Graph Polynomials from Principal Pivoting

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Abstract

In a recent paper, Arratia, Bollobás and Sorkin introduced the *interlace polynomial*, a graph polynomial defined in terms of a new pivoting operation on undirected, unweighted graphs. In this paper, we employ pivoting operations on matrices to understand and extend their work. Specifically, we show that Arratia *et al.*'s pivoting operation on a graph is indeed equivalent to a principal pivoting transform on a corresponding adjacency matrix, provided that all calculations are performed in the Galois field $\mathbb{F}_2$. The observations that adjacency matrices of undirected graphs are antisymmetric in $\mathbb{F}_2$ and that the principal pivoting transforms employed preserve antisymmetry in *all* fields suggest to extend Arratia *et al.*'s pivoting operation to fields other than $\mathbb{F}_2$. As a consequence, the original interlace polynomial is extended to the class of *gain* graphs, namely bidirected edge-weighted graphs whereby reversed edges carry weights that differ only by their sign. For infinite fields it is shown that the extended interlace polynomial does not depend on the (non-zero) gains, as long as they obey a non-singularity condition. Thus, a new graph polynomial for undirected graphs is defined. Its recursive calculation can be done such that all ends of the recursion correspond to independent sets. Moreover, its degree equals the independence number. However, the new graph polynomial is different from the independence polynomial.

1 Introduction

Motivated by a problem arising from DNA sequencing by hybridization [1], Arratia, Bollobás and Sorkin introduced the *interlace polynomial* q, a graph polynomial defined for all undirected graphs G. In the special case of G being a *circle graph*, i.e. an (undirected) graph derived from a 2-in, 2-out directed graph D, $q(G)$ was shown to count circuit partitions of D [2]. The authors conclude with the sentence "Again, though, the real question is this: What does $q(G)$ compute, say when G is not a circle graph?". Indeed, computing $q(G)$ one has to iterate a new kind of pivoting operations, each of which takes an edge and toggles others (from a non-edge to an edge or vice versa). The situation after only two pivoting operations is already "obscure" (see the paragraph before the proof of Lemma 14 in [2]).

The aim of this paper is threefold. Firstly, we show that pivoting on an edge $e = \{u, v\}$ of an undirected graph G, as defined in [2, 3], is equivalent to a sequence of two classical pivoting steps on an adjacency matrix of G, provided that the calculations are performed in the Galois field $\mathbb{F}_2$. If a and b denote the row and column numbers of u and v in an adjacency matrix A of G, pivoting on e is equivalent to classical matrix pivoting on $A(a, b)$, followed by classical matrix pivoting on $A(b, a)$. This sequence of pivoting operations, in turn, is equivalent to a principal pivoting transform of A on the non-singular 2×2 principal submatrix $A[\{a, b\}]$, followed by swapping row a with row b and column a with column b. Here, $A[\{a, b\}]$ denotes the principal submatrix of A given by the rows and columns a, b. The principal pivoting transform can be understood as a generalization of classical pivoting, where pivoting is performed on non-singular principal submatrices instead of single non-zero entries [6].

Our second contribution comes from the observation that the principal pivoting transformation on $A[\{a, b\}]$ preserves antisymmetry in *all* fields. Note also that A is antisymmetric in $\mathbb{F}_2$. Hence, by extending the pivoting operation from $\mathbb{F}_2$ to arbitrary fields, we extend the interlace polynomial to gain graphs, with gains from an arbitrary field, i.e. bidirected graphs, the edges of which have attributes $g(\cdot)$ such that $g(r, s) = -g(s, r) \neq 0$ for all edges (r, s). In [2] the key to prove the existence of the interlace polynomial is a lemma about two consecutive pivoting operations on undirected graphs. This lemma turns out to be a special case, i.e. $\mathbb{F} = \mathbb{F}_2$, of an equality easily formulated and proven in terms of principal pivoting on antisymmetric matrices.

The final part of the paper is devoted to finding a variant p of q that again takes undirected unweighted graphs G. The plan is to define $p(G) := q(G')$, where G' is from a class of gain graphs that have the same vertices and the same edges (apart from the directions) as G. In addition, the class of gain graphs is restricted by a non-singularity condition on the gains. The latter also ensures that $q(G')$ can be computed such that none of the edges in the original gain graph is removed by pivoting. Thus, the vertex sets of the edge-less graphs arising from selected calculations of the extended interlace polynomial are always independent sets of the original gain graphs. It turns out to be crucial that the gains come from an infinite field. The new graph polynomial is called *pivoting polynomial*. Its degree equals the independence number, whereas the degree of $q(G)$ is merely an upper bound of the independence number.

The plan of the paper is as follows. Sec. 2 and Sec. 3 deal with antisymmetric matrices only. In Sec. 2 we introduce double pivots of antisymmetric adjacency matrices over an arbitrary field $\mathbb{F}$. Sec. 3 reveals the relation between double pivoting and principal pivoting. This section also provides the key lemma to prove the existence of the extended interlace polynomial. The transition from pivoting on antisymmetric matrices to pivoting on gain graphs is done in Sec. 4. The purpose of Sec. 5 is to show that the pivoting operation proposed in [2] corresponds to the special case of pivoting operations on gain graphs with gains from $\mathbb{F}_2$. The extended interlace polynomial is defined in Sec. 6. Sec. 7 is devoted to the characterization of independent sets in terms of linear complementarity problems. Here we also sketch the idea of how to arrive at the pivoting polynomial. Technically, this is done by the classes of matrices specified in Sec. 8. Finally, the pivoting polynomial is defined in Sec. 9.

2 Double pivots of antisymmetric matrices

Pivoting is a standard method in linear and quadratic optimization. Given a field $\mathbb{F}$, consider q, x, $x' \in \mathbb{F}^n$ and $M = (M(i,j))_{i,j=1}^n \in \mathbb{F}^{n\times n}$. The *simple pivot* on $M(a,b) \neq 0$ of the system $Mx = q$ is the system

$$M^{(a,b)}x' = q^{(a,b)},$$

where $q^{(a,b)} \in \mathbb{F}^n$ and $M^{(a,b)} \in \mathbb{F}^{n\times n}$ are defined as follows [4]

$$\begin{aligned}
q_a^{(a,b)} &= -q_a/M(a,b) \\
q_i^{(a,b)} &= q_i - (M(i,b)/M(a,b))q_a \quad i \neq a \\
M^{(a,b)}(a,b) &= 1/M(a,b) \\
M^{(a,b)}(i,b) &= M(i,b)/M(a,b) \quad i \neq a \\
M^{(a,b)}(a,j) &= -M(a,j)/M(a,b) \quad j \neq b \\
M^{(a,b)}(i,j) &= M(i,j) - (M(i,b)/M(a,b))M(a,j) \quad i \neq a, j \neq b.
\end{aligned}$$

From now on we require M to be antisymmetric, i.e. $M(i,j) = -M(j,i)$ and $M(i,i) = 0$ for all i,j. The former requirement implies the latter whenever $\mathbb{F} \neq \mathbb{F}_2$. From $M(a,b) \neq 0$ it follows $a \neq b$, and

$$\begin{aligned}
M^{(a,b)}(b,j) &= M(b,j) \quad \forall j, \\
M^{(a,b)}(i,a) &= M(i,a) \quad \forall i.
\end{aligned}$$

In particular, $M^{(a,b)}(b,a) = M(b,a) = -M(a,b) \neq 0$ and thus $(M^{a,b})^{(b,a)}$ is well defined. By symmetry, $(M^{(b,a)})^{(a,b)}$ is also well defined. In the following we write $M^{(a,b)(c,d)}$ instead of $(M^{(a,b)})^{(c,d)}$.

Prop. 2.1 (Double pivot, antisymmetry) *Let $M \in \mathbb{F}^{n\times n}$ be antisymmetric with $M(a,b) \neq 0$. Then, $M^{(a,b)(b,a)} = M^{(b,a)(a,b)}$. Moreover, $M^{(a,b)(b,a)}$ is again antisymmetric.*

Proof: Distinguish between the following cases.

$$\begin{aligned}
M^{(a,b)(b,a)}(i,j) &= M^{(a,b)}(i,j) - \frac{M^{(a,b)}(i,a)}{M^{(a,b)}(b,a)}M^{(a,b)}(b,j) \\
&= M(i,j) - \frac{M(i,b)}{M(a,b)}M(a,j) - \frac{M(i,a)}{M(b,a)}M(b,j) \\
&= M^{(b,a)(a,b)}(i,j) = -M^{(a,b)(b,a)}(j,i),
\end{aligned}$$

$$
\begin{aligned}
& \qquad \text{if } \{i,j\} \cap \{a,b\} = \emptyset, \\
M^{(a,b)(b,a)}(i,b) &= M^{(a,b)}(i,b) = \frac{M(i,b)}{M(a,b)} = \frac{M^{(b,a)}(i,b)}{M(a,b)} \\
&= M^{(b,a)(a,b)}(i,b), \quad \text{if } i \neq a, \\
M^{(a,b)(b,a)}(b,i) &= -\frac{M^{(a,b)}(b,i)}{M^{(a,b)}(b,a)} = -\frac{M(b,i)}{M(b,a)} = M^{(b,a)}(b,i) \\
&= M^{(b,a)(a,b)}(b,i) = -M^{(a,b)(b,a)}(i,b), \quad \text{if } i \neq a, \\
M^{(a,b)(b,a)}(a,j) &= M^{(a,b)}(a,j) = -\frac{M(a,j)}{M(a,b)} = -\frac{M^{(b,a)}(a,j)}{M(a,b)} \\
&= M^{(b,a)(a,b)}(a,j), \quad \text{if } j \neq b, \\
M^{(a,b)(b,a)}(j,a) &= \frac{M^{(a,b)}(j,a)}{M^{(a,b)}(b,a)} = \frac{M(j,a)}{M(b,a)} = M^{(b,a)}(j,a) \\
&= M^{(b,a)(a,b)}(j,a) = -M^{(a,b)(b,a)}(a,j), \quad \text{if } j \neq b, \\
M^{(a,b)(b,a)}(a,b) &= M^{(a,b)}(a,b) = \frac{1}{M(a,b)} = \frac{1}{M^{(b,a)}(a,b)} \\
&= M^{(b,a)(a,b)}(a,b), \\
M^{(a,b)(b,a)}(b,a) &= \frac{1}{M^{(a,b)}(b,a)} = \frac{1}{M(b,a)} = M^{(b,a)}(b,a) \\
&= M^{(b,a)(a,b)}(b,a) = -M^{(a,b)(b,a)}(a,b).
\end{aligned}
$$

□

Definition 2.2 (Double pivot $M^{\{a,b\}}$) *Let $M \in \mathbb{F}^{n\times n}$ be antisymmetric and let $M(a,b) \neq 0$. Then the antisymmetric matrix*

$$M^{\{a,b\}} := M^{(a,b)(b,a)} \tag{1}$$

is called the double pivot of M on $\{a,b\}$.

3 Double pivoting in terms of PPT

For the double pivot we will derive the analogue of Lemma 10(ii) in [2], the key to prove the existence of the interlace polynomial. The plan is to express the double pivot in terms of the principal pivot transform. Using well known theorems about iterated principal pivot transforms we will then be able to derive the analogue of Lemma 10(ii). In the following, n denotes a positive integer and M is from $\mathbb{F}^{n\times n}$. The next definitions are from [6]:

- $\langle n\rangle := \{1,2,\ldots,n\}$. For any $\alpha \subseteq \langle n\rangle$ the set $\langle n\rangle \setminus \alpha$ is denoted by $\overline{\alpha}$.
- $M[\alpha,\beta]$ is the submatrix of M whose rows and columns are indexed by α and β, respectively. The submatrix $M[\alpha,\alpha]$ of M is written as $M[\alpha]$. The matrix $M[\alpha]$ is a so called *principal* submatrix of M.

- The *Schur complement* $M \setminus M[\alpha]$ of a non-singular principal submatrix $M[\alpha]$ in M is defined by

$$M \setminus M[\alpha] := M[\overline{\alpha}] - M[\overline{\alpha}, \alpha]M[\alpha]^{-1}M[\alpha, \overline{\alpha}]. \tag{2}$$

Definition 3.1 (Principal pivot transform $ppt(M, \alpha)$) *Let $\alpha \subseteq \langle n \rangle$ be such that $M[\alpha]$ is non-singular. The principal pivot transform $ppt(M, \alpha)$ of $M \in \mathbb{F}^{n \times n}$ is obtained from M through replacing*

$M[\alpha]$ by $M[\alpha]^{-1}$,
$M[\overline{\alpha}]$ by $M \setminus M[\alpha]$,
$M[\alpha, \overline{\alpha}]$ by $-M[\alpha]^{-1}M[\alpha, \overline{\alpha}]$, and
$M[\overline{\alpha}, \alpha]$ by $M[\overline{\alpha}, \alpha]M[\alpha]^{-1}$.

Double and principal pivots turn out to differ only by a swap of two rows and the corresponding columns. Such a swap will be denoted by a subscript indicating the numbers of the rows (columns) being swapped.

Prop. 3.2 (Double and principal pivots) *Let $a \neq b \in \langle n \rangle$ be such that $M[\{a, b\}]$ is non-singular. Then $M^{\{a,b\}} = (ppt(M, \{a, b\}))_{ab}$.*

Proof: Denoting $\{a, b\}$ by α we get

- $M[\alpha]^{-1}(a, a) = 0 = M^{\{a,b\}}(a, a)$,
  $M[\alpha]^{-1}(b, b) = 0 = M^{\{a,b\}}(b, b)$.
- $M[\alpha]^{-1}(a, b) = \frac{1}{M(b,a)} = M^{\{a,b\}}(b, a)$,
  $M[\alpha]^{-1}(b, a) = \frac{1}{M(a,b)} = M^{\{a,b\}}(a, b)$.
- $(-M[\alpha]^{-1}M[\alpha, \overline{\alpha}])(a, j) = -\frac{M(b,j)}{M(b,a))} = M^{\{a,b\}}(b, j)$ for $j \neq a, b$.
- $(-M[\alpha]^{-1}M[\alpha, \overline{\alpha}])(b, j) = -\frac{M(a,j)}{M(a,b))} = M^{\{a,b\}}(a, j)$ for $j \neq a, b$.
- $(M[\overline{\alpha}, \alpha]M[\alpha]^{-1})(i, a) = \frac{M(i,b)}{M(a,b)} = M^{\{a,b\}}(i, b)$ for $i \neq a, b$.
- $(M[\overline{\alpha}, \alpha]M[\alpha]^{-1})(i, b) = -\frac{M(i,a)}{M(a,b)} = M^{\{a,b\}}(i, a)$ for $i \neq a, b$.
- $(M \setminus M[\alpha])(i, j) = M^{\{a,b\}}(i, j)$ for $\{i, j\} \cap \{a, b\} = \emptyset$.

$\square$

The following theorems differ from Theorem 3.1 and Theorem 3.2 in [6] only in that they are formulated for arbitrary fields $\mathbb{F}$ and not just for the field of complex numbers. However, the proofs given in [6] depend merely on the axioms for fields. The two theorems will serve to derive rules for iterated double pivots.

Theorem 3.3 *Let $M \in \mathbb{F}^{n\times n}$ and $\alpha \subseteq \langle n\rangle$ such that $M[\alpha]$ is non-singular. For each $x, y \in \mathbb{F}^n$ define $u = u(x,y), v = v(x,y) \in \mathbb{F}^n$ by $u_i = y_i$, $v_i = x_i$ for all $i \in \alpha$, and $u_j = x_j$, $v_j = y_j$ for all $j \in \overline{\alpha}$. Then $N = ppt(M,\alpha)$ is the unique matrix with the property that*

$$Mx = y \;\leftrightarrow\; Nu = v \; . \tag{3}$$

Theorem 3.4 *Let $M \in \mathbb{F}^{n\times n}$ and let $\alpha = \bigcup_{i=1}^{k} \alpha_i \subseteq \langle n\rangle$ for some k such that $\alpha_i \cap \alpha_j = \emptyset$ for all $i \neq j$. If the sequence of matrices*

$$M_0 = M,\;\; M_i = ppt(M_{i-1}, \alpha_i),\;\; i = 1, \ldots, k$$

is well defined, i.e. if all $M_{i-1}[\alpha_i]$ are non-singular, then $ppt(M,\alpha) = M_k$.

In the following we write $M^{\{a,b\}\{a,c\}}$ instead of $(M^{\{a,b\}})^{\{a,c\}}$.

Lemma 3.5 (First analogue of Lemma 10(ii) in [2]) *Let $M \in \mathbb{F}^{n\times n}$ be antisymmetric with $M(a,b), M(a,c) \neq 0$. Then,*

$$M^{\{a,b\}\{a,c\}} = (M^{\{a,c\}})_{bc} \; . \tag{4}$$

Proof: First we prove that $ppt(ppt(M,\{a,b\}),\{b,c\}) = ppt(M,\{a,c\})$. Applying Theorem 3.3 first to principal pivoting on $\{a,b\}$ and then to principal pivoting on $\{b,c\}$, it follows that $Mx = y \;\leftrightarrow\; Nu = v$, where $N = ppt(ppt(M,\{a,b\}),\{b,c\})$ is unique and u, v are defined by

$$u_i = \begin{cases} y_i & \text{if } i \in \{a,c\}, \\ x_i & \text{otherwise} \end{cases} \qquad v_i = \begin{cases} x_i & \text{if } i \in \{a,c\}, \\ y_i & \text{otherwise.} \end{cases} \tag{5}$$

Note that x_b and y_b have been exchanged twice. Applying Theorem 3.3 to principal pivoting on $\{a,c\}$ only, it follows that $Mx = y \;\leftrightarrow\; N'u = v$, where $N' = ppt(M,\{a,c\})$ is unique and u, v are as above. Hence, $N = N'$.
Finally, using Prop. 3.2 and the result just proven, we have:

$$\begin{aligned} M^{\{a,b\}\{a,c\}} &= (ppt((ppt(M,\{a,b\}))_{ab},\{a,c\}))_{ac} \\ &= (ppt(ppt(M,\{a,b\}),\{b,c\})_{ab})_{ac} = ((ppt(M,\{a,c\}))_{ab})_{ac} \\ &= (((M^{\{a,c\}})_{ac})_{ab})_{ac} = (M^{\{a,c\}})_{bc} \end{aligned}$$

□

4 Double pivots of gain graphs

For the rest of the paper, let $G = (V, E, g)$ denote a gain graph with non-zero gains. Formally, the edge set E is a subset of $(V \times V) \setminus \{(v,v) \mid v \in V\}$ such that $(u,v) \in E$ implies $(v,u) \in E$, and the gains are given by a mapping $g(\cdot)$ from E to $\mathbb{F} \setminus \{0\}$ such that $g(u,v) = -g(v,u)$ for all $(u,v) \in E$.

In this section we define double pivots of G by means of double pivots of an (antisymmetric) adjacency matrix of G. In order to show that the double pivots of G do not depend on the choice of the adjacency matrix, we first define adjacency matrices in terms of functions $l(\cdot)$ that specify the correspondence between the vertices of G and the rows (columns) of the adjacency matrices.

Definition 4.1 (Adjacency matrix M_l of G) *Let $l : V \to \{1, \ldots, |V|\}$ be bijective. The adjacency matrix M_l of G is defined by*

$$M_l(i,j) = \begin{cases} g(l^{-1}(i), l^{-1}(j)) & \text{if } (l^{-1}(i), l^{-1}(j)) \in E, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Note that $M_l(l(u), l(v))$ is independent of the labeling function $l(\cdot)$. From the definition of $M^{\{a,b\}}$ it follows that $M_l^{\{l(r),l(s)\}}(l(u), l(v))$ is also independent of $l(\cdot)$. Finally, independence of $l(\cdot)$ extends to the following.

Definition 4.2 (Double pivot of G) *Let M_l denote an adjacency matrix of G and let $(r,s) \in E$. The graph $G^{\{r,s\}} := (V^{\{r,s\}}, E^{\{r,s\}}, g^{\{r,s\}})$ defined by*

- $V^{\{r,s\}} := V$,
- $E^{\{r,s\}} := \{(u,v) \mid u, v \in V,\ M_l^{\{l(r),l(s)\}}(l(u), l(v)) \neq 0\}$,
- $g^{\{r,s\}}(u,v) := M_l^{\{l(r),l(s)\}}(l(u), l(v))$ *for all* $(u,v) \in E^{\{r,s\}}$.

is called the double pivot of G on r and s.

In particular, Lemma 3.5 extends to

$$G^{\{r,s\}\{r,t\}} = (G^{\{r,t\}})_{st} \quad \forall (r,s), (r,t) \in E, \tag{7}$$

where the subscript indicates that the vertex names r and s have been exchanged. In accordance with Prop. 3.2 we may define

$$ppt(G, \{r,s\}) := (G^{\{r,s\}})_{rs} \quad \forall (r,s) \in E. \tag{8}$$

If $\sum_{v \in V} g(u,v) = 0$ for all $u \in V$, then $g(\cdot)$ is a flow on G. Hence,

$$\sum_j M_l(i,j) = 0 \quad \forall i$$

holds for any adjacency matrix M_l of G. Let (r,s) be an edge of G and let $l(r) = a$, $l(s) = b$. Due to the definition of $M_l^{\{a,b\}}$ via simple pivoting it holds that

$$\sum_j M_l^{\{a,b\}}(i,j) = 0 \quad \forall i.$$

Since $M_l^{\{a,b\}}$ is antisymmetric and since $g^{\{a,b\}}(u,v) \neq 0$ for all $(u,v) \in E^{\{r,s\}}$, it follows that $g^{\{r,s\}}$ is a flow on $G^{\{r,s\}}$. In other words, Kirchhoff's conservation law is preserved by double pivoting.

(a) $\begin{matrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{matrix}$ (b) $\begin{matrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{matrix}$ (c) $\begin{matrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{matrix}$

(d) (e) (f)

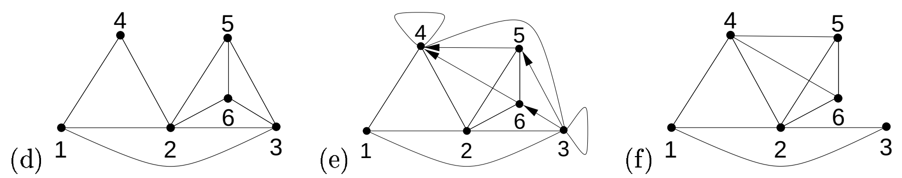

Figure 1: (a) Adjacency Matrix M_l. The pivoting element $(1,2)$ is circuited. Changes will take place in the rectangle. (b) $M_l^{(1,2)}$. The new pivoting element is $(2,1)$. (c) $M_l^{\{1,2\}}$. (d), (e), (f) Graphs defined by (a), (b), and (c), respectively.

5 Double pivots in $\mathbb{F}_2$

In case of $\mathbb{F} = \mathbb{F}_2$, the Galois field containing only the numbers 0 and 1, the simple pivot of M_l on $M_l(a,b) \neq 0$, i.e. on $M_l(a,b) = 1$, takes the form

$$M_l^{(a,b)}(i,j) = \begin{cases} 1 - M_l(i,j) & \text{if } i \neq a, j \neq b \text{ and } M_l(i,b) = M_l(a,j) = 1, \\ M_l(i,j) & \text{otherwise.} \end{cases}$$

In Fig.1 the matrix $M_l^{(a,b)}$ is interpreted as a directed graph with self-loops. Double pivoting on $\{a,b\}$ (see again Fig. 1) yields $M_l^{\{a,b\}}(i,j) = M_l^{(a,b)(b,a)}(i,j)$

$$= \begin{cases} 1 - M_l^{(a,b)}(i,j) & \text{if } i \neq b, j \neq a \text{ and } M_l^{(a,b)}(i,a) = M_l^{(a,b)}(b,j) = 1, \\ M_l^{(a,b)}(i,j) & \text{otherwise,} \end{cases}$$

$$= \begin{cases} 1 - M_l^{(a,b)}(i,j) & \text{if } i \neq b, j \neq a \text{ and } M_l(i,a) = M_l(b,j) = 1, \\ M_l^{(a,b)}(i,j) & \text{otherwise.} \end{cases}$$

Comparing M_l with $M_l^{\{a,b\}}$, the entry at (i,j) has changed, if and only if the following boolean expression is true.

$$(i \neq a, j \neq b, M_l(i,b) = M_l(a,j) = 1) \text{ Xor } (i \neq b, j \neq a, M_l(i,a) = M_l(b,j) = 1) \tag{9}$$

Let $G_u = (V, E_u)$ be an undirected simple graph, i.e. E_u is a collection of 2-element subsets of V. G_u corresponds to a bidirected graph $G = (V, E, g)$ with

Table 1: Eq. (9) depending on the classes of the nodes $l^{-1}(i)$, $l^{-1}(j)$.

	$l^{-1}(j) \in C_1^{r,s}$	$l^{-1}(j) \in C_2^{r,s}$	$l^{-1}(j) \in C_3^{r,s}$	$l^{-1}(j) \in C_4^{r,s}$
$l^{-1}(i) \in C_1^{r,s}$	0 Xor 0	0 Xor 1	0 Xor 1	0 Xor 0
$l^{-1}(i) \in C_2^{r,s}$	1 Xor 0	0 Xor 0	1 Xor 0	0 Xor 0
$l^{-1}(i) \in C_3^{r,s}$	1 Xor 0	0 Xor 1	1 Xor 1	0 Xor 0
$l^{-1}(i) \in C_4^{r,s}$	0 Xor 0	0 Xor 0	0 Xor 0	0 Xor 0

gains in $\mathbb{F}_2$ by $E = \{(u,v) \mid \{u,v\} \in E_u\}$ and $g(e) = 1$ for all $e \in E$. Let (r,s) be an edge of G and let M_l be an antisymmetric adjacency matrix of G with $l(r) = a$, $l(s) = b$. Partitioning $V \setminus \{r,s\}$ into the classes

1) $C_1^{r,s} := \{t \in V \setminus \{r,s\} \mid (t,r) \in E, (t,s) \notin E\}$,
2) $C_2^{r,s} := \{t \in V \setminus \{r,s\} \mid (t,s) \in E, (t,r) \notin E\}$,
3) $C_3^{r,s} := \{t \in V \setminus \{r,s\} \mid (t,r), (t,s) \in E\}$,
4) $C_4^{r,s} := \{t \in V \setminus \{r,s\} \mid (t,r), (t,s) \notin E\}$.

and evaluating the boolean expression (9) according to the membership of $l^{-1}(i)$ and $l^{-1}(j)$ to the classes $C_1^{r,s}$ to $C_4^{r,s}$ yields Tab. 1. Note that $G^{\{r,s\}}$ corresponds to another undirected simple graph. Thus, double pivoting on gain graphs with non-zero gains in $\mathbb{F}_2$ induces an operation on the corresponding undirected simple graphs which may be characterized as follows (see Tab. 1). An edge $\{u,v\}$ of G_u is toggled (between a non-edge and an edge), if and only if u and v belong to different classes other than class $C_4^{r,s}$. This characterization coincides with the definition of the pivoting operation in [2].

6 The extended interlace polynomial

Let $\mathcal{G}$ denote the class of gain graphs and let $G \in \mathcal{G}$. The order of G is written as $|G|$. The following is analogous to [2] (an undirected edge in [2] corresponds to a pair of reversed edges in the gain graph).

Theorem 6.1 (Interlace polynomial)
There is a unique map $q : \mathcal{G} \to \mathbf{Z}[\mathbf{x}]$, $\mathbf{G} \mapsto \mathbf{q}(\mathbf{G})$, such that

$$q(G) = \begin{cases} q(G-r) + q(G^{\{r,s\}} - s) & \text{for any edge } (r,s) \text{ of } G, \\ x^{|G|} & \text{if } G \text{ has no edges} \end{cases} \tag{10}$$

Proof: Using Eq. (7) instead of Lemma 10(ii) in [2], the proof is the same as the one in [2]. □

Using Eq. (8) the recursive part of Eq. (10) can be rewritten as

$$q(G) = q(G-r) + q(ppt(G, \{r,s\}) - r). \tag{11}$$

7 LCP for independent sets

In this section a one-to-one-correspondence is established between the [maximal] independent sets of an undirected simple graph G_u and the solutions of an appropriately defined [strict] linear complementarity problem. Formally, $G_u = (V, E_u)$, where the elements of E_u are two-element subsets of V. Given a matrix $A \in \mathbb{R}^{n\times n}$ and a vector $q \in \mathbb{R}^n$, the linear complementarity problem LCP $[q,A]$ is to find vectors $x \in \mathbb{R}^n$ such that x and $y := Ax + q$ fulfill the constraints [4]

$$x_i \geq 0 \quad \forall i, \quad y_i \geq 0 \quad \forall i, \quad \text{and} \quad \sum_{i=1}^{n} x_i y_i = 0. \tag{12}$$

In our case $A = A_l = A_l(G_u)$ will be the symmetric adjacency matrix of G_u with respect to the bijective mapping $l : V \to \{1, \ldots, |V|\}$, i.e.

$$A_l(i,j) = \begin{cases} 1 & \text{if } \{l^{-1}(i), l^{-1}(j)\} \in E_u, \\ 0 & \text{otherwise,} \end{cases} \tag{13}$$

and q will be the zero-vector o. In order to formulate a correspondence between the solutions of the LCP $[o,A_l]$ and the independent sets of G_u, we define the mapping $supp_l^V : \mathbb{R}^n \to 2^V$ by

$$supp_l^V(x) := \{l^{-1}(i) \mid x_i > 0\}, \quad x \in \mathbb{R}^n \ . \tag{14}$$

Prop. 7.1 (Complementarity of independent sets in G_u) *Let A_l be a symmetric adjacency matrix of an undirected simple graph G_u. Then, $x \in \mathbb{R}^n$ solves the LCP $[o,A_l]$, if and only if $supp_l^V(x)$ is an independent set of G_u.*

Proof: Since $(A_l x)_i = \sum_{j=1}^{n} A_l(i,j)x_j = \sum_{j|x_j>0} A_l(i,j)x_j$, the vector x solves the LCP $[o,A_l]$, if and only if the implication

$$x_i > 0 \quad \to \quad A_l(i,j) = 0 \quad \text{for all } j \text{ such that } x_j > 0 \tag{15}$$

holds for all i. This implication, in turn, is equivalent to $supp_l^V(x)$ being an independent set. □

For $U \subseteq V$ we define the *characteristic vector* $x^{U,l} = (x_1^{U,l}, \ldots, x_n^{U,l})$ by

$$x_i^{U,l} := \begin{cases} 1 & \text{if } l^{-1}(i) \in U, \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

Let x be a strict solution of the LCP $[o,A_l]$, i.e.

$$x_i = 0 \quad \to \quad (A_l x)_i > 0 \quad \text{for all } i \ . \tag{17}$$

Then $U := supp_l^V(x)$ is a *maximal* independent set. Indeed, from Prop. 7.1 we know that U is an independent set. Furthermore, if v is not in U (i.e. $x_{l(v)}^{U,l} = 0$), consider the node set $S := U \cup \{v\}$ (note that $x_{l(v)}^{S,l} = 1$). Since A_l is from $\mathbb{R}_{\geq 0}^{n\times n}$, it holds that $(A_l x^{S,l})_i \geq (A_l x^{U,l})_i$ for all $1 \leq i \leq n$. Hence, implication (17) yields $x^{S,l}_{l(v)}(A_l x^{S,l})_{l(v)} > 0$, i.e. $x^{S,l}$ does not solve the LCP [o,A_l] and S is not an independent set (Prop. 7.1).

Conversely, assume that $U = supp_l^V(x)$ is an independent set and that implication 17 does not hold. Then there exists i, $1 \leq i \leq n$, such that $x_i = (A_l x)_i = 0$. Set $v := l^{-1}(i)$, $S := U \cup \{v\}$. From $(A_l x^{U,l})_i = \sum_{j | l^{-1}(j) \in U} A_l(i,j) x_j^{U,l} = 0$ it follows that $A_l(i,j) = 0$ for all j with $l^{-1}(j) \in U$. Hence, S is an independent set larger than U. Prop. 7.2 summarizes the results.

Prop. 7.2 (Maximal Independent sets) *The vector $x \in \mathbb{R}^n$ is a strict solution of the LCP [o,A_l], if and only if $supp_l^V(x)$ is a maximal independent set.*

How can the above characterization of [maximal] independent sets be extended from symmetric adjacency matrices A of undirected simple graphs G_u to antisymmetric adjacency matrices M of gain graphs G with gains from $\mathbb{F} \setminus \{0\}$? Assume that $\mathbb{F} = \mathbb{R}$ and $\overline{M} = A$, where $\overline{M}$ is defined as follows.

Definition 7.3 (Template $\overline{M}$, supremum of matrices from $\{0,1\}^{n \times n}$) *Let $M \in \mathbb{F}^{n \times n}$. The template $\overline{M} \in \{0,1\}^{n \times n}$ of M is defined by*

$$\overline{M}(i,j) = \begin{cases} 1 & \textit{if } M(i,j) \neq 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Let $\mathcal{M}$ be a collection of matrices from $\{0,1\}^{n \times n}$ and let $S \in \mathcal{M}$. S is called supremum of $\mathcal{M}$, if $S(i,j) = 0$ implies that $M(i,j) = 0$ for all $M \in \mathcal{M}$.

From M being antisymmetric it follows that every $x \in \mathbb{R}^n_{\geq 0}$ solves the LCP [o,M]. Hence, the solutions of the LCP [o,A] and the LCP [o,M] coincide *only* if every entry of A (and thus every entry of M) is zero. Note that the recursive computation of $q(G)$ yields edge-less graphs, i.e. graphs with this kind of adjacency matrices. If the computation of $q(G)$ can be done such that the pivoting operations do not destroy edges of G, the vertices of the resulting edge-less graphs will certainly form independent sets of G.

8 Classes of antisymmetric adjacency matrices

The plan for the rest of the paper is to derive a graph polynomial for undirected simple graphs $G_u = (V, E_u)$ via the extended interlace polynomial of "corresponding" gain graphs. The term "corresponding" implies that the gain graphs take the form $G = (V, E, g(\cdot))$, where $E = \{(u,v) \mid \{u,v\} \in E_u\}$ and $g(e) \neq 0$ for all $e \in E$. We will have to make sure that from G and G' both corresponding to G_u, it always follows that $q(G) = q(G')$. In this section the problem of finding "corresponding" gain graphs is approached via classes of adjacency matrices.

Given a symmetric adjacency matrix $A \in \{0,1\}^{n \times n}$ of G_u, we are looking for a class sup^A of antisymmetric adjacency matrices from $\mathbb{F}^{n \times n}$ such that

(1) $\overline{M} = A$ for all $M \in sup^A$ (see Def. 7.3).

(2) Let $M, M' \in sup^A$ be antisymmetric adjacency matrices of the gain graphs G and G'. Then, $q(G) = q(G')$.

Provided that $\mathbb{F}$ is infinite, the non-empty class sup^A defined at the end of this section will turn out to solve the problem.

Moreover, all terms $a_k x^k$ of $q(G)$ with $a_k \neq 0$ will be shown to correspond to independent sets of cardinality k in G, and thus in G_u (Sec. 9).

In the following we will consider Schur complements $M \setminus M[\alpha]$. Varying the non-zero elements outside the non-singular principal submatrix $M[\alpha]$, we will maximize the number of non-zero entries in the Schur complement.

Formally, we set

$$\mathcal{V}_{M,\alpha} := \{N \in \mathbb{F}^{n\times n} \mid N \text{ is antisymmetric, } \overline{N} = \overline{M} \text{ and } N[\alpha] = M[\alpha]\} \quad (18)$$

and determine $S \in \mathcal{V}_{M,\alpha}$ such that $\overline{S \setminus S[\alpha]}$ is the supremum (see Def. 7.3) of

$$\mathcal{T}_{M,\alpha} := \{\overline{N \setminus N[\alpha]} \mid N \in \mathcal{V}_{M,\alpha}\} \ . \quad (19)$$

Moreover, the elements in $\mathcal{V}_{M,\alpha}$ that yield non-supremum Schur complement templates will turn out to be singular (see the proof of Prop. 8.1).

Prop. 8.1 (Maximal template) *Let $\mathbb{F}$ be infinite, let $M \in \mathbb{F}^{n\times n}$ be antisymmetric and let $\alpha \in \langle n\rangle$ be such that $M[\alpha]$ is non-singular. Then, $\mathcal{T}_{M,\alpha}$ has a supremum.*

Proof: For $N \in \mathcal{V}_{M,\alpha}$ it holds that $(N \setminus N[\alpha])(i,j)$

$$\begin{aligned} &= N[\overline{\alpha}](i,j) - N[\overline{\alpha},\alpha]\ N[\alpha]^{-1}\ N[\alpha,\overline{\alpha}] \\ &= N[\overline{\alpha}](i,j) - \sum_{k\in\alpha}(N[\overline{\alpha},\alpha](i,k)\ (\sum_{m\in\alpha} N[\alpha]^{-1}(k,m)\ N[\alpha,\overline{\alpha}](m,j))) \end{aligned}$$

for all $i,j \in \overline{\alpha}$. Note that $N[\alpha]^{-1} = M[\alpha]^{-1}$. Hence, if the sum over all k (containing the sum over all m) is non-zero, there exist k and m such that $N[\overline{\alpha},\alpha](i,k)$, $N[\alpha]^{-1}(k,m)$, and $N[\alpha,\overline{\alpha}](m,j)$ are non-zero. Thus, for $i \neq j$, in any matrix $N' \in \mathcal{V}_{M,\alpha}$ the fact whether the sum over all k is zero or not can be controlled by modifying $N'[\alpha,\overline{\alpha}](m,j)$ and $N'[\overline{\alpha},\alpha](j,m)$ only (note that the modified matrix also has to be antisymmetric). Moreover, there exists at most one pair of values for $N'[\alpha,\overline{\alpha}](m,j)$ and $N'[\overline{\alpha},\alpha](j,m)$ such that the sum over all k amounts to zero.

Consider the case that $N[\overline{\alpha}](i,j) \neq 0$ (note that this implies $i \neq 0$) and let $N' \in \mathcal{V}_{M,\alpha}$. Then, modifying $N'[\overline{\alpha}](i,j)$ and $N'[\overline{\alpha}](j,i)$ only suffices to control the fact whether $(N' \setminus N'[\alpha])(i,j)$ is zero or not. Moreover, there exists at most one pair of values for $N'[\overline{\alpha}](i,j)$ and $N'[\overline{\alpha}](j,i)$ such that $(N' \setminus N'[\alpha])(i,j)$ is zero.

Now consider the more general case that $(N \setminus N[\alpha])(i,j)$ is non-zero (note that this also implies $i \neq 0$) and let $N' \in \mathcal{V}_{M,\alpha}$. In the case $N[\overline{\alpha}](i,j) = 0$

the sum over all k is non-zero. Hence, all but one modification of the pair $N'[\alpha,\overline{\alpha}](m,j)$ and $N'[\overline{\alpha},\alpha](j,m)$ yields $(N' \setminus N'[\alpha])(i,j) \neq 0$ (see above). In the case $N[\overline{\alpha}](i,j) \neq 0$ all but one modification of the pair $N'[\overline{\alpha}](i,j)$ and $N'[\overline{\alpha}](j,i)$ yields $(N' \setminus N'[\alpha])(i,j) \neq 0$.

Hence, if there exists $N \in \mathcal{V}_{M,\alpha}$ with $(N \setminus N[\alpha])(i,j) \neq 0$, and if $N' \in \mathcal{V}_{M,\alpha}$, then it suffices to touch (i.e. arbitrarily change) at most one pair of entries in order to make $(N' \setminus N'[\alpha])(i,j)$ non-zero. Since $\mathbb{F}$ is infinite, touching of pairs can be continued until one arrives at a matrix $S \in \mathcal{V}_{M,\alpha}$ such that $\overline{S \setminus S[\alpha]}$ is the supremum of $\mathcal{T}_{M,\alpha}$. □

The last proposition suggests to define the following class of antisymmetric adjacency matrices.

Definition 8.2 (Matrix class $sup_{M,\alpha}$) *Let $\mathbb{F}$ be infinite, let $\alpha \in \langle n \rangle$ and let $M \in \mathbb{F}^{n\times n}$ be antisymmetric such that $M[\alpha]$ is non-singular. Set*

$$sup_{M,\alpha} := \{S \in \mathcal{V}_{M,\alpha} \mid \overline{S \setminus S[\alpha]} \text{ is the supremum of } \mathcal{T}_{M,\alpha}\} \ . \tag{20}$$

In the following we will specify those principal submatrices of a symmetric adjacency matrix A that have a non-singular counterpart in an antisymmetric adjacency matrix M with $\overline{M} = A$.

Definition 8.3 (Regular with respect to A, n_A) *Let $\mathbb{F}$ be infinite and let $A \in \{0,1\}^{n\times n}$ be symmetric with $A(i,i) = 0$ for all i. The set $\alpha \in \langle n \rangle$ is called regular with respect to A, if there exists antisymmetric $M \in \mathbb{F}^{n\times n}$ such that $\overline{M} = A$ and $M[\alpha]$ is non-singular. Set*

$$n_A := \{\alpha \in \langle n \rangle \mid \alpha \text{ is regular with respect to } A\} \ .$$

Finally, the matrix class sup^A with respect to $\mathbb{F}$ is defined as follows.

Definition 8.4 (Matrix class sup^A with respect to $\mathbb{F}$) *Let $\mathbb{F}$ be infinite and let $A \in \{0,1\}^{n\times n}$ be symmetric with $A(i,i) = 0$ for all i. The class sup^A of A is defined by*

$$sup^A := \bigcap_{\alpha\in n_A} \left(\bigcup (sup_{N,\alpha} \mid \overline{N} = A \text{ and } N[\alpha] \text{ is non-singular}) \right) \ .$$

Looking into the proof of Prop. 8.1 once again, we get the following.

Prop. 8.5 *Let $M \in sup^A$ and let $M[\alpha]$ be a non-singular principal submatrix of M. Then, $(ppt(M,\alpha))(i,j) = 0$ implies $M(i,j) = 0$ for all $i,j \in \overline{\alpha}$.*

9 The pivoting polynomial

The following definition will allow to extend the results of Sec. 8 to classes of gain graphs.

Definition 9.1 (Supremum gain graphs from G_u)
Let G_u be an undirected simple graph with vertex set V. Moreover, let $l(\cdot)$ be a bijective mapping from V to $\{1, \ldots, |V|\}$, and let A be the symmetric adjacency matrix of G_u with respect to l. A gain graph with vertex set V is called supremum gain graph from G_u, if its antisymmetric adjacency matrix with respect to $l(\cdot)$ is in sup^A.

To see that the above definition is independent of $l(\cdot)$ consider an edge $\{u, v\}$ of G_u. It corresponds to the entries $A(l(u), l(v)) = A(l(v), l(u)) = 1$ of A, i.e. to the entries $M(l(u), l(v)) = -M(l(v), l(u)) \neq 0$ of *all* matrices $M \in sup^A$. Those entries, in turn, correspond to the edges (u, v) and (v, u) of *all* supremum gain graphs from G_u.

In the following we will see that the extended interlace polynomials of supremum gain graphs from G_u are identical.

A gain graph G' is called *first-order-descendant* of G, if G has an edge (a, b) such that $G' = G - a$ or $G' = G^{\{a,b\}} - b$. A *descendant of order* $n + 1$ is a first-order-descendant of an an n-th order descendant. It will turn out to be convenient, if G may be referred to as its descendant of order 0. G' is called a *descendant* of G, if G' is an n-th order descendant of G for some $n \in \mathbb{N}_0$. The vertex set of a descendant of $G = (V, E, g)$ is always a subset of V. A descendant of G is called *terminal*, if it has no edges (and thus no descendants).

Prop. 9.2 (Descendants of $q(G)$) *Let G be a gain graph. Then, the recursive computation of $q(G)$ can be done such that each descendant is a subgraph of G or a subgraph of some $G^{U_1 \cdots U_m} \setminus U$, where $U = \bigcup_{i=1}^{m} U_i \subset V$ and each U_i has cardinality 2.*

Proof: We may assume that G has at least one non-isolated vertex r_1. Consider the following scheme for computing $q(G)$.

- **Step** $i = 1$. As long as r_1 is a non-isolated vertex of D (note that $D = G$ in the beginning), continue the calculation of $q(D)$ via the descendants $D - r_1$ and $D^{\{r_1,s_1\}} - s_1$ for some s_1. The descendant $D - r_1$ has no further descendants at this stage of the scheme. If r_1 is not isolated in $D^{\{r_1,s_1\}} - s_1$, the scheme transforms $D^{\{r_1,s_1\}} - s_1$ into the descendants $D^{\{r_1,s_1\}} - r_1 - s_1$ and $(D^{\{r_1,s_1\}} - s_1)^{\{r_1,s_2\}} - s_2$ for some s_2. Again, the first descendant has no further descendants at this stage and, using Eq. (7), the second descendant can be written as $(D^{\{r_1,s_1\}} - s_1)^{\{r_1,s_2\}} - s_2 =$

$$(D^{\{r_1,s_1\}})^{\{r_1,s_2\}} - s_1 - s_2 = D^{\{r_1,s_2\}} - s_1 - s_2.$$

  In the following let d denote the degree of r_1 in D. Continued expansion of each second descendant eventually leaves us with the descendant $D - r$, descendants of the form

$$D^{\{r_1,s_k\}} - r_1 - s_1 - s_2 - \ldots - s_k \quad (k < d), \tag{21}$$

  and the descendant $D^{\{r_1,s_d\}} - s_1 - \ldots - s_d$ with isolated r_1. The expression of the latter descendant can be simplified as follows. Pivoting on $\{r_1, s_d\}$

has no effect on the neighborhood of r_1. Hence, $s_1, \ldots, s_d$ are precisely the neighbors of r_1 in D and s_d is the only neighbor of r_1 in $D - s_1 - \ldots - s_{d-1}$. Thus, $D^{\{r_1,s_d\}} - s_1 - \ldots - s_d = (D - s_1 - \ldots - s_{d-1})^{\{r_1,s_d\}} - s_d =$

$$D - s_1 - \ldots - s_d \; .$$

Hence, each descendant is a subset of D or it has the form (21).

- **Step** $i > 1$. We may assume that each descendant D is a subgraph of G or a subgraph of $G^{U_1,\ldots,U_j} - \bigcup_{k=1}^{j} U_k$ for some U_k, $1 \leq k \leq j \leq i-1$. If D has a non-isolated vertex r_i, we apply the first step of the scheme to r_i instead of r_1. All descendants still have the required form. □

Prop. 9.3 (Identical polynomials, independent sets) *Let $G = (V, E, g(\cdot))$ and G' be supremum gain graphs from G_u. Then $q(G) = q(G')$. Furthermore, for each term $a_k x^k$ of $q(G)$ with $a_k \neq 0$ there exist independent sets of G_u with cardinality k.*

Proof: According to Prop. 9.2 the polynomial $q(G)$ can be computed such that every descendant is a subgraph of G or a subgraph of some $G^{U_1 \ldots U_m} \setminus U$, where $U = \bigcup_{i=1}^{m} U_i$ and each U_i has cardinality 2.

Let A be a the template of an antisymmetric adjacency matrix of G with respect to the bijective mapping $l : V \to \{1, \ldots |V|\}$. From Theorem 3.4 it follows that the adjacency matrix of $G^{U_1 \ldots U_m} \setminus U$ with respect to $l(\cdot)$ is $M \setminus M[\alpha]$ for some $M \in \overline{sup^A}$, where $\alpha = \bigcup_{i=1}^{m} \alpha_i$, $\alpha_i = l(U_i)$ for all i.

From $\overline{M \setminus M[\alpha]}$ being unique it follows that the templates of the terminal descendants are also unique. Moreover, Prop. 8.5 implies that the edge set of $G^{U_1 \ldots U_m} \setminus U$ is a superset of the edge set of $G \setminus U$. Hence, the vertex sets of the (edge-less) terminal descendants of $q(G)$ are unique independent sets of G_u. □

Prop. 9.3 implies that the following is well defined.

Definition 9.4 (Pivoting polynomial $p(G_u)$) *Let G_u be an undirected simple graph and let G be a supremum gain graph from G_u. The pivoting polynomial $p(G_u)$ of G_u is defined by $p(G_u) := q(G)$.*

The following consideration is from [2]. Take a vertex a_1. If a_1 is an isolated vertex of G, then $q(G) = xq(G - a_1)$. Otherwise, for any edge (a_1, b) it holds that $q(G) = q(G - a_1) + q(G^{\{a_1,b\}} - b)$. In any case, since all coefficients of all $q(\cdot)$ are non-negative, it follows that the degree of $q(G)$ is not smaller than the degree of $q(G - a_1)$. By subsequently taking a_i that are not contained in a fixed maximum independent set, we get that the degree of $q(G)$ is an upper bound for the independence number.

From Prop. 9.3 it follows that the degree of $q(G)$ is a lower bound for the independence number, if G is a supremum gain graph from some G_u. Hence, we get the following.

Prop. 9.5 (Degree = independence number) *The degree of $p(G_u)$ is equal to the independence number of G_u.*

The example $p(K^3) = 4x$ shows that the pivoting polynomial, in contrast to the independence polynomial [5], may list an independent set more than once.

10 Conclusions

We defined a new graph polynomial for undirected simple graphs, the *pivoting polynomial*. It was derived from the interlace polynomial by extending it to gain graphs over arbitrary fields. Provided that the underlying field is infinite, gain graphs with identical vertex and edge sets were then shown to yield the same extended interlace polynomial, as long as they fulfill a non-singularity condition formulated in terms of their adjacency matrices. Finally, the pivoting polynomial was defined by means of gain graphs that fulfill the non-singularity condition. The recursive computation of the pivoting polynomial can be done such that each end of the recursion corresponds to an independent set, one of which is maximum.

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11 Appendix

Alternative proof of Lemma 3.5: We first derive analogous results for simple pivoting and then extend the proof to double pivoting.

(1) Let $j \neq b, c$. If $i \neq a$, then $M^{(a,b)(a,c)}(i,j)$

$$\begin{aligned}
&= M^{(a,b)}(i,j) - \frac{M^{(a,b)}(i,c)}{M^{(a,b)}(a,c)} M^{(a,b)}(a,j) \\
&= M(i,j) - \frac{M(i,b)}{M(a,b)} M(a,j) - (M(i,c) \\
&\quad - \frac{M(i,b)}{M(a,b)} M(a,c)) \frac{M(a,b)}{M(a,c)} \frac{M(a,j)}{M(a,b)} \\
&= M(i,j) + M(a,j)(-\frac{M(i,b)}{M(a,b)} - \frac{M(i,c)}{M(a,c)} + \frac{M(i,b)}{M(a,b)}) \\
&= M(i,j) - M(a,j)(\frac{M(i,c)}{M(a,c)}) \\
&= M^{(a,c)}(i,j). \text{ Otherwise,}
\end{aligned}$$

$$M^{(a,b)(a,c)}(a,j) = -\frac{M^{(a,b)}(a,j)}{M^{(a,b)}(a,c)} = -\frac{M(a,j)}{M(a,c)} = M^{(a,c)}(a,j).$$

(2) Let $i \neq b, c$ and let M^T denote the transpose of M. Then, $M^{(b,a)(c,a)}(i,j) = ((M^T)^{(a,b)(a,c)})(j,i) = ((M^T)^{(a,c)})(j,i) = M^{(c,a)}(i,j)$.

(3) For $\{i,j\} \cap \{b,c\} = \emptyset$ it holds that $M^{\{a,b\}\{a,c\}}(i,j)$

$$\begin{aligned}
&= M^{(a,b)(b,a)(a,c)(c,a)}(i,j) = M^{(b,a)(a,b)(a,c)(c,a)}(i,j) \\
&= M^{(b,a)(a,c)(c,a)}(i,j) = M^{(b,a)(c,a)(a,c)}(i,j) \\
&= M^{(c,a)(a,c)}(i,j) \\
&= M^{\{a,c\}}(i,j).
\end{aligned}$$

(4) Let $i \neq a, b, c$. Then $M^{\{a,b\}\{a,c\}}(i,b)$

$$\begin{aligned}
&= M^{(a,b)(b,a)}(i,b) - (\frac{M^{(a,b)(b,a)}(i,c)}{M^{(a,b)(b,a)}(a,c)} M^{(a,b)(b,a)}(a,b)) \\
&\quad - \frac{M^{(a,b)(b,a)}(i,a)}{M^{(a,b)(b,a)}(c,a)} M^{(a,b)(b,a)}(c,b) \\
&= M^{(a,b)}(i,b) - (\frac{M^{(a,b)(b,a)}(i,c)}{M^{(a,b)}(a,c)} M^{(a,b)}(a,b)) - \frac{M(i,a)}{M(c,a)} M^{(a,b)}(c,b) \\
&= \frac{M(i,b)}{M(a,b)} + \frac{1}{M(a,b)} M^{\{a,b\}}(i,c) - \frac{M(i,a)}{M(c,a)} \frac{M(c,b)}{M(a,b)} \\
&= \frac{M(i,b)}{M(a,b)} + \frac{1}{M(a,b)} (M(i,c) - \frac{M(i,b)}{M(a,b)} M(a,c)) - \frac{M(i,a)}{M(c,a)} \frac{M(c,b)}{M(a,b)} \\
&= \frac{M(i,b)}{M(a,b)} + \frac{M(i,c)}{M(a,b)} - \frac{M(i,a)}{M(b,a)} \frac{M(b,c)}{M(a,c)} - \frac{M(i,a)}{M(c,a} \frac{M(c,b)}{M(a,b)} \\
&= M^{\{a,c\}}(i,c).
\end{aligned}$$

Furthermore (the cases $i = a, b, c$),

$$\begin{aligned}
M^{\{a,b\}\{a,c\}}(a,b) &= M^{\{a,b\}(a,c)}(a,b) = -\frac{M^{\{a,b\}}(a,b)}{M^{\{a,b\}}(a,c)} = -\frac{M^{(a,b)}(a,b)}{M^{(a,b)}(a,c)} \\
&= \frac{1}{M^{(a,c)}} = M^{(a,c)}(a,c) = M^{\{a,c\}}(a,c), \\
M^{\{a,b\}\{a,c\}}(b,b) &= 0 = M^{\{a,c\}}(c,c), \\
M^{\{a,b\}\{a,c\}}(c,b) &= M^{\{a,b\}(c,a)}(c,b) = -\frac{M^{(a,b)(b,a)}(c,b)}{M^{(b,a)(a,b)}(c,a)} \\
&= -\frac{M^{(a,b)}(c,b)}{M^{(b,a)}(c,a)} = \frac{M(c,b)}{M(c,a)} = -M(c,a)(c,b) \\
&= -M^{\{a,c\}}(c,b) = M^{\{a,c\}}(b,c).
\end{aligned}$$

(5) Let $i \neq a, b, c$. Then $M^{\{a,b\}\{a,c\}}(i,c)$

$$\begin{aligned}
&= \frac{M^{\{a,b\}}(i,c)}{M^{(a,b)(b,a)}(a,c)} = \frac{M^{\{a,b\}}(i,c)}{M^{(a,b)}(a,c)} = -\frac{M(a,b)}{M(a,c)}(M^{\{a,b\}}(i,c)) \\
&= -\frac{M(a,b)}{M(a,c)}(m(i,c) - \frac{M(i,b)}{M(a,b)}M(a,c) - \frac{M(i,a)}{M(b,a}M(b,c)) \\
&= M(i,b) - \frac{M(i,c)}{M(a,c)}M(a,b) - \frac{M(i,a)}{M(a,c)}M(b,c) \\
&= M(i,b) - \frac{M(i,c)}{M(a,c)}M(a,b) - \frac{M(i,a)}{M(c,a)}M(c,b) \\
&= M^{\{a,c\}}(i,b).
\end{aligned}$$

Furthermore (cases $i = (a, b, c)$),

$$\begin{aligned}
M^{\{a,b\}\{a,c\}}(a,c) &= M^{\{a,b\}(a,c)}(a,c) = \frac{1}{M^{\{a,b\}}(a,c)} \\
&= \frac{1}{M^{(a,b)}(a,c)} = -\frac{M(a,b)}{M(a,c)} = M^{(a,c)}(a,b) \\
&= M^{\{a,c\}}(a,b), \\
M^{\{a,b\}\{a,c\}}(b,c) &= M^{\{a,b\}(b,c)} = \frac{M^{\{a,b\}}(b,c)}{M^{\{a,b\}}(a,c)} = \frac{M^{(b,a)}(b,c)}{M^{(a,b)}(a,c)} \\
&= -M^{(a,c)}(b,c) = -M^{\{a,c\}}(b,c) = M^{\{a,c\}}(c,b), \\
M^{\{a,b\}\{a,c\}}(c,c) &= 0 = M^{\{a,b\}\{a,c\}}(b,b).
\end{aligned}$$

(6) The remaining part follows from the preservation of antisymmetry under double pivoting.

$$\begin{aligned}
M^{\{a,b\}\{a,c\}}(b,i) &= -M^{\{a,b\}\{a,c\}}(i,b) = -M^{\{a,c\}}(i,c) = M^{\{a,c\}}(c,i), \\
M^{\{a,b\}\{a,c\}}(c,i) &= -M^{\{a,b\}\{a,c\}}(i,c) = -M^{\{a,c\}}(i,b) = M^{\{a,c\}}(b,i).
\end{aligned}$$