proposition 2.9 we may as well show that C_i' is a closed subtree of $\overline{T_i}$ with root r_i. By construction, C_i' is a subtree of $\overline{T_i}$ with root r_i. To see that C_i' is a closed subgraph of $\overline{T_i}$ it suffices to prove the following.

(i) For each edge $e_i = (v_i, p_i(target(e^a))) \in \overline{E_i}$ with $v_i \in V_i^C$ there exists a path from v_i to $p_i(target(e^a))$ on G'. Indeed, since C_i is a closed subgraph of $\overline{T_i}$ (see Proposition 2.9 again), one can go from v_i to $p_i(source(e^a))$ and then one can go on to $p_i(target(e^a))$.

(ii) For each edge $e_i = (p_i(target(e^a)), v_i) \in \overline{E_i}$ with $v_i \in V_i^C$ there exists a path from $p_i(target(e^a))$ to v_i on G'. The proof is analogous to the one in (i). □

Definition 4.6 (Half-roots with respect to T_1 and T_2, $V_{nhr}(T_1, T_2)$)
Let T_i be rooted trees with vertex sets V_i and roots r_i, $i = 1, 2$. An element $v \in V_1 \times V_2$ is called half-root with respect to T_1 and T_2, if

$$p_i(v) = r_i \ \wedge \ p_{3-i}(v) \neq r_{3-i}$$

for $i = 1$ or $i = 2$. Moreover, $V_{nhr}(T_1, T_2)$ denotes the set of elements in $V_1 \times V_2$ that are not half-roots with respect to T_1 and T_2.

Definition 4.7 (E_b^*, E_l^*, E^*, $K^*(T_1, T_2)$)
Define $E_b^*, E_l^* \subset V_1 \times V_2$ by

$$\begin{array}{rcl} E_b^* & := & \{e \in \overline{E}_K \mid e \textit{ is basic in } K(\overline{T_1}, \overline{T_2})\} \ , \\ E_h^* & := & \{(h, h') \mid \ h, h' \textit{ are half roots of } K(\overline{T_1}, \overline{T_2}) \textit{ with} \\ & & p_1(h) = p_1(h') \wedge p_2(h) \prec p_2(h') \textit{ or} \\ & & p_2(h) = p_2(l') \wedge p_1(h) \prec p_1(h')\} \ , \textit{ and} \\ E_l^* & := & \{(l, l') \mid \ l, l' \textit{ are leafs of } K(\overline{T_1}, \overline{T_2}) \textit{ with} \\ & & p_1(l) = p_1(l') \wedge p_2(l) \prec p_2(l') \textit{ or} \\ & & p_2(l) = p_2(l') \wedge p_1(l) \prec p_1(l')\}. \end{array}$$

Setting $E^* := E_b^* \cup E_h^* \cup E_l^*$, the subgraph of $(V_1 \times V_2, C(V_1, V_2))$ (see Definition 2.1) induced by E^* is denoted by $K^*(T_1, T_2)$.

Definition 4.8 ((In)compatible vertices in $K^*(T_1, T_2)$)
The vertices $v \neq w \in V_1 \times V_2$ are called incompatible in $K^*(T_1, T_2)$ if $p_i(v) \neq p_i(w)$ $(i = 1, 2)$ and if there exist $e_v, e_w \in E^*$ such that

- *$target(e_v) = v$, $target(e_w) = w$, and*
- *$source(e_v) = source(e_w)$.*

Otherwise, v and w are called compatible.

Proposition 4.9 (Pairwise compatibility $\iff$ order preservation)
Let M_i be a minor of T_i with edge set E_i^M $(i = 1, 2)$ and let β: $E_1^M \mapsto E_2^M$ be a bijection with $E_\beta \subseteq E^*$. Then β is an OP path bijection if and only if the vertices from V_β are pairwise compatible in $K^*(T_1, T_2)$.

Proof: "$\longrightarrow$"
Let β: $E_1^M \mapsto E_2^M$ be an OP path bijection and assume that there exist incompatible $v, w \in V_\beta$. Then there exists $u \in V_1 \times V_2$ such that $e_v := (u, v), e_w := (u, w) \in E^*$. From $p_i(v) \neq p_i(w)$, $(i = 1, 2)$ (see Definition 4.8) and $e_v, e_w \in E^*$ it follows that $p_i(e_v) \in E_i$ for exactly one i and that $p_i(e_w) \in E_{3-i}$. Without lack of generality we assume $p_1(e_v) \in E_1$, $p_2(e_v) \in \overline{E_2} \setminus E_2$, $p_1(e_w) \in \overline{E_1} \setminus E_1$, and $p_2(e_w) \in E_2$. Then $p_1(v) \prec p_1(w)$ in T_1 and $p_2(w) \prec p_2(v)$ in T_2 - a contradiction to item 3. of Proposition 3.6.
"$\longleftarrow$"
Conversely, assume that β is not an OP path bijection. Without lack of generality there exist $v, w \in V_\beta$ such that $p_1(v) \prec p_1(w)$ in T_1 and $p_2(w) \prec p_2(v)$ in T_2. Without lack of generality we consider the following cases and show that they cannot occur.

- Both v and w are not leafs of $K(\overline{T_1}, \overline{T_2})$. Then there exist $e_1 \in E_1$, $e_2 \in E_2$ such that $target(e_1) = p_1(v)$ and $target(e_2) = p_2(w)$. Set $u := (source(e_1), source(e_2))$, $e_v := (u, v)$, and $e_w := (u, w)$. Clearly, $p_1(e_v) = e_1 \in E_1$ and $p_2(e_w) = e_2 \in E_2$. Moreover, $p_1(u) \prec p_1(w)$ and $p_2(u) \prec p_2(v)$ imply that $p_1(e_w) \in \overline{E_1}$ and $p_2(e_v) \in \overline{E_2}$, respectively. Hence, $e_v, e_w \in E_b^*$ and thus we get that v and w are incompatible in $K^*(T_1, T_2)$ - a contradiction.
- v is a leaf of $K(\overline{T_1}, \overline{T_2})$ and w is not. Without lack of generality we assume that $p_1(v)$ is a leaf of T_1. Clearly, there exists $e_2 \in E_2$ with $target(e_2) = p_2(w)$. Set $u := (p_1(v), source(e_2))$. Then $e_v := (u, v) \in E_l^*$ and $e_w := (u, w) \in E_b^*$ fulfill the two items in Definition 4.8 - a contradiction.
- Both v and w are leafs of $K(\overline{T_1}, \overline{T_2})$. Set $u := (p_1(v), p_2(w))$ Then $e_v := (u, v) \in E_l^*$ and $e_w := (u, w) \in E_l^*$ fulfill the two items in Definition 4.8 - a contradiction. $\square$

The following is a straight forward consequence of Propositions 3.6 and 4.9.

Proposition 4.10 (OP path bijections from subgraphs of $K^*(T_1, T_2)$)
Let $S = (V_S, E_S)$ be a subgraph of $K^*(T_1, T_2)$ such that

(a) $p_i(e) \neq p_i(e')$ for all $e, e' \in E_S$ $(i = 1, 2)$,

(b) $p_i(S)$ *is a minor of* $\overline{T_i}$ *for* $i = 1, 2$, *and*

(c) *any two vertices from* V_S *are compatible in* $K^*(T_1, T_2)$.

Then $S = G_\beta$ *for some* $\beta \in \mathcal{B}(T_1, T_2)$. *Conversely, let* β *be an OP path bijection with* $E_\beta \subseteq E^*$. *Then there exists a subgraph* S *of* $K^*(T_1, T_2)$ *which fulfills the three items above.*

5 OP path bijections from matroid intersection

Assumption 5.1 *From now on we assume* $|Con(E_i)| \geq 2$, $i = 1, 2$.

Definition 5.1 (Equivalence relations $\sim_i$ on $Con(E^*)$, $[e]_i$)
Let $i \in \{1, 2\}$. *An equivalence relation* $\sim_i$ *on* $Con(E^*)$ *(see Definitions 4.3 and 4.7) is defined by*

$$e \sim_i e' \ :\leftrightarrow\ p_i(source(e)) = p_i(source(e')). \tag{7}$$

The equivalence class of e *with respect to* $\sim_i$ *is denoted by* $[e]_i$.

Definition 5.2 (Independent subset of $Con(E^*)$, $\mathcal{I}(T_1, T_2)$)
A subset E_{Con} *of* $Con(E^*)$ *is called independent, if* $[e]_i \cap E_{Con} = \{e\}$ *for all* $e \in E_{Con}$ *and for* $i = 1, 2$. *The collection of all independent subsets of* $Con(E^*)$ *is denoted by* $\mathcal{I}(T_1, T_2)$.

Definition 5.3 (Refinement $R^*(E_{Con})$ of $E_{Con} \in \mathcal{I}(T_1, T_2)$)
Let $E_{Con} \in \mathcal{I}(T_1, T_2)$. *Then*

$$R^*(E_{Con}) := \{e^* \in E^* \mid \ e^* \lhd e\}$$

is called refinement of E_{Con}.

Definition 5.4 (Compatible sub-collection $\mathcal{J}(T_1, T_2)$)
The compatible sub-collection $\mathcal{J}(T_1, T_2)$ *of* $\mathcal{I}(T_1, T_2)$ *consists of all* $E_{Con} \in \mathcal{I}(T_1, T_2)$ *such that the subgraph of* $K^*(T_1, T_2)$ *induced by* $R^*(E_{Con})$ *contains no incompatible vertices.*

Proposition 5.5 (Characterization of OP path bijections in $K^*(T_1, T_2)$)
Let β *be an OP path bijection between* T_1 *and* T_2 *with* $E_\beta \subseteq E^*$. *Then* $E_\beta = R^*(E_{Con})$ *for some* $E_{Con} \in \mathcal{J}(T_1, T_2)$. *Conversely, let* $E_{Con} \in \mathcal{J}(T_1, T_2)$. *Then* $R^*(E_{Con}) = E_\beta$ *for some OP path bijection* β *between* T_1 *and* T_2.

Proof: "$\longrightarrow$"
Let β be an OP path bijection between T_1 and T_2 with $E_\beta \subseteq E^*$.

(1) First we show that $p_i(source(e)) \neq p_i(source(e'))$ for all $e \neq e' \in Con(E_\beta)$, $i = 1, 2$. We assume the opposite, i.e. that there exist $e \neq e' \in Con(E_\beta)$ such that $p_i(source(e)) = p_i(source(e'))$ for $i = 1$ or 2. Let $e_a, e_b, e'_a, e'_b \in E_\beta$ be such that e and e' are concatenations of e_a, e_b and e'_a, e'_b, respectively. If $e_a \neq e'_a$, then $p_i(e_a) \neq p_i(e'_a)$ and $p_i(source(e_a)) = p_i(source(e'_a))$. Otherwise, $e_b \neq e'_b$, $p_i(e_b) \neq p_i(e'_b)$ and $p_i(source(e_b)) = p_i(source(e'_b))$. In any case $p_i(G_\beta)$ is not a minor of T_i - a contradiction to Proposition 3.6.

(2) From item (1) it follows that $Con(E_\beta) = E_{Con}$ for some $E_{Con} \in \mathcal{J}(T_1, T_2)$ and that $R^*(Con(E_\beta)) = E_\beta$. Hence, $R^*(E_{Con}) = R^*(Con(E_\beta)) = E_\beta$.

"$\longleftarrow$"

Conversely, let $E_{Con} \in \mathcal{J}(T_1, T_2)$, let $R^*(E_{Con})$ be denoted by E_S and let S be the subgraph of $K^*(T_1, T_2)$ induced by E_S. It suffices to show that the items (a) and (b) in Proposition 4.10 are fulfilled.

(a) Let $e \neq e' \in E_S$ and let $i = 1$ or $i = 2$. Assume that $p_i(e) = p_i(e')$ for $i = 1$ or $i = 2$. From $e \neq e'$ it follows that either $source(p_i(e))$ is not a leaf of T_i or $target(p_i(e)) \neq r_i$ (otherwise, $e \lhd e_c$, $e' \lhd e'_c$ for some $e_c, e'_c \in E_{Con}$ would imply $e_c \sim_i e'_c$ - a contradiction to $e \neq e' \in E_S$).

  (A) In the first case there exist $d_i, c_i \in E_i$ such that c_i is the concatenation of d_i and $p_i(e)$. Assumption 5.1 implies that there exist $e^c_{3-i} \neq f^c_{3-i} \in Con(E_{3-i})$. Let c and c' be the (unique) edges of $Con(E^*)$ such that $p_i(c) = c_i$, $p_{3-i}(c) = e^c_{3-i}$, and $p_i(c') = c_i$, $p_{3-i}(c') = f^c_{3-i}$, respectively. From $c \neq c'$, $p_i(c) = p_i(c')$ it now follows that $c \in Con(E^*) \setminus E_{Con}$ or $c' \in Con(E^*) \setminus E_{Con}$. Hence, e and e' cannot both be contained in $E_S = R^*(E_{Con})$ - a contradiction.

  (B) In the second case there exist $f^i, c^i \in E_i$ such that c^i is the concatenation of $p^i(e)$ and f^i. As in item (A) we can derive a contradiction to $e, e' \in E_S$. $\square$

(b) Without lack of generality assume that $M_1 = (V_1^M, E_1^M) := p_1(S)$ is not a minor of T_1. Note that $E_1^M = p_1(E_S)$. Then there exist $e_1 \in E_1^M$ and $v_1 \in V_1^M$ such that $source(e_1) \prec v_1 \prec target(e_1)$ in T_1. Consequently, there exist $e'_1 \in E_1$ such that $target(e'_1) = v_1$ and $e''_1 \in \overline{E_1}$ such that $source(e''_1) = source(e'_1)$, $target(e''_1) = target(e_1)$. Note that $e''_1 \in \overline{E_1} \setminus E_1$. Since $e'_1 \sim_1 e''_1$ we get that either v or w must be contained in V^x (see Definition 5.3) - a contradiction to $v_1, target(e_1) \in V_1^M$. $\square$

Definition 5.6 ($K^*(R^*(E_{Con}), T_1, T_2)$, σ-maximum $E_{Con} \in \mathcal{I}(T_1, T_2)$ $[\mathcal{J}(T_1, T_2)]$) *Let $E_{Con} \in \mathcal{I}(T_1, T_2)$ $[\mathcal{J}(T_1, T_2)]$ and let the subgraph of $K^*(T_1, T_2)$ induced by $R^*(E_{Con})$ be denoted by $K^*(R^*(E_{Con}), T_1, T_2)$. If*

$$\sigma(K^*(R^*(E_{Con}), T_1, T_2)) \geq \sigma(K^*(R^*(E'_{Con}), T_1, T_2))$$

for all $E'_{Con} \in \mathcal{I}(T_1, T_2)$ $[\mathcal{J}(T_1, T_2)]$, then E_{Con} is called σ-maximum.

The next proposition is a consequence of Proposition 5.5. Moreover, Proposition 4.5 ensures that there always exists a σ-maximum OP path bijection β between T_1 and T_2 with $E_\beta \subseteq E^*$.

Proposition 5.7 (σ-maximum $\beta \leftrightarrow$ σ-maximum independent E_{Con}) *Let β be a σ-maximum OP path bijection between T_1 and T_2 with $E_\beta \subseteq E^*$. Then $E_\beta = R^*(E_{Con})$ for some σ-maximum $E_{Con} \in \mathcal{J}(T_1, T_2)$. Conversely, let $E_{Con} \in \mathcal{I}(T_1, T_2)$ be σ-maximum. Then $R^*(E_{Con}) = E_\beta$ for some σ-maximum OP path bijection β between T_1 and T_2.*

Proposition 5.8 (Incompatible vertices in $R^*(E_{Con})$)
Let $E_{Con} \in \mathcal{I}(T_1, T_2)$ and let G_{Con} be the subgraph of $K^*(T_1, T_2)$ induced by $R^*(E_{Con})$. Furthermore, let v, w be incompatible vertices of G_{Con}. Then there exists no $e^v \in R^*(E_{Con})$ with $source(e^v) = v$.

Proof: Let $u, v \in V_1 \times V_2$ be incompatible in $K^*(T_1, T_2)$. In particular, there exist $e_v, e_w \in E^*$ such that both items in Definition 4.8 are fulfilled. Without lack of generality we assume $p_1(e_v) \in E_1$, $p_2(e_v) \in \overline{E_2} \setminus E_2$, $p_1(e_w) \in \overline{E_1} \setminus E_1$, and $p_2(e_w) \in E_2$. In the remaining part of the proof we denote the common source of e_v and e_w by u. Assume that there exists $e^v \in E_S$ with $source(e^v) = v$. From w being a vertex of G_{Con} it follows that at least one of the following cases holds.

- There exists $e^w \in E^*$ with $source(e^w) = w$. Let c_v and c_w denote the concatenation of e_v, e^v and e_w, e^w, respectively. Then $c_v \neq c_w$, $c_v \sim_1 c_w$ - a contradiction to $e^v, e^w \in R^*(E_{Con})$.
- There exists $e^w \in E^*$ with $target(e^w) = w$. We distinguish between the following two cases.
  - If $p_1(e^w) \in E_1$, then $p_2(e^w) \in \overline{E_2} \setminus E_2$ and thus $p_1(v) \prec p_1(source(e^w))$ in T_1 and $p_2(source(e^w)) \prec p_2(v)$ in T_2. Hence, the vertices $source(e^w)$ and v are incompatible and we arrived at the case studied above (i.e. the sources of e^v and e^w are incompatible).
  - If $p_1(e^w) \in \overline{E_1} \setminus E_1$, then $p_2(e^w) \in E_2$ and thus $p_2(source(e^w)) = p_2(u)$. Moreover, there exists $e'_w \in E^*$ with $source(e'_w) = w = target(e'_w)$. Let c_v and c_w denote the concatenations of e_v, e^v and e^w, e'_w, respectively. Then $c_v \neq c_w$, $c_v \sim_2 c_w$ - a contradiction to $e^v, e^w \in R^*(E_{Con})$.

Proposition 5.9 (Incompatibility in $K^*(R^*(E_{Con}), T_1, T_2)$)
Incompatibility in $K^*(R^*(E_{Con}), T_1, T_2)$ is an equivalence relation.

Propositions 4.5 and 5.7 allow us to reduce the problem of finding a σ-maximum OP path bijection to the problem of finding $E_{Con} \in \mathcal{J}(T_1, T_2)$ such that the positive real number $\sigma(K^*(R^*(E_{Con}), T_1, T_2))$ is maximum. To find $E_{Con} \in \mathcal{J}(T_1, T_2)$ with a high $\sigma(K^*(R^*(E_{Con}), T_1, T_2))$ (we cannot prove maximality), we proceed as follows.

1. For all $e \in Con(E^*)$, $e^a, e^b \in E^*$, such that e is the concatenation of e^a and e^b, we set $\sigma_c(e) := \sigma(e^a) + \sigma(e^b)$.
2. Calculate σ-maximum $E_{Con} \in \mathcal{I}(T_1, T_2)$.
3. From each equivalence class of $K^*(R^*(E_{Con}), T_1, T_2)$ choose a vertex v such that $\sum(\sigma_c(e) \mid \; e \in E_{Con}, \; target(e) = v)$ is maximal and delete all other vertices from the equivalence class. Proposition 5.8 ensures that the refinement of the remaining concatenations contains no more incompatible vertices.

The collection $\mathcal{I}(T_1, T_2)$, in turn, is a common independent set of the matroids $\mathcal{M} = (Con(E^*), \mathcal{I}_i)$ [3], where

$$\mathcal{I}_i = \{E_{Con} \in Con(E^*) \mid \; [e]_i = \{e\} \text{ for all } e \in E_{Con}\}. \tag{8}$$

Thus, the problem of finding a σ-maximum $E_{Con} \in \mathcal{I}(T_1, T_2)$ becomes a so-called weighted matroid intersection problem [1].

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