Partial orders, Lattices, etc.
In our context…

• We aim at computing properties on programs
• How can we represent these properties? Which kind of algebraic features have to be satisfied on these representations?
• Which conditions guarantee that this computation terminates?
Motivating Example (1)

- Consider the renovation of the building of a firm. In this process several tasks are undertaken
  - Remove asbestos
  - Replace windows
  - Paint walls
  - Refinish floors
  - Assign offices
  - Move in office furniture
  - ...

Motivating Example (2)

• Clearly, some things had to be done before others could begin
  – Asbestos had to be removed before anything (except assigning offices)
  – Painting walls had to be done before refinishing floors to avoid ruining them, etc.

• On the other hand, several things could be done concurrently:
  – Painting could be done while replacing the windows
  – Assigning offices could be done at anytime before moving in office furniture

• This scenario can be nicely modeled using partial orderings
Partial Orderings: Definitions

• **Definitions:**
  – A relation \( R \) on a set \( S \) is called a **partial order** if it is
    • Reflexive
    • Antisymmetric
    • Transitive
  – A set \( S \) together with a partial ordering \( R \) is called a **partially ordered set** (poset, for short) and is denote \((S, R)\)

• Partial orderings are used to give an order to sets that may not have a natural one

• In our renovation example, we could define an ordering such that \((a,b) \in R\) if ‘a must be done before b can be done’
Partial Orderings: Notation

- We use the notation:
  - $a \prec b$, when $(a,b) \in R$
  - $a \not\prec b$, when $(a,b) \in R$ and $a \neq b$

- The notation $\prec$ is not to be mistaken for “less than” ($\prec$ versus $\leq$)
- The notation $\prec$ is used to denote any partial ordering
Comparability: Definition

• **Definition:**
  - The elements $a$ and $b$ of a poset $(S, \prec)$ are called **comparable** if either $a \prec b$ or $b \prec a$.
  - When for $a, b \in S$, we have neither $a \prec b$ nor $b \prec a$, we say that $a, b$ are **incomparable**.

• Consider again our renovation example
  - Remove Asbestos $\prec a_i$ for all activities $a_i$ except assign offices
  - Paint walls $\prec$ Refinish floors
  - Some tasks are incomparable: Replacing windows can be done before, after, or during the assignment of offices
Total orders: Definition

- **Definition:**
  - If \((S, \prec)\) is a poset and every two elements of \(S\) are comparable, \(S\) is called a **totally ordered set**.
  - The relation \(\prec\) is said to be a **total order**

- **Example**
  - The relation “less than or equal to” over the set of integers \((\mathbb{Z}, \leq)\)
    since for every \(a, b \in \mathbb{Z}\), it must be the case that \(a \leq b\) or \(b \leq a\)
  - What happens if we replace \(\leq\) with \(<\)?

The relation \(<\) is not reflexive, and \((\mathbb{Z}, <)\) is not a poset
Hasse Diagrams

• Like relations and functions, partial orders have a convenient graphical representation: Hasse Diagrams
  – Consider the digraph representation of a partial order
  – Because we are dealing with a partial order, we know that the relation must be reflexive and transitive
  – Thus, we can simplify the graph as follows
    • Remove all self loops
    • Remove all transitive edges
    • Remove directions on edges assuming that they are oriented upwards
  – The resulting diagram is far simpler
Hasse Diagram: Example
Hasse Diagrams: Example (1)

• Of course, you need not always start with the complete relation in the partial order and then trim everything.

• Rather, you can build a Hasse Diagram directly from the partial order

• Example: Draw the Hasse Diagram
  – for the following partial ordering: \{(a,b) \mid a|b \}
  – on the set \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}
  – (these are the divisors of 60 which form the basis of the ancient Babylonian base-60 numeral system)
Example

\[ L = \{a, b, c, d, e, f, g\} \]
\[ \prec = \{(a, c), (a, e), (b, d), (b, f), (c, g), (d, g), (e, g), (f, g)\}^{RT} \]

\((L, \prec)\) is a partial order
Example

$L = \mathbb{N}$ (natural numbers)

$\alpha = \{(0,1), (1,2), (2,3), (3,4), (4,5), \ldots\}_{RT}$

$(L, \prec)$ is a totally ordered set (infinite)
Example

$L = \mathbb{N}$ (natural numbers)

$\prec = \{(n, m) : \exists k \text{ such that } m = n \cdot k\}$

$(L, \prec)$ is a partially ordered set (infinite)
Example

• On the same set $E=\{1,2,3,4,6,12\}$ we can define different partial orders:
Example

- All possible partial orders on a set of three elements (modulo renaming)
Extremal Elements: Summary

We will define the following terms:

- A maximal/minimal element in a poset \((S, \prec)\)
- The maximum (greatest)/minimum (least) element of a poset \((S, \prec)\)
- An upper/lower bound element of a subset \(A\) of a poset \((S, \prec)\)
- The greatest lower/least upper bound element of a subset \(A\) of a poset \((S, \prec)\)
Extremal Elements: Maximal

- **Definition**: An element $a$ in a poset $(S, \prec)$ is called **maximal** if it is not less than any other element in $S$. That is: $\neg (\exists b \in S (a \prec b))$

- If there is one **unique** maximal element $a$, we call it the **maximum** element (or the **greatest** element)
Extremal Elements: Minimal

• **Definition:** An element $a$ in a poset $(S, \prec)$ is called **minimal** if it is not greater than any other element in $S$. That is: $\neg(\exists b \in S \ (b \prec a))$

• If there is one **unique** minimal element $a$, we call it the **minimum** element (or the **least** element)
Extremal Elements: Upper Bound

• **Definition:** Let \((S, \prec)\) be a poset and let \(A \subseteq S\). If \(u\) is an element of \(S\) such that \(a \prec u\) for all \(a \in A\) then \(u\) is an **upper bound of** \(A\).

• An element \(x\) that is an upper bound on a subset \(A\) and is less than all other upper bounds on \(A\) is called the **least upper bound on** \(A\). We abbreviate it as lub.
Extremal Elements: Lower Bound

- **Definition:** Let \((S, \preceq)\) be a poset and let \(A \subseteq S\). If \(l\) is an element of \(S\) such that \(l \preceq a\) for all \(a \in A\) then \(l\) is an lower bound of \(A\).

- An element \(x\) that is a lower bound on a subset \(A\) and is greater than all other lower bounds on \(A\) is called the greatest lower bound on \(A\). We abbreviate it glb.
Example

\[(x_1, y_1) \leq_{\mathbb{N} \times \mathbb{N}} (x_2, y_2) \iff x_1 \leq_{\mathbb{N}} x_2 \land y_1 \leq_{\mathbb{N}} y_2\]
Example

\[(x_1, y_1) \leq_{\mathbb{N} \times \mathbb{N}} (x_2, y_2) \iff x_1 \leq_{\mathbb{N}} x_2 \land y_1 \leq_{\mathbb{N}} y_2\]
Extremal Elements: Example 1

What are the minimal, maximal, minimum, maximum elements?

- Minimal: \{a,b\}
- Maximal: \{c,d\}
- There are no unique minimal or maximal elements, thus no minimum or maximum
Extremal Elements: Example 2

Give lower/upper bounds & glb/lub of the sets:

\{d,e,f\}, \{a,c\} and \{b,d\}

\{d,e,f\}
- Lower bounds: \emptyset, thus no glb
- Upper bounds: \emptyset, thus no lub

\{a,c\}
- Lower bounds: \emptyset, thus no glb
- Upper bounds: \{h\}, lub: h

\{b,d\}
- Lower bounds: \{b\}, glb: b
- Upper bounds: \{d,g\}, lub: d because d \prec g
Extremal Elements: Example 3

• Minimal/Maximal elements?
  • Minimal & Minimum element: a
  • Maximal elements: b,d,i,j

• Bounds, glb, lub of \{c,e\}?  
  • Lower bounds: \{a,c\}, thus glb is c
  • Upper bounds: \{e,f,g,h,i,j\}, thus lub is e

• Bounds, glb, lub of \{b,i\}?  
  • Lower bounds: \{a\}, thus glb is c
  • Upper bounds: \emptyset, thus lub DNE
Lattices

• A special structure arises when every pair of elements in a poset has an lub and a glb

• **Definition:** A lattice is a partially ordered set in which every pair of elements has both
  – a least upper bound and
  – a greatest lower bound
Lattices: Example 1

• Is the example from before a lattice?

• No, because the pair \{b,c\} does not have a least upper bound
Lattices: Example 2

- What if we modified it as shown here?

- Yes, because for any pair, there is an lub & a glb
A Lattice Or Not a Lattice?

• To show that a partial order is not a lattice, it suffices to find a pair that does not have an lub or a glb (i.e., a counter-example)

• For a pair not to have an lub/glb, the elements of the pair must first be incomparable (Why?)

• You can then view the upper/lower bounds on a pair as a sub-Hasse diagram: If there is no maximum/minimum element in this sub-diagram, then it is not a lattice
Complete lattices

- **Definition:**
  A lattice $A$ is called a complete lattice if every subset $S$ of $A$ admits a glb and a lub in $A$.

- **Exercise:**
  Show that for any (possibly infinite) set $E$, $(P(E), \subseteq)$ is a complete lattice.
  ($P(E)$ denotes the powerset of $E$, i.e. the set of all subsets of $E$).
Example

$L = \{a, b, c, d, e, f, g\}$

$\leq = \{(a, c), (a, e), (b, d), (b, f), (c, g), (d, g), (e, g), (f, g)\}^T$

$(L, \leq)$ is not a lattice:

- $a$ and $b$ are lower bounds of $Y$, but $a$ and $b$ are not comparable.
Exercise

• Prove that “Every finite lattice is a complete lattice”.

Example

- $L = \emptyset(\{1,2,3\})$
- $\subseteq$
- $\text{lub}(Y) = \cup Y$
- $\text{glb}(Y) = \cap Y$
Example

\[ L = \mathbb{Z} \cup \{ T, \bot \} \]

\[ \forall n \in \mathbb{Z} : \bot \prec n \prec T \]
Example

$L = \mathbb{Z}_+$

\(<\) total order on $\mathbb{Z}_+$

lub = max

glb = min

It is a lattice, but not complete:

For instance, the set of even numbers has no lub
Example

$L = \mathbb{Z}_+ \cup \{T\}$

\(\prec\) total order on \(\mathbb{Z}_+ \cup \{T\}\)

lub = max

glb = min

This is a complete lattice
Examples

- **L=R** (real numbers) with $< = \leq$ (total order)
- $(R, \leq)$ is not a complete lattice:
  - for instance $\{x \in R \mid x > 2\}$ has no lub

On the other hand,
- for each $x < y$ in $R$, $([x, y], \leq)$ is a complete lattice

- **L=Q** (rational numbers) with $< = \leq$ (total order)
- $(Q, \leq)$ is not a complete lattice
- The set $\{x \in Q \mid x^2 < 2\}$ has upper bounds but there is no least upper bound in $Q$. 
• **Theorem:**
  Let \((L, \preceq)\) be a partial order. The following conditions are equivalent:
  1. \(L\) is a complete lattice
  2. Each subset of \(L\) has a least upper bound
  3. Each subset of \(L\) has a greatest lower bound

• **Proof:**
  – \(1 \Rightarrow 2 \text{ e } 1 \Rightarrow 3\) by definition
  – In order to prove that \(2 \Rightarrow 1\), let us define for each \(Y \subseteq L\)
    \[
    \text{glb}(Y) = \text{lub}\{l \in L \mid \forall l' \in Y : l \leq l'\}
    \]
\[ \text{glb}(Y) = \text{lub}(\{ l \in L \mid \forall l' \in Y : l \leq l' \}) \]

\[ Z = \{ l \in L \mid \forall l' \in Y : l \leq l' \} \]

upper bounds of Z

\[ \text{lub}(Z) \]
Functions on partial orders

- Let \((P, \leq_P)\) and \((Q, \leq_Q)\) two partial orders. A function \(\varphi\) from \(P\) to \(Q\) is said:

  - **monotone** (order preserving) if
    \[ p_1 \leq_P p_2 \implies \varphi(p_1) \leq_Q \varphi(p_2) \]

  - **embedding** if
    \[ p_1 \leq_P p_2 \iff \varphi(p_1) \leq_Q \varphi(p_2) \]

  - **Isomorphism** if it is a surjective embedding
Examples

\( \varphi_1(a) \)
\( \varphi_1(d) \)
\( \varphi_1(b) = \varphi_1(c) \)

- \( \varphi_1 \) is not monotone

\( \varphi_2(d) = \varphi_2(e) \)
\( \varphi_2(b) = \varphi_2(c) \)
\( \varphi_2(a) \)

- \( \varphi_2 \) is monotone, but it is not an embedding: \( \varphi_2(b) \leq_Q \varphi_2(c) \) but it is not true that \( b \leq_P c \)
Examples

- $\varphi_3$ is monotone but it is not an embedding: $\varphi_3(b) \leq Q \varphi_3(c)$ but it is not true that $b \leq_P c$

- $\varphi_4$ is an embedding, but not an isomorphism.
Isomorphism
Monotone? Embedding? Isomorphism?

□ \( \varphi \) from \((\mathbb{Z}, \leq)\) to \((\mathbb{Z}, \leq)\), defined by: \( \varphi(x) = x + 1 \)

□ \( \varphi \) from \((\mathcal{P}(S), \subseteq)\) to \((\mathcal{P}(\mathbb{Z}), \subseteq)\), defined by:
\( \varphi(U) = 1 \) if \( U \) is nonempty, \( \varphi(\emptyset) = 0 \).

□ \( \varphi \) from \((\mathcal{P}(\mathbb{Z}), \subseteq)\) to \((\mathcal{P}(\mathbb{Z}), \subseteq)\), defined by:
\( \varphi(U) = \{1\} \) if \( 1 \in U \)
\( \varphi(U) = \{2\} \) if \( 2 \in U \) and \( 1 \) does not belong to \( U \)
\( \varphi(U) = \emptyset \) otherwise
Ascending chains

- A sequence \((l_n)_{n \in \mathbb{N}}\) of elements in a partial order \(L\) is an ascending chain if
  \[ n \leq m \Rightarrow l_n \leq l_m \]

- A sequence \((l_n)_{n \in \mathbb{N}}\) converges if and only if
  \[ \exists n_0 \in \mathbb{N} : \forall n \in \mathbb{N} : n_0 \leq n \Rightarrow l_{n_0} = l_n \]

- A partial order \((L, \leq)\) satisfies the ascending chain condition (ACC) iff each ascending chain converges.
Example

- The set of even natural numbers satisfies the descending chain condition, but not the ascending chain condition.
Example

- Infinite set
- Satisfies both ACC and DCC
Lattices and ACC

• If P is a lattice, it has a bottom element and satisfies ACC, then it is a complete lattice

• If P is a lattice without infinite chains, then it is complete
Continuity

- In Calculus, a function is continuous if it preserves the limits.
- Given two partial orders \((P, \leq_P)\) and \((Q, \leq_Q)\), a function \(\varphi\) from \(P\) to \(Q\) is continuous if for every chain \(S\) in \(P\)

\[
\varphi(\text{lub}(S)) = \text{lub}\{ \varphi(x) \mid x \in S \}
\]
Fixpoints

- Consider a monotone function $f: (P, \leq_P) \rightarrow (P, \leq_P)$ on a partial order $P$.
- An element $x$ of $P$ is a fixpoint of $f$ if $f(x) = x$.
- The set of fixpoints of $f$ is a subset of $P$ called $\text{Fix}(f)$:

$$\text{Fix}(f) = \{ l \in P \mid f(l) = l \}$$
Fixpoint on Complete Lattices

• Consider a \textit{monotone} function \( f: \mathbb{L} \to \mathbb{L} \) on a \textit{complete lattice} \( \mathbb{L} \).

• \( \text{Fix}(f) \) is also a complete lattice:

\[
\begin{align*}
\text{lfp}(f) & = \text{glb}(\text{Fix}(f)) \quad \in \text{Fix}(f) \\
\text{gfp}(f) & = \text{lub}(\text{Fix}(f)) \quad \in \text{Fix}(f)
\end{align*}
\]

• \textbf{Tarski Theorem:}
  Let \( \mathbb{L} \) be a complete lattice. If \( f: \mathbb{L} \to \mathbb{L} \) is \textit{monotone} then

\[
\begin{align*}
\text{lfp}(f) & = \text{glb}\{ l \in \mathbb{L} \mid f(l) \leq l \} \\
\text{gfp}(f) & = \text{lub}\{ l \in \mathbb{L} \mid l \leq f(l) \}
\end{align*}
\]
Fixpoints on Complete Lattices

\[ \text{Fix}(f) = \{ l \in L \mid f(l) = l \} \]

\[ \{ l \in L \mid f(l) \leq_P l \} \]

\[ \{ l \in L \mid l \leq_P f(l) \} \]

\[ \text{gfp}(f) = \text{lub}\{ l \in L \mid l \leq f(l) \} \]

\[ \text{lfp}(f) = \text{glb}\{ l \in L \mid f(l) \leq l \} \]
Kleene Theorem

Let $f$ be a monotone function: $(P,\leq_P) \to (P,\leq_P)$ on a complete lattice $P$. Let $\alpha=\bigcup_{n\geq 0} f^n(\bot)$

- If $\alpha \in \text{Fix}(f)$ then $\alpha=\text{lfp}(f)$

- **Kleene Theorem**
  
  If $f$ is continuous then the least fixpoint of $f$ exists, and it is equal to $\alpha$