Game Theory in Computer Vision and Pattern Recognition

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Tutorial at *CVPR 2011*, Colorado Springs, CO, June 20, 2010
Web page: http://www.dsi.unive.it/~atorsell/cvpr2011tutorial

Course Outline

Part I (08:00 – 10:00) – *M. Pelillo*

- ✓ Introduction to game theory
- ✓ Polymatrix games, contextual pattern recognition (relaxation labeling), and graph transduction
- ✓ Evolutionary games and data clustering (dominant sets)

Coffee break

Part II (10:30 – 12:30) – *A. Torsello*

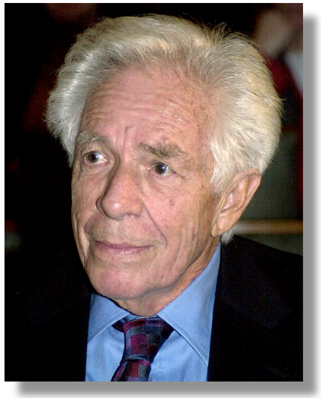
- ✓ Applications
- ✓ Repeated games and online learning

Introduction to Game Theory

What is Game Theory?

“Game theory is to games of strategy what probability theory is to games of chance. And just as probability theory far transcends its role as the logical basis of rational gambling, so does game theory transcende its original guise as the logical basis of parlor games.”

Anatol Rapoport
Two-Person Game Theory (1966)



“The central problem of game theory was posed by von Neumann as early as 1926 in Göttingen. It is the following: If n players, $P_1,\ldots, P_n$, play a given game Γ, how must the i^{th} player, P_i, play to achieve the most favorable result for himself?”

Harold W. Kuhn
Lectures on the Theory of Games (1953)

A Few Cornerstones in Game Theory

1921–1928: Emile Borel and John von Neumann give the first modern formulation of a mixed strategy along with the idea of finding minimax solutions of normal-form games.

1944, 1947: John von Neumann and Oskar Morgenstern publish *Theory of Games and Economic Behavior*.

1950–1953: In four papers John Nash made seminal contributions to both non-cooperative game theory and to bargaining theory.

1972–1982: John Maynard Smith applies game theory to biological problems thereby founding “evolutionary game theory.”

late 1990’s –: Development of algorithmic game theory…

Nobel prizes in economics awarded to game theorists:

1994: John Nash, John Harsanyi and Reinhard Selten: "for their pioneering analysis of equilibria in the theory of non-cooperative games”

2005: Robert Aumann and Thomas Schelling: "for having enhanced our understanding of conflict and cooperation through game-theory analysis”

2007: Leonid Hurwicz, Eric Maskin and Roger Myerson: “for having laid the foundations of mechanism design theory”

Normal-form Games

We shall focus on finite, non-cooperative, simultaneous-move games in **normal form**, which are characterized by:

- ✓ A set of **players**: $I = \{1, 2, \ldots, n\}$ ($n \geq 2$)
- ✓ A set of **pure strategy profiles**: $S = S_1 \times S_2 \times \ldots \times S_n$ where each $S_i = \{1, 2, \ldots, m_i\}$ is the (finite) set of pure strategies (actions) available to the player i
- ✓ A **payoff function**: $\pi : S \rightarrow \Re^n$, $\pi(s) = (\pi_1(s),\ldots,\pi_n(s))$, where $\pi_i(s)$ (i=1…n) represents the "payoff" (or utility) that player i receives when strategy profile s is played

Each player is to choose one element from his strategy space in the absence of knowledge of the choices of the other players, and "payments" will be made to them according to the function $\pi_i(s)$.

Players' goal is to maximize their own returns.

Two Players

In the case of two players, payoffs can be represented as two m_1 x m_2 matrices (say, A for player 1 and B *for player 2*):

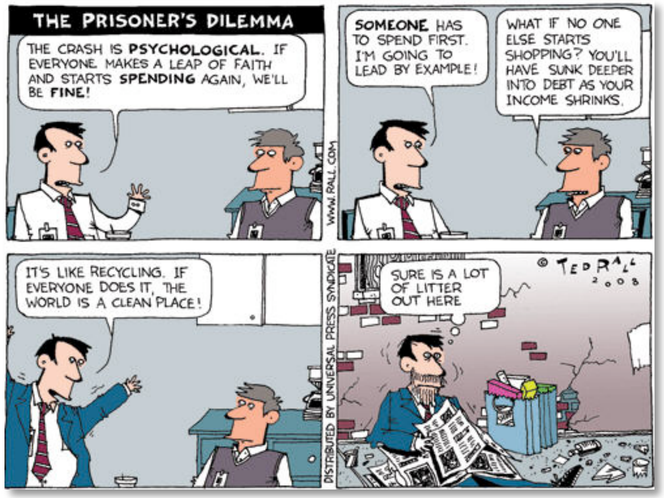
$$A = (a_{hk}) \qquad a_{hk} = \pi_1(h,k)$$

$$B = (b_{hk}) \qquad b_{hk} = \pi_2(h,k)$$

Special cases:

- ✓ Zero-sum games: $A + B = 0$ ($a_{hk} = -b_{hk}$ for all h and k)
- ✓ Symmetric games: $B^T = A$
- ✓ Doubly-symmetric games: $A = A^T = B^T$

Example 1:
Prisoner's Dilemma



		Prisoner 2	
		Confess (defect)	Deny (cooperate)
Prisoner 1	Confess (defect)	-10 , -10	-1 , -25
	Deny (cooperate)	-25 , -1	-3 , -3

Example 2:
Battle of the Sexes

		Wife	
		Soccer	Ballet
Husband	Soccer	2 , 1	0 , 0
	Ballet	0 , 0	1 , 2

Example 3:
Rock-Scissors-Paper

		You		
		Rock	Scissors	Paper
Me	Rock	0 , 0	1 , -1	-1 , 1
	Scissors	-1 , 1	0 , 0	1 , -1
	Paper	1 , -1	-1 , 1	0 , 0

Mixed Strategies

A **mixed strategy** for player i is a probability distribution over his set S_i of pure strategies, which is a point in the $(m_i$-1)-dimensional **standard simplex**:

$$\Delta_i = \left\{ x_i \in R^{m_i} \ : \ \forall h = 1 \ldots m_i : x_{ih} \geq 0, \text{ and } \sum_{h=1}^{m_i} x_{ih} = 1 \right\}$$

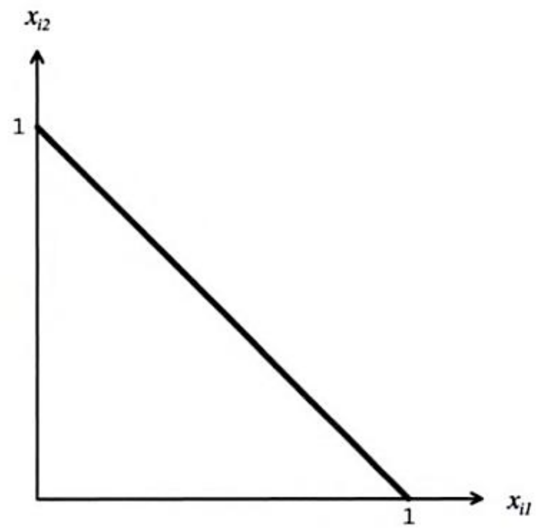
The set of pure strategies that is assigned positive probability by mixed strategy $x_i \in \Delta_i$ is called the **support** of x_i:

$$\sigma(x_i) = \{ h \in S_i : x_{ih} > 0 \}$$

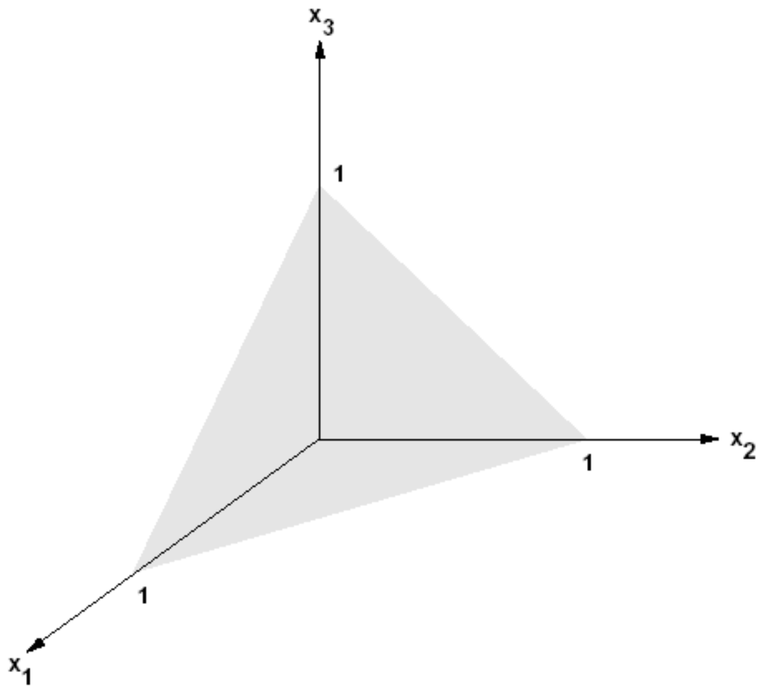
A **mixed strategy profile** is a vector $x = (x_1, \ldots, x_n)$ where each component $x_i \in \Delta_i$ is a mixed strategy for player $i \in I$.

The **mixed strategy space** is the multi-simplex $\Theta = \Delta_1 \times \Delta_2 \times \ldots \times \Delta_n$

Standard Simplices



$m_i = 2$



$m_i = 3$

Note: Corners of standard simplex correspond to pure strategies.

Mixed-Strategy Payoff Functions

In the standard approach, all players' randomizations are assumed to be independent.

Hence, the probability that a pure strategy profile $s = (s_1,\dots,s_n)$ will be used when a mixed-strategy profile x is played is:

$$x(s) = \prod_{i=1}^{n} x_{is_i}$$

and the expected value of the payoff to player i is:

$$u_i(x) = \sum_{s\in S} x_i(s)\pi_i(s)$$

In the special case of two-players games, one gets:

$$u_1(x) = \sum_{h=1}^{m_1}\sum_{k=1}^{m_2} x_{1h}a_{hk}x_{2k} = x_1^T A x_2 \qquad u_2(x) = \sum_{h=1}^{m_1}\sum_{k=1}^{m_2} x_{1h}b_{hk}x_{2k} = x_1^T B x_2$$

where A and B are the payoff matrices of players 1 and 2, respectively.

Dominance Relations

Notational shortcut.

Here, and in the sequel, if $z \in \Theta$ and $x_i \in \Delta_i$, the notation (x_i, z_{-i}) stands for the strategy profile in which player $i \in I$ plays strategy x_i, while all other players play according to z.

A strategy $y_i \in \Delta_i$ is said to **weakly dominate** another strategy $x_i \in \Delta_i$ if the first strategy never earns a lower payoff than the second (and sometimes earns a higher payoff).

Formally:

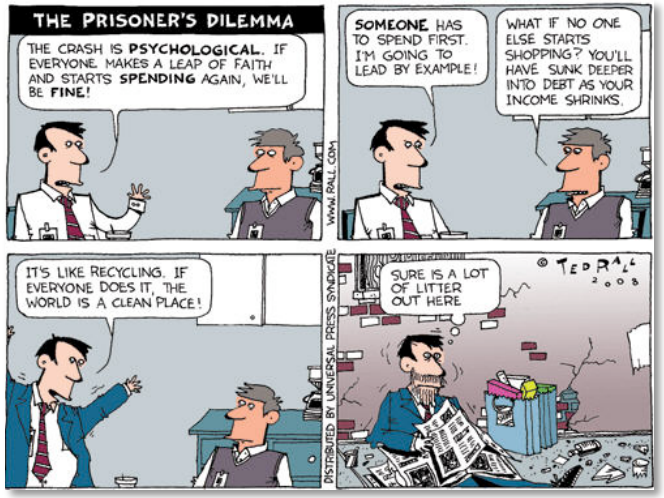
$$u_i(y_i, z_{-i}) \geq u_i(x_i, z_{-i}) \quad \text{for all } z \in \Theta$$

with strict inequality for some $z \in \Theta$.

Strategy y_i is said to **strictly dominate** x_i if:

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i}) \quad \text{for all } z \in \Theta$$

Dominance Solvability



Dominated (pure) strategy !

		Prisoner 2	
		Confess (defect)	Deny (cooperate)
Prisoner 1	Confess (defect)	-10 , -10	-1 , -25
	Deny (cooperate)	-25 , -1	-3 , -3

Dominated (pure) strategy !

Best Replies

Player i's **best reply** to the strategy profile x_{-i} is a mixed strategy $x_i^* \in \Delta_i$ such that

$$u_i(x_i^*, x_{-i}) \geq u_i(x_i, x_{-i})$$

for all strategies $x_i \in \Delta_i$.

The best reply is not necessarily unique. Indeed, except in the extreme case in which there is a unique best reply that is a pure strategy, the number of best replies is always infinite.

Indeed:

- ✓ When the support of a best reply x^* includes two or more pure strategies, any mixture of these strategies must also be a best reply

- ✓ Similarly, if there are two pure strategies that are individually best replies, any mixture of the two is necessarily also a best reply

Nash Equilibrium

The Nash equilibrium concept is motivated by the idea that a theory of rational decision-making should not be a self-destroying prophecy that creates an incentive to deviate for those who believe it.

A strategy profile $x \in \Theta$ is a **Nash equilibrium** if it is a best reply to itself, namely, if:

$$u_i(x_i, x_{-i}) \geq u_i(z_i, x_{-i})$$

for all $i = 1 \ldots n$ and all strategies $z_i \in \Delta_i$.

If strict inequalities hold for all $z_i \neq x_i$ then x is said to be a **strict Nash equilibrium**.

Theorem. A strategy profile $x \in \Theta$ is a Nash equilibrium if and only if for every player $i \in I$, every pure strategy in the support of x_i is a best reply to x_{-i}.

It follows that every pure strategy in the support of any player's equilibrium mixed strategy yields that player the same payoff.

Finding Pure-strategy Nash Equilibria

		Player 2		
		Left	Middle	Right
Player 1	Top	3 , 1	2 , 3	10 , 2
	High	4 , 5	3 , 0	[illegible] , 4
	Low	2 , 2	5 , 4	12 , 3
	Bottom	5 , 6	4 , 5	9 , 7

Nash equilibrium!

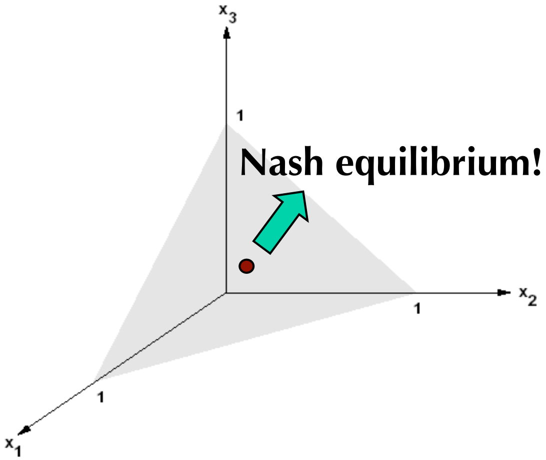
Multiple Equilibria in Pure Strategies

Nash equilibrium!

Nash equilibrium!

		Wife	
		Soccer	Ballet
Husband	Soccer	2 , 1	0 , 0
	Ballet	0 , 0	1 , 2

No Equilibrium in Pure Strategies



ium!

No Nas

		You		
		Rock	Scissors	Paper
Me	Rock	0 , 0	1 , -1	-1 , 1
	Scissors	-1 , 1	0 , 0	1 , -1
	Paper	1 , -1	-1 , 1	0 , 0

Existence of Nash Equilibria

Theorem (Nash, 1951). Every finite normal-form game admits a mixed-strategy Nash equilibrium.

Idea of proof.

1. Define a continuous map T on Θ such that the fixed points of T are in one-to-one correspondence with Nash equilibria.
2. Use Brouwer's theorem to prove existence of a fixed point.

Note. For symmetric games, Nash proved that there always exists a **symmetric Nash equilibrium**, namely a Nash equilibrium where all players play the same (possibly mixed) strategy.

Finding Nash Equilibria: Two-Player Zero-Sum Games

"As far as I can see, there could be no theory of games […] without that theorem […]
I thought there was nothing worth publishing until the Minimax Theorem was proved"

John von Neumann (1928)

Minimax Theorem (von Neumann, 1928). In any finite, two-player, zero-sum game (with payoff matrices *A* and –*A*) in any Nash equilibrium each player receives a payoff that is equal to both its maximim and his minimax value:

$$\max_x \min_y x^T Ay = \min_y \max_x x^T Ay \ .$$

Solved by standard linear programming methods!

The Complexity of Finding Nash Equilibria

"Together with factoring, *the complexity of finding a Nash equilibrium is in my opinion the most important concrete open question on the boundary of P today.*"

Christos Papadimitriou

Algorithms, games, and the internet (2001)

At present, no known reduction exists from our problem to a decision problem that is NP-complete, nor has our problem been shown to be easier.

An intuitive stumbling block is that *every* game has at least one Nash equilibrium, whereas known *NP*-complete problems are expressible in terms of decision problems that do not always have solutions.

Variations on Theme

Theorem (Gilboa and Zemel, 1989). The following are *NP*-complete problems, even for symmetric games.

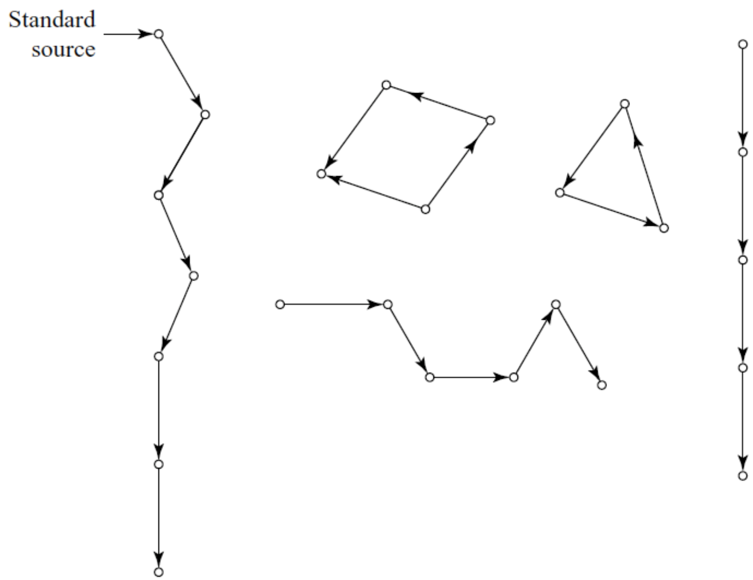
Given a two-player game in normal form, does it have:

1. at least two Nash equilibria?
2. a Nash equilibrium in which player 1 has payoff at least a given amount?
3. a Nash equilibrium in which the two players have a total payoff at least a given amount?
4. a Nash equilibrium with support of size greater than a give number?
5. a Nash equilibrium whose support contains a given strategy?
6. a Nash equilibrium whose support does not contain a given strategy?
7. etc.

The Complexity Class PPAD
(Polynomial Parity Argument on Directed graphs)

Consider the class of (directed) graphs characterized by:

1. A finite but exponentially large set of vertices
2. Each vertex has indegree and outdegree at most one
3. Given a string, it is computationaly simple to *(a)* tell if it is a vertex of the graph, and if so to *(b)* find its neighbors, and to *(c)* tell which one is the predecessor and/or which one is the successor
4. There is one known source (the "standard source")



PPAD is the class of all problems whose solution space can be set up as the set of all sinks and all nonstandard sources in a directed graph with the properties displayed above.

NASH is PPAD-complete

Theorem (Daskalakis *et al*. 2005; Chen and Deng, 2005, 2006). The problem of finding a sample Nash equilibrium of a general-sum finite game with two or more players is PPAD-complete.

As of today, PPAD includes some 25 problems from such different areas as:

- ✓ Fixed point analysis
- ✓ Economics and market equilibria
- ✓ Topology
- ✓ Combinatorics and graph theory
- ✓ Social networks

See: http://www.cc.gatech.edu/~kintali/ppad.html

It is not known whether or not P = PPAD. However, it is generally believed that the two classes are not equivalent.

The common belief is that, in the worst case, computing a sample Nash equilibrium will take time that is exponential in the size of the game.

Algorithms for Finding a Nash Equilibrium

1. **Linear-complementarity approach** (Lemke and Howson, 1964)
   Only two players; uses pivoting; it may take exponentially many steps.

2. **Simplicial subdivision** (van der Laan *et al.*, 1987)
   Approximate a fixed point of a function by triangulating the multi-simplex and traversing the triangulation along a fixed path.

3. **Continuation methods** (Govindan and Wilson, 2003)
   Perturb a game to one that has a known equilibrium, and then trace the solution back to the original game.

4. **Enumeration of support** (Mangasarian, 1964; Dickaut and Kaplan, 1991; Porter, E. Nudelman and Y. Shoham, 2008)
   Enumerate all possible supports and solve a feasibity program; equipped with simple heuristics, outperform more sophisticated algorithms.

5. **Evolutionary dynamics** (Taylor and Jonker, 1978; Rota Bulò and Bomze, 2010)
   Inspired by population-based interpretations of game theory.

References

Texts on (classical) game theory

J. von Neumann and O. Morgerstern. *Theory of Games and Economic Behavior*. Princeton University Press (1944, 1953).

D. Fudenberg and J. Tirole. *Game Theory*. MIT Press (1991).

R. Myerson. *Game Theory: Analysis of Conflict*. Harvard University Press (1991).

M. J. Osborne and A. Rubinstein. *A Course in Game Theory*. MIT Press (1994).

Computationally-oriented texts

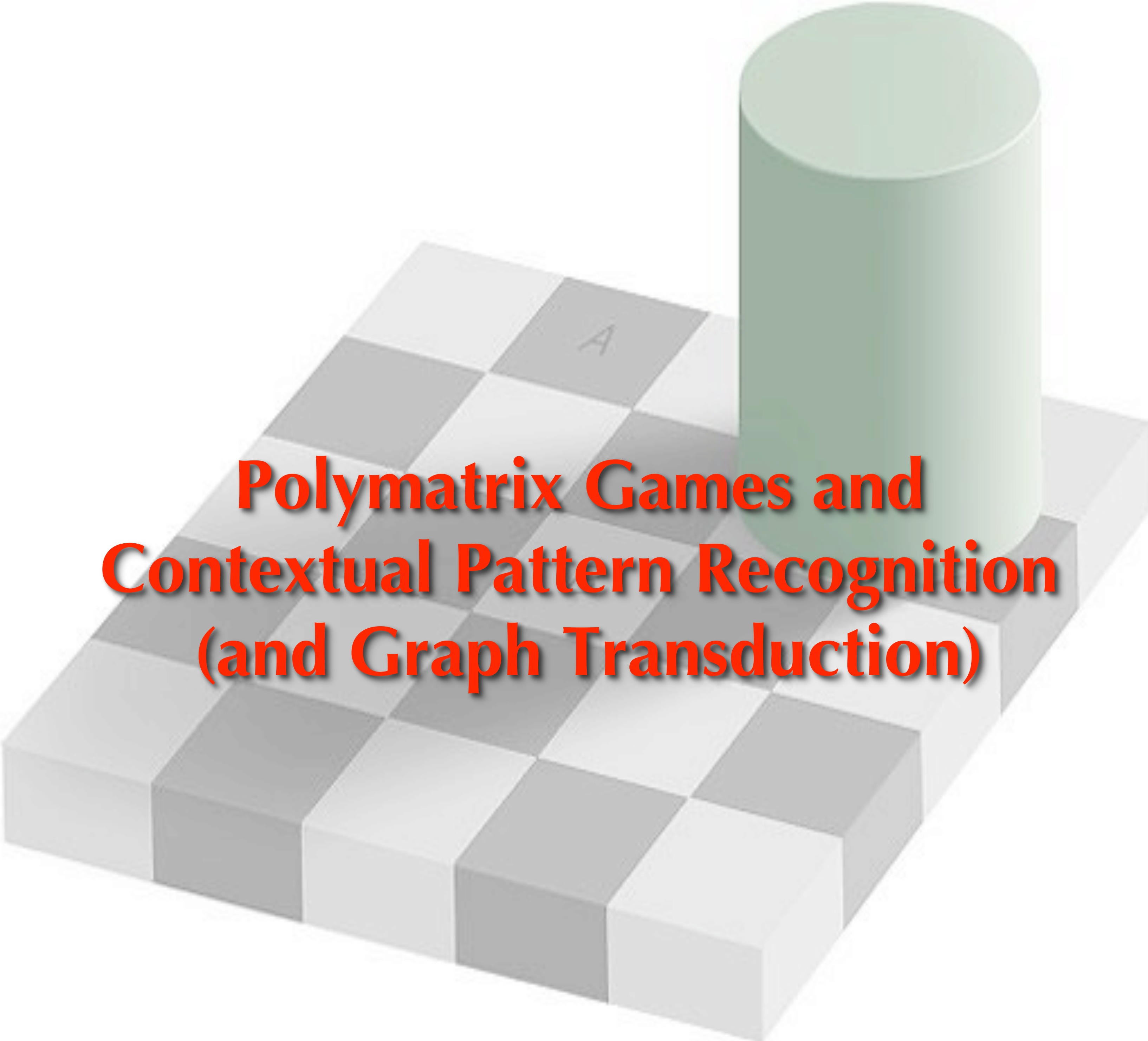
N. Nisan, T. Roughgarden, E. Tardos, and V. Vazirani (Eds.) *Algorithmic Game Theory*. Cambridge University Press (2007).

Y. Shoham and K. Leyton-Brown. *Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations*. Cambridge University Press (2009).

On-line resources

http://gambit.sourceforge.net/	a library of game-theoretic algorithms
http://gamut.stanford.edu/	a suite of game generators for testing game algorithms

Polymatrix Games and Contextual Pattern Recognition (and Graph Transduction)



Succinct Games

Describing a game in normal form entails listing all payoffs for all players and strategy combinations. In a game with n players, each facing m pure strategies, one need to store nm^n numbers!

A **succinct game** (or a **succinctly representable game**) is a game which may be represented in a size much smaller than its normal-form representation.

Examples.

Graphical games. The payoffs of each player depends on the actions of very few (at most d) other players. The number of payoffs needed to describe this game is nm^{d+1}.

Sparse games. Most of the payoffs are zero (special case of graphical games).

Symmetric games. All players are identical, so in evaluating the payoff of a combination of strategies, all that matters is how many of the n players play each of the s strategies.

Polymatrix Games

A **polymatrix game** (a.k.a. **multimatrix game**) is a non-cooperative game in which the relative influence of the selection of a pure strategy by any one player on the payoff to any other player is always the same, regardless of what the rest of the players do.

Formally:

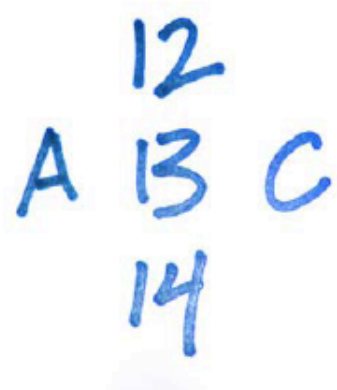
- ✓ There are n players each of whom can use m pure strategies
- ✓ For each pair (i,j) of players there is an m x m payoff matrix A^{ij}
- ✓ The payoff of player i for the strategy combination $s_1,\ldots,s_n$ is given by

$$u_i(s_1,\cdots,s_n) = \sum_{j \neq i} A^{ij}_{s_i s_j}$$

The number of payoff values required to represent such a game is $O(n^2m^2)$.

The problem of finding a Nash equilibrium in a polymatrix game is PPAD-complete.

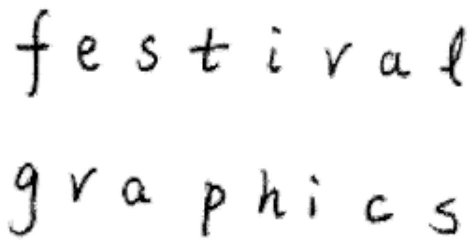
Context helps …



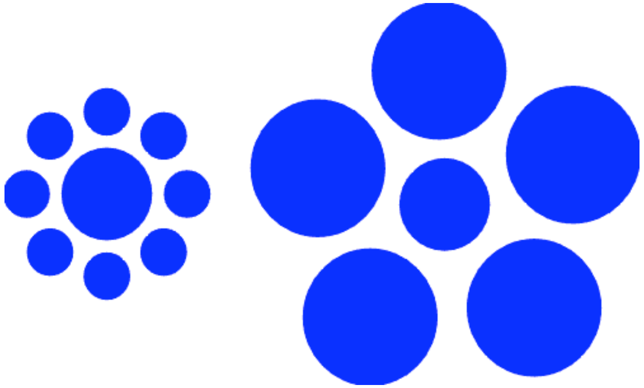
c → cat
→ circus

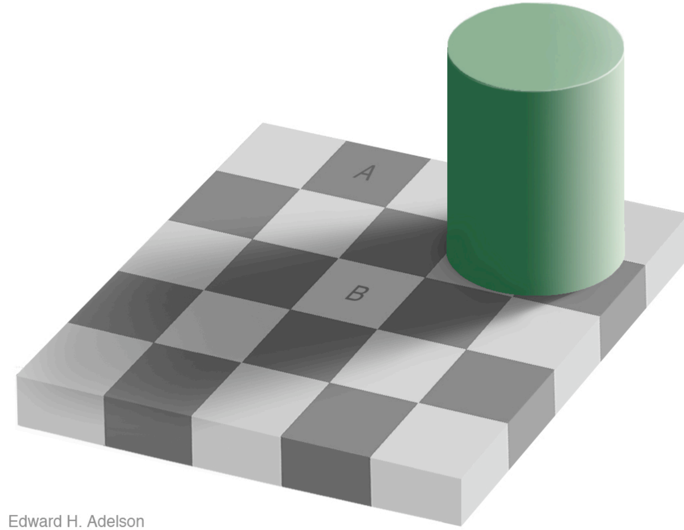
i → sin
→ fine

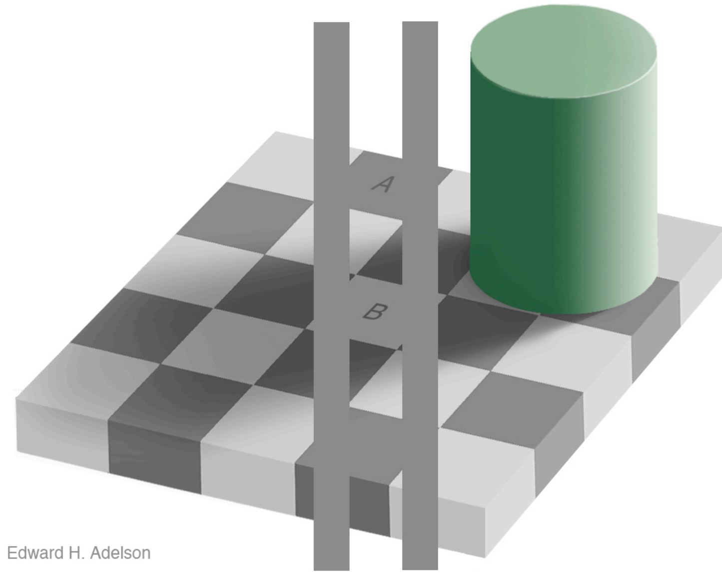
e → red
→ read



… but can also deceive!







What do you see?

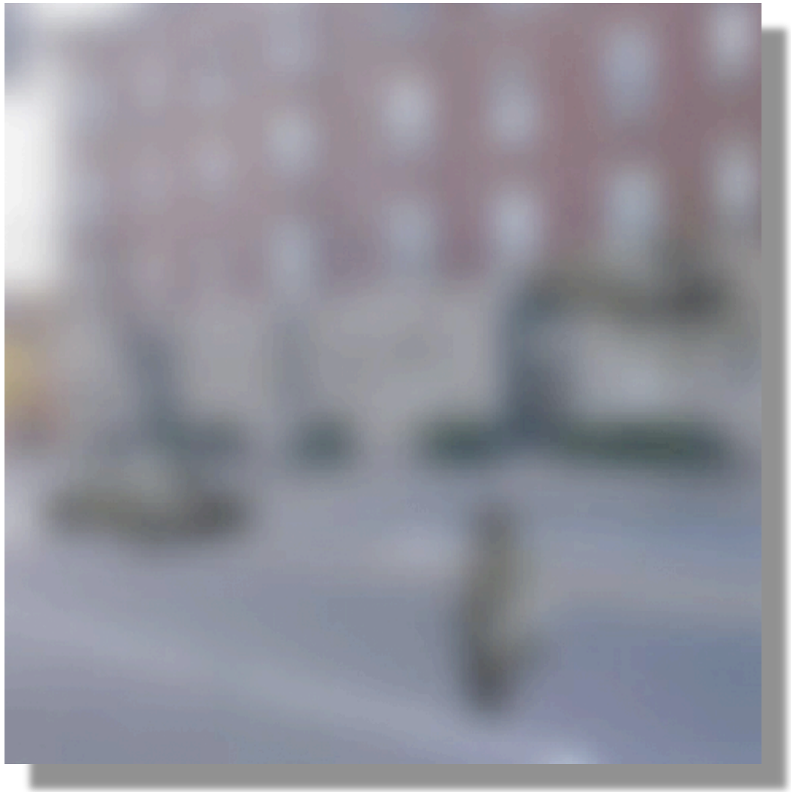
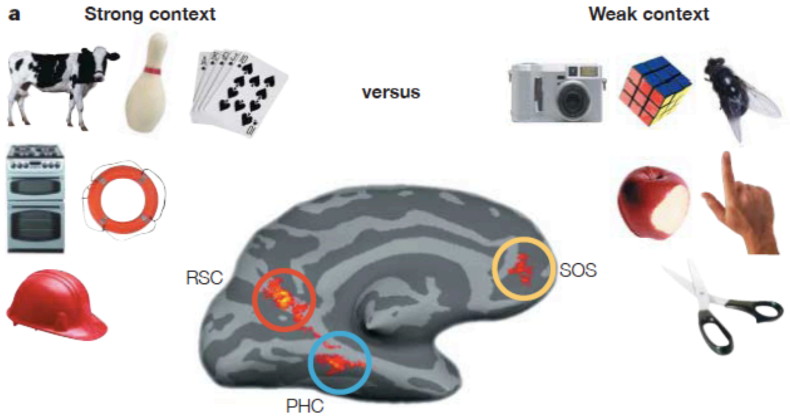


Figure 2. The strength of context. The visual system makes assumptions regarding object identities according to their size and location in the scene. In this picture, observers describe the scene as containing a car and pedestrian in the street. However, the pedestrian is in fact the same shape as the car, except for a 90° rotation. The atypicality of this orientation for a car within the context defined by the street scene causes the car to be recognized as a pedestrian.

From: A. Oliva and A. Torralba, “The role of context in object recognition”, *Trends in Cognitive Sciences*, 2007.

Context and the Brain



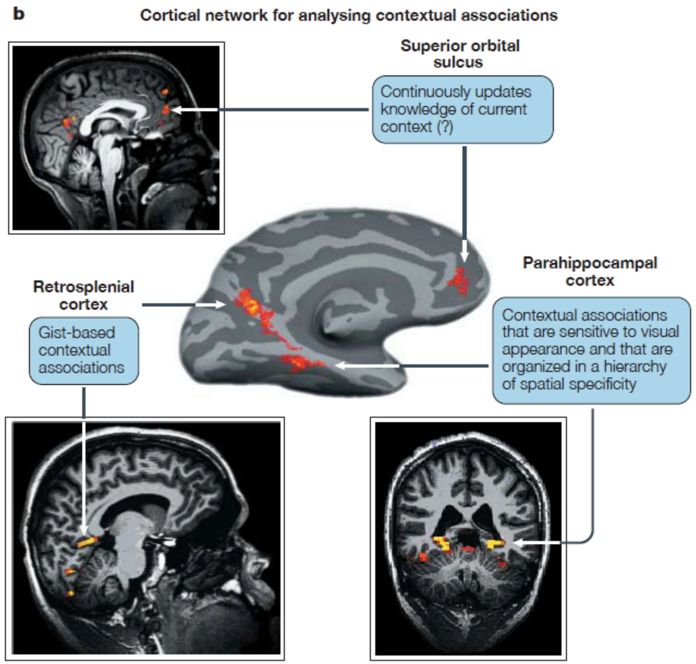


Figure 3 | **Cortical areas involved in processing context. a** | A functional magnetic resonance imaging (fMRI) statistical activation map representing the difference between perceiving objects that are strongly associated with a specific context and perceiving objects that are not associated with a unique context. This is a medial view of the left hemisphere, shown using a precise computer reconstruction where the sulci have been exposed by 'inflation'. The parahippocampal cortex (PHC) is circled in blue; the retrosplenial cortex (RSC) is circled in red; the superior orbital sulcus (SOS) is circled in yellow. Note that in all experimental conditions, subjects viewed similar looking colour photographs of meaningful, everyday common objects that were equally recognizable. Consequently, activation due to low-level processes was presumably subtracted out, and the differential activation map shown here represents only processes that are related to the level of contextual association. **b** | The cortical network for contextual associations among visual objects, suggested on the basis of existing evidence. Other types of context might involve additional regions (for example, hippocampus for navigation[125] and Broca's area for language-related context). Modified, with permission, from REF. 12 © (2003) Elsevier Science.

From: M. Bar, "Visual objects in context", *Nature Reviews Neuroscience*, August 2004.

The (Consistent) Labeling Problem

A **labeling problem** involves:

- ✓ A set of n **objects** $B = \{b_1,\ldots,b_n\}$
- ✓ A set of m **labels** $\Lambda = \{1,\ldots,m\}$

The goal is to label each object of B with a label of Λ.

To this end, two sources of information are exploited:

- ✓ Local measurements which capture the salient features of each object viewed in isolation
- ✓ Contextual information, expressed in terms of a real-valued $n^2 \times m^2$ matrix of **compatibility coefficients** $R = \{r_{ij}(\lambda, \mu)\}$.

The coefficient $r_{ij}(\lambda, \mu)$ measures the strenght of compatibility between the two hypotheses: "b_i is labeled λ" and "b_j is labeled μ".

Relaxation Labeling Processes

The initial local measurements are assumed to provide, for each object $b_i \in B$, an m-dimensional (probability) vector:

$$p_i^{(0)} = \left(p_i^{(0)}(1), \cdots, p_i^{(0)}(m)\right)^T$$

with $p_i^{(0)}(\lambda) \geq 0$ and $\sum_\lambda p_i^{(0)}(\lambda) = 1$. Each $p_i^{(0)}(\lambda)$ represents the initial, non-contextual degree of confidence in the hypothesis "b_i is labeled λ".

By concatenating vectors $p_1^{(0)}, \ldots, p_n^{(0)}$ one obtains an (initial) **weighted labeling assignment** $p^{(0)} \in \Re^{nm}$.

The space of weighted labeling assignments is

$$\mathrm{IK} = \underbrace{\Delta \times \ldots \times \Delta}_{m \text{ times}}$$

where each Δ is the standard simplex of $\Re^n$. Vertices of IK represent unambiguous labeling assignments

A relaxation labeling process takes the initial labeling assignment $p^{(0)}$ as input and iteratively updates it taking into account the compatibility model R.

Relaxation Labeling Processes

In a now classic 1976 paper, Rosenfeld, Hummel, and Zucker introduced heuristically the following update rule (assuming a non-negative compatibility matrix):

$$p_i^{(t+1)}(\lambda) = \frac{p_i^{(t)}(\lambda) q_i^{(t)}(\lambda)}{\sum_\mu p_i^{(t)}(\mu) q_i^{(t)}(\mu)}$$

where

$$q_i^{(t)}(\lambda) = \sum_j \sum_\mu r_{ij}(\lambda,\mu) p_i^{(t)}(\mu)$$

quantifies the support that context gives at time t to the hypothesis "b_i is labeled with label λ".

See (Pelillo, 1997) for a rigorous derivation of this rule in the context of a formal theory of consistency.

Applications

Since their introduction in the mid-1970's relaxation labeling algorithms have found applications in virtually all problems in computer vision and pattern recognition:

- ✓ Edge and curve detection and enhancement
- ✓ Region-based segmentation
- ✓ Stereo matching
- ✓ Shape and object recognition
- ✓ Grouping and perceptual organization
- ✓ Graph matching
- ✓ Handwriting interpretation
- ✓ …

Further, intriguing similarities exist between relaxation labeling processes and certain mechanisms in the early stages of biological visual systems (see Zucker, Dobbins and Iverson, 1989, for physiological and anatomical evidence).

Hummel and Zucker's Consistency

In 1983, Bob Hummel and Steve Zucker developed an elegant theory of consistency in labeling problem.

By analogy with the unambiguous case, which is easily understood, they define a weighted labeling assignment $p \in \mathbb{K}$ **consistent** if:

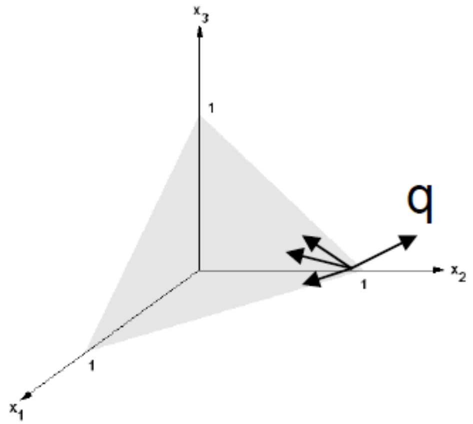
$$\sum_{\lambda} p_i(\lambda) q_i(\lambda) \geq \sum_{\lambda} v_i(\lambda) q_i(\lambda) \qquad i = 1 \ldots n$$

for all labeling assignments $v \in \mathbb{K}$.

If strict inequalities hold for all $v \neq p$, then p is said to be **strictly consistent**.

Geometrical interpretation.

The support vector q points away from all tangent vectors at p (it has null projection in $\mathbb{K}$).



Characterizations

Theorem (Hummel and Zucker, 1983). A labeling $p \in$ IK is consistent if and only if, for all i = 1…n, the following conditions hold:

1. $q_i(\lambda) = c_i$ whenever $p_i(\lambda) > 0$
2. $q_i(\lambda) \leq c_i$ whenever $p_i(\lambda) = 0$

for some constants $c_1 \ldots c_n$.

The "average local consistency" of a labeling $p \in$ IK is defined as:

$$A(p) = \sum_i \sum_\lambda p_i(\lambda) q_i(\lambda)$$

Theorem (Hummel and Zucker, 1983). If the compatibility matrix R is symmetric, i.e., $r_{ij}(\lambda, \mu) = r_{ji}(\mu, \lambda)$, then any local maximizer $p \in$ IK of A is consistent.

Understanding the "1976-rule"

Using the Baum-Eagon inequality it is easy to prove the following result, concerning the original Rosenfeld-Hummel-Zucker (RHZ) update rule.

Theorem (Pelillo, 1997). The RHZ relaxation operator is a "growth transformation" for the average local consistency A, provided that compatibility coefficients are symmetric. In other words, the algorithm strictly increases the average local consistency on each iteration, i.e.,

$$A(p^{(t+1)}) > A(p^{(t)})$$

for $t = 0,1,\ldots$ until a fixed point is reached.

Theorem (Elfving and Eklundh, 1982; Pelillo, 1997). Let $p \in \mathbb{K}$ be a strictly consistent labeling. Then p is an asymptotically stable equilibrium point for the RHZ relaxation scheme, whether or not the compatibility matrix is symmetric.

Relaxation Labeling and Polymatrix Games

As observed by Miller and Zucker (1991) the consistent labeling problem is equivalent to a polymatrix game.

Indeed, in such formulation we have:

- ✓ Objects = players
- ✓ Labels = pure strategies
- ✓ Weighted labeling assignments = mixed strategies
- ✓ Compatibility coefficients = payoffs

and:

- ✓ Consistent labeling = Nash equilibrium
- ✓ Strictly consistent labeling = strict Nash equilibrium

Further, the RHZ update rule corresponds to discrete-time multi-population "replicator dynamics" used in evolutionary game theory (see next part).

Graph Transduction

Given a set of data points grouped into:

- ✓ labeled data: $\{(\mathbf{x}_1, y_1), \ldots, \mathbf{x}_\ell, y_\ell)\}$
- ✓ unlabeled data: $\{\mathbf{x}_{\ell+1}, \ldots, \mathbf{x}_n\}$ $\qquad \ell \ll n$

Express data as a graph *G=(V,E)*

- ✓ *V* : nodes representing labeled and unlabeled points
- ✓ *E* : pairwise edges between nodes weighted by the similarity between the corresponding pairs of points

Goal: Propagate the information available at the labeled nodes to unlabeled ones in a "consistent" way.

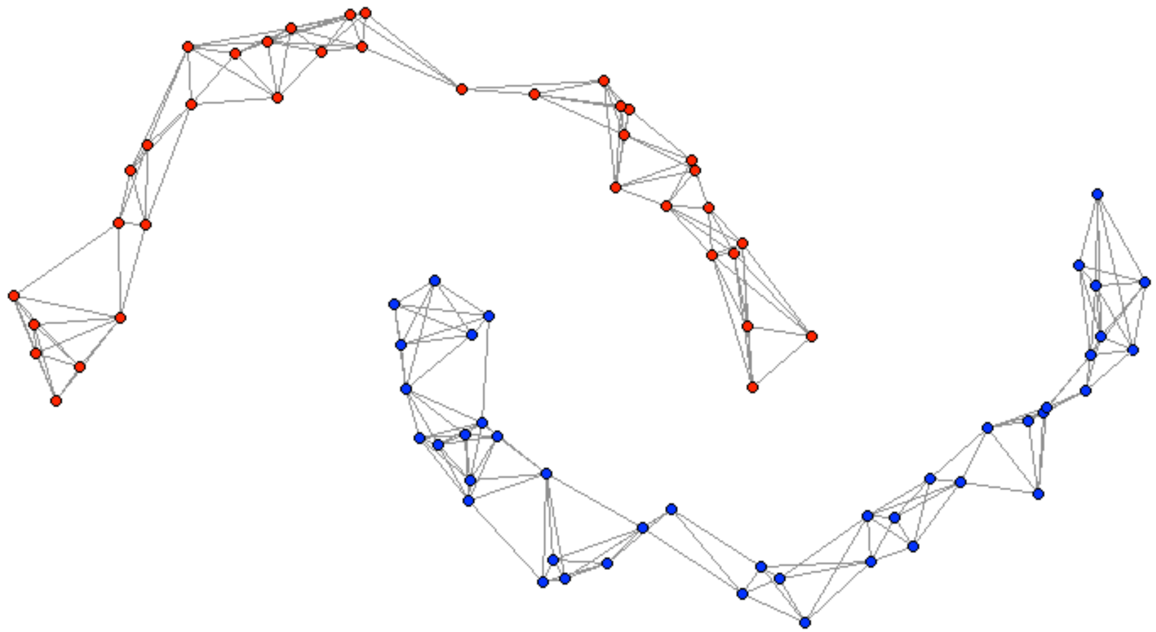
Cluster assumption:

- ✓ The data form distinct clusters
- ✓ Two points in the same cluster are expected to be in the same class

A Special Case:
Unweighted Undirected Graphs

A simple case of graph transduction in which the graph G is an unweighted undirected graph:

- ✓ An edge denotes perfect similarity between points
- ✓ The adjacency matrix of G is a 0/1 matrix



The cluster assumption: Each node in a connected component of the graph should have the same class label.

A Special Case: Unweighted Undirected Graphs

This toy problem can be formulated as a (binary) **constraint satisfaction problem** (CSP) as follows:

- ✓ The set of variables: $V = \{v_1, \ldots, v_n\}$
- ✓ Domains: $D_{v_i} = \begin{cases} \{y_i\} & \text{for all } 1 \le i \le l \\ Y & \text{for all } l+1 \le i \le n \end{cases}$
- ✓ Binary constraints: $\forall\, i,j$: if $a_{ij} = 1$, then $v_i = v_j$
  e.g. for a 2-class problem $R_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Each assignment of values to the variables satisfying all the constraints is a solution of the CSP, thereby providing a consistent labeling for the unlabeled points.

Goal: Generalize to real-valued (soft) constraints

Idea: Use consistency criterion of relaxation labeling (= Nash equilibrium)

The Graph Transduction Game

Assume:

- ✓ the players participating in the game correspond to the vertices of the graph
- ✓ the set of strategies available to each player denote the possible hypotheses about its class membership

- labeled players $\mathcal{I}_\ell = \{\mathcal{I}_{\ell|1}, \ldots, \mathcal{I}_{\ell|c}\}$
- unlabeled players $\mathcal{I}_u$

Labeled players choose their strategies at the outset.

- each player $i \in \mathcal{I}_{l|k}$ always play its k^{th} pure strategy.

The transduction game is in fact played among the unlabeled players to choose their memberships according.

By assuming that only pairwise interactions are allowed, we obtain a polymatrix game that can be solved used standard relaxation labeling / replicator algorithms.

In short…

Graph transduction can be formulated as a (polymatrix) non-cooperative game (i.e., a consistent labeling problem).

The proposed game-theoretic framework can cope with **symmetric, negative and asymmetric similarities** (none of the existing techniques is able to deal with all three types of similarities).

Experimental results on standard datasets show that our approach is not only more general but also competitive with standard approaches.

A. Erdem and M. Pelillo. Graph transduction as a non-cooperative game. *GbR 2011* (journal version under review).

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A. Rosenfeld, R. A. Hummel, and S. W. Zucker. Scene labeling by relaxation operations. *IEEE Trans. Syst. Man. Cybern* (1976).

R. A. Hummel and S. W. Zucker. On the foundations of relaxation labeling processes. *IEEE Trans. Pattern Anal. Machine Intell.* (1983).

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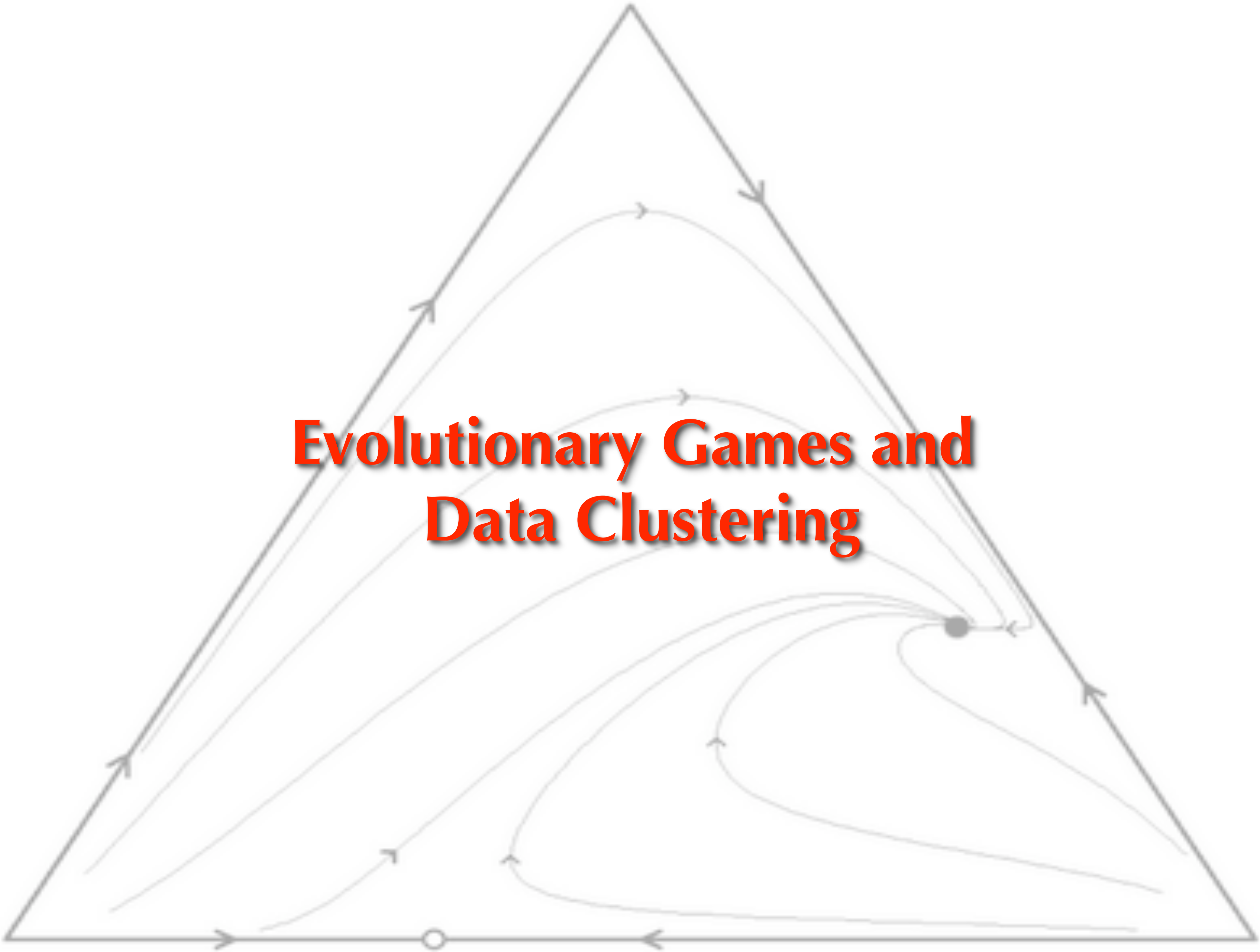
Evolutionary Games and Data Clustering



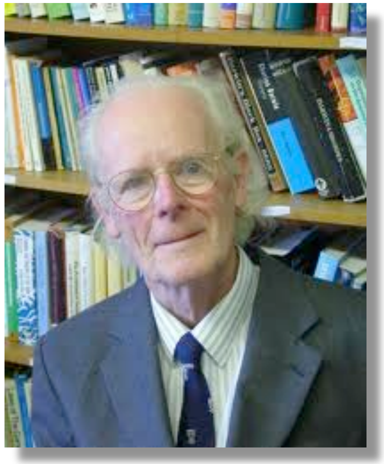
Evolution and the Theory of Games

"We repeat most emphatically that our theory is thoroughly static. A dynamic theory would unquestionably be more complete and therefore preferable.

But there is ample evidence from other branches of science that it is futile to try to build one as long as the static side is not thoroughly understood."

John von Neumann and Oskar Morgenstern

Theory of Games and Economic Behavior (1944)



"Paradoxically, it has turned out that game theory is more readily applied to biology than to the field of economic behaviour for which it was originally designed."

John Maynard Smith

Evolution and the Theory of Games (1982)

Evolutionary Games

Introduced by John Maynard Smith (1973, 1974, 1982) to model the evolution of behavior in animal conflicts.

Assumptions:

- ✓ A large population of individuals belonging to the same species which compete for a particular limited resource
- ✓ This kind of conflict is modeled as a symmetric two-player game, the players being pairs of randomly selected population members
- ✓ Players do not behave "rationally" but act according to a pre-programmed behavioral pattern (pure strategy)
- ✓ Reproduction is assumed to be asexual
- ✓ Utility is measured in terms of Darwinian fitness, or reproductive success

Evolutionary Stability

A strategy is **evolutionary stable** if it is resistant to invasion by new strategies.

Formally, assume:

- ✓ A small group of "invaders" appears in a large populations of individuals, all of whom are pre-programmed to play strategy $x \in \Delta$
- ✓ Let $y \in \Delta$ be the strategy played by the invaders
- ✓ Let ε be the share of invaders in the (post-entry) population ($0 < \varepsilon < 1$)

The payoff in a match in this bimorphic population is the same as in a match with an individual playing mixed strategy:

$$w = \varepsilon y + (1 - \varepsilon)x \in \Delta$$

hence, the (post-entry) payoffs got by the incumbent and the mutant strategies are $u(x,w)$ and $u(y,w)$, respectively.

Evolutionary Stable Strategies

Definition. A strategy $x \in \Delta$ is said to be an **evolutionary stable strategy** (ESS) if for all $y \in \Delta - \{x\}$ there exists $\delta \in (0,1)$, such that for all $\varepsilon \in (0, \delta)$ we have:

$$u[x, \varepsilon y + (1-\varepsilon)x] > u[y, \varepsilon y + (1-\varepsilon)x]$$

Theorem. A strategy $x \in \Delta$ is an ESS if and only if it meets the following first- and second-order best-reply conditions:

1. $u(y,x) \leq u(x,x)$ for all $y \in \Delta$
2. $u(y,x) = u(x,x) \Rightarrow u(y,y) < u(x,y)$ for all $y \in \Delta - \{x\}$

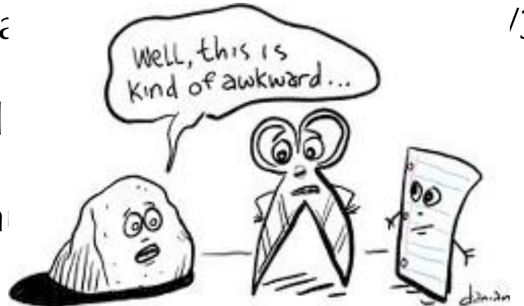
Note. From the conditions above, we have:

- ✓ $\Delta^{ESS} \subseteq \Delta^{NE}$
- ✓ If $x \in \Delta$ is a strict Nash equilibrium, then x is an ESS

Existence of ESS's

Unlike Nash equilibria existence of ESS's is not guaranteed.

- ✓ Unique Na /3,1/3)^T
- ✓ Hence, all
- ✓ Let the "m
- ✓ But $u(y,y) = u(x,y)$, hence $\Delta^{ESS} = \emptyset$



		You		
		Rock	Scissors	Paper
Me	Rock	0 , 0	1 , -1	-1 , 1
	Scissors	-1 , 1	0 , 0	1 , -1
	Paper	1 , -1	-1 , 1	0 , 0

Complexity Issues

Two questions of computational complexity naturally present themselves:

- ✓ What is the complexity of determining whether a given game has an ESS (and of finding one)?
- ✓ What is the complexity of recognizing whether a given x is an ESS for a given game?

Theorem (Etessami and Lochbihler, 2004). Determining whether a given two-player symmetric game has an ESS is both NP-hard and coNP-hard.

Theorem (Nisan, 2006). Determining whether a (mixed) strategy x is an ESS of a given two-player symmetric game is coNP-hard.

Replicator Dynamics

Let $x_i(t)$ the population share playing pure strategy i at *time* t. The **state** of the population at time t is: $x(t)= (x_1(t),\ldots,x_n(t))\in\Delta$.

Replicator dynamics (Taylor and Jonker, 1978) are motivated by Darwin's principle of natural selection:

$$\frac{\dot{x}_i}{x_i} \propto \text{payoff of pure strategy i} - \text{average population payoff}$$

which yields:

$$\begin{aligned}\dot{x}_i &= x_i\left[u(e^i,x) - u(x,x)\right] \\ &= x_i\left[(Ax)_i - x^T Ax\right]\end{aligned}$$

Notes.

- ✓ Invariant under positive affine transformations of payoffs (i.e., $u \leftarrow \alpha u+\beta$, with $\alpha>0$)
- ✓ Standard simplex Δ is invariant under replicator dynamics, namely, $x(0)\in\Delta \Rightarrow x(t)\in\Delta$, for all $t > 0$ (so is its interior and boundary)

Replicator Dynamics and Dominated Strategies

Theorem (Akin, 1980). Let i be a strictly dominated pure strategy of a two-player symmetric game.

If $x \in \Delta$ is the limit point of a replicator dynamics trajectory starting from int(Δ), then $x_i = 0$.

Theorem (Samuelson and Zhang, 1992). Let i be an iteratively strictly dominated pure strategy of a two-player symmetric game.

If $x \in \Delta$ is the limit point of a replicator dynamics trajectory starting from int(Δ), then $x_i = 0$.

Replicator Dynamics and Nash Equilibria

Proposition. If $x \in \Delta$ is a Nash equilibrium, then x is an equilibrium point of replicator dynamics.

Note. The contrary need not be true (e.g., all corners of Δ are equilibria).

Theorem (Bomze, 1986). If $x \in \Delta$ is (Lyapunov) stable under replicator dynamics, then x is a Nash equilibrium.

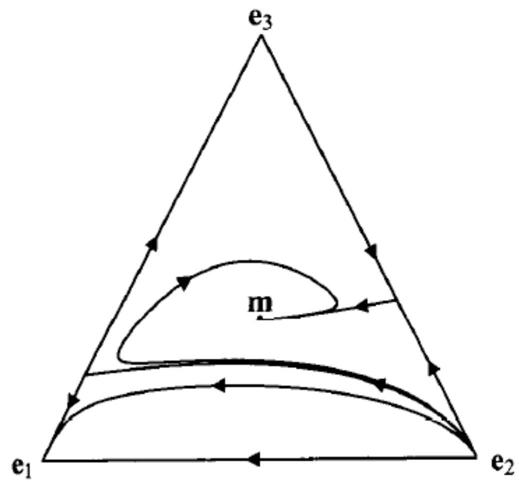
Theorem (Nachbar, 1990). A point $x \in \Delta$ is a Nash equilibrium if and only if x is the limit point of a replicator dynamics trajectory starting from int(Δ).

Replicator Dynamics and ESS's

Theorem (Taylor and Jonker, 1978). If $x \in \Delta$ is an ESS, then it is an asymptotically stable equilibrium point for the replicator dynamics.

The opposite need not be true.

$$A = \begin{bmatrix} 0 & 6 & -4 \\ -3 & 0 & 5 \\ -1 & 3 & 0 \end{bmatrix}$$



- ✓ The point $m=(1/3,1/3,1/3)^T$ is asymptotically stable (its eigenvalues have negative parts).
- ✓ But $e^1=(1,0,0)^T$ is an ESS.
- ✓ Hence m cannot be an ESS (being in the interior, it would have to be the unique ESS).

Proposition. If $x \in \text{int}(\Delta)$ is an ESS, then all replicator trajectories starting from $\text{int}(\Delta)$ will eventually converge to x.

In summary …

Let $x \in \Delta$ be a mixed strategy of a two-player symmetric game.

Then:

- ✓ x ESS $\Rightarrow$ x asymptotically stable
- ✓ x asymptotically stable $\Rightarrow$ x Nash
- ✓ x Nash $\Rightarrow$ x stationary

More formally:

$$\Delta^{ESS} \subseteq \Delta^{Asymp.\ Stable} \subseteq \Delta^{NE} \subseteq \Delta^{Stationary}$$

Doubly Symmetric Games

In a doubly symmetric (or partnership) game, the payoff matrix A is symmetric ($A = A^{\mathrm{T}}$).

Fundamental Theorem of Natural Selection (Losert and Akin, 1983).

For any doubly symmetric game, the average population payoff $f(x) = x^{\mathrm{T}}Ax$ is strictly increasing along any non-constant trajectory of replicator dynamics, namely, $d/dt f(x(t)) \geq 0$ for all $t \geq 0$, with equality if and only if $x(t)$ is a stationary point.

Characterization of ESS's (Hofbauer and Sigmund, 1988)

For any doubly simmetric game with payoff matrix A, the following statements are equivalent:

a) $x \in \Delta^{ESS}$

b) $x \in \Delta$ is a strict local maximizer of $f(x) = x^{\mathrm{T}}Ax$ over the standard simplex Δ

c) $x \in \Delta$ is asymptotically stable in the replicator dynamics

Discrete-time Replicator Dynamics

A well-known discretization of replicator dynamics, which assumes non-overlapping generations, is the following (assuming a non-negative A):

$$x_i(t+1) = x_i(t)\frac{A\big(x(t)\big)_i}{x(t)^T Ax(t)}$$

which inherits most of the dynamical properties of its continuous-time counterpart (*e.g.*, the fundamental theorem of natural selection).

MATLAB implementation

```
distance=inf;
while distance>epsilon
    old_x=x;
    x = x.*(A*x);
    x = x./sum(x);
    distance=pdist([x,old_x]');
end
```

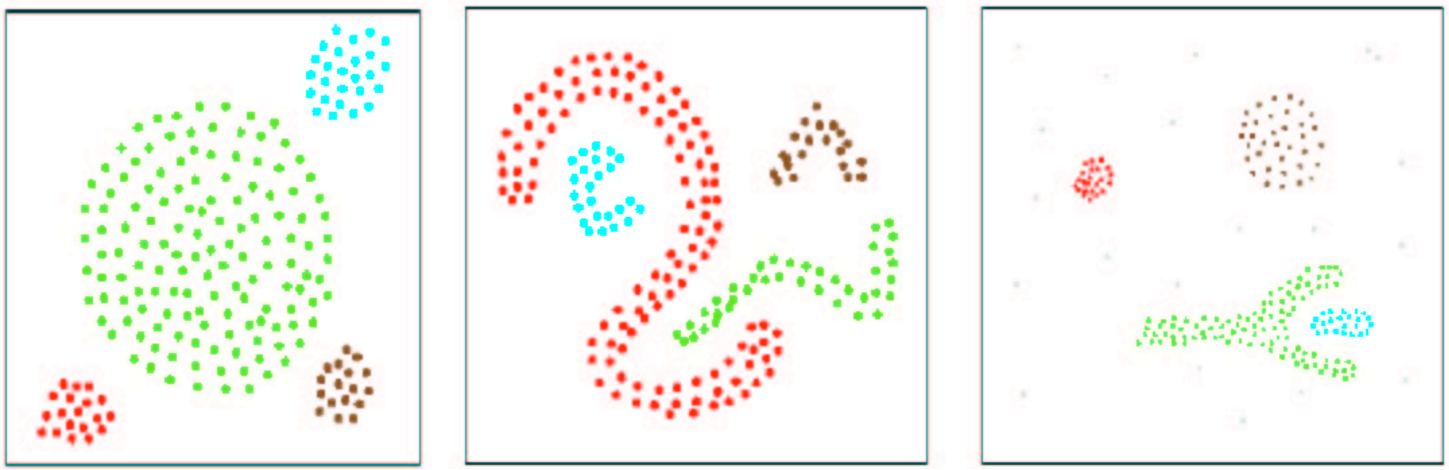
The "Classical" (Pairwise) Clustering Problem

Given:

- a set of n "objects"
- an $n \times n$ matrix of pairwise (dis)similarities

Goal: <u>*Partition*</u> the input objects into maximally homogeneous groups (a.k.a. clusters).

For a review see, e.g., A. K. Jain, "Data clustering: 50 years beyond K-means," *Pattern Recognition Letters* 31(8):651-666, 2010.



The Need for Non-exhaustive Clusterings

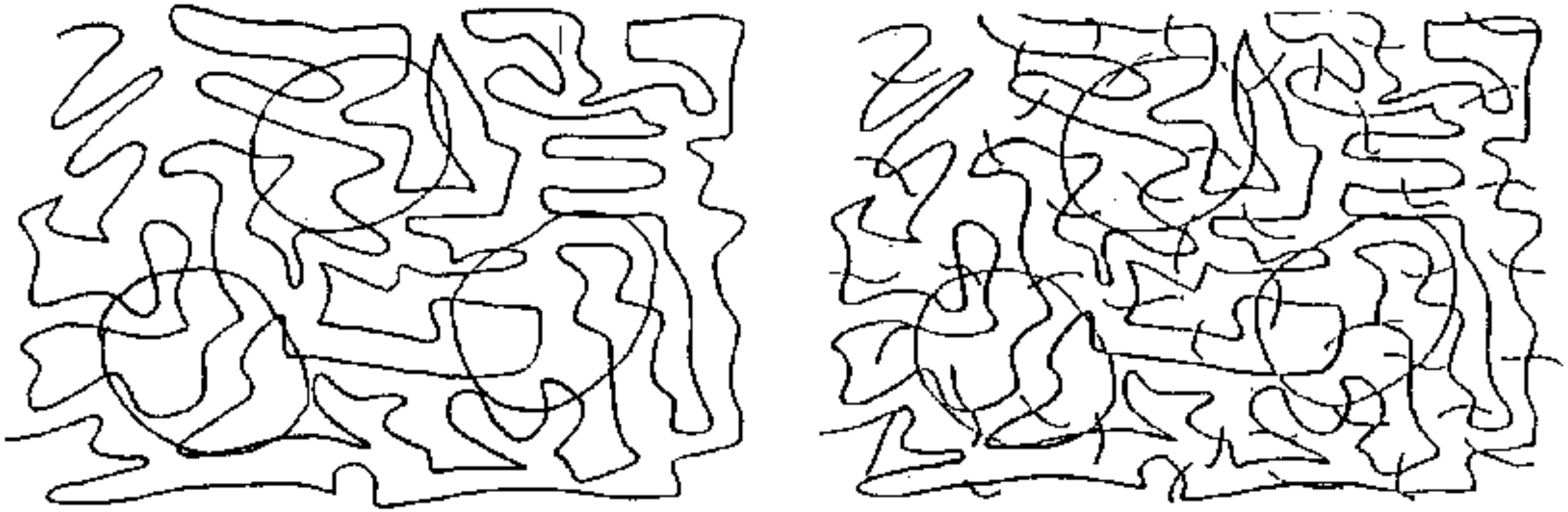


Figure 1a. Three prominent blobs are perceived immediately and with little effort. Locally, the blobs are similar to the background contours. (adopted from Mahoney (1986))

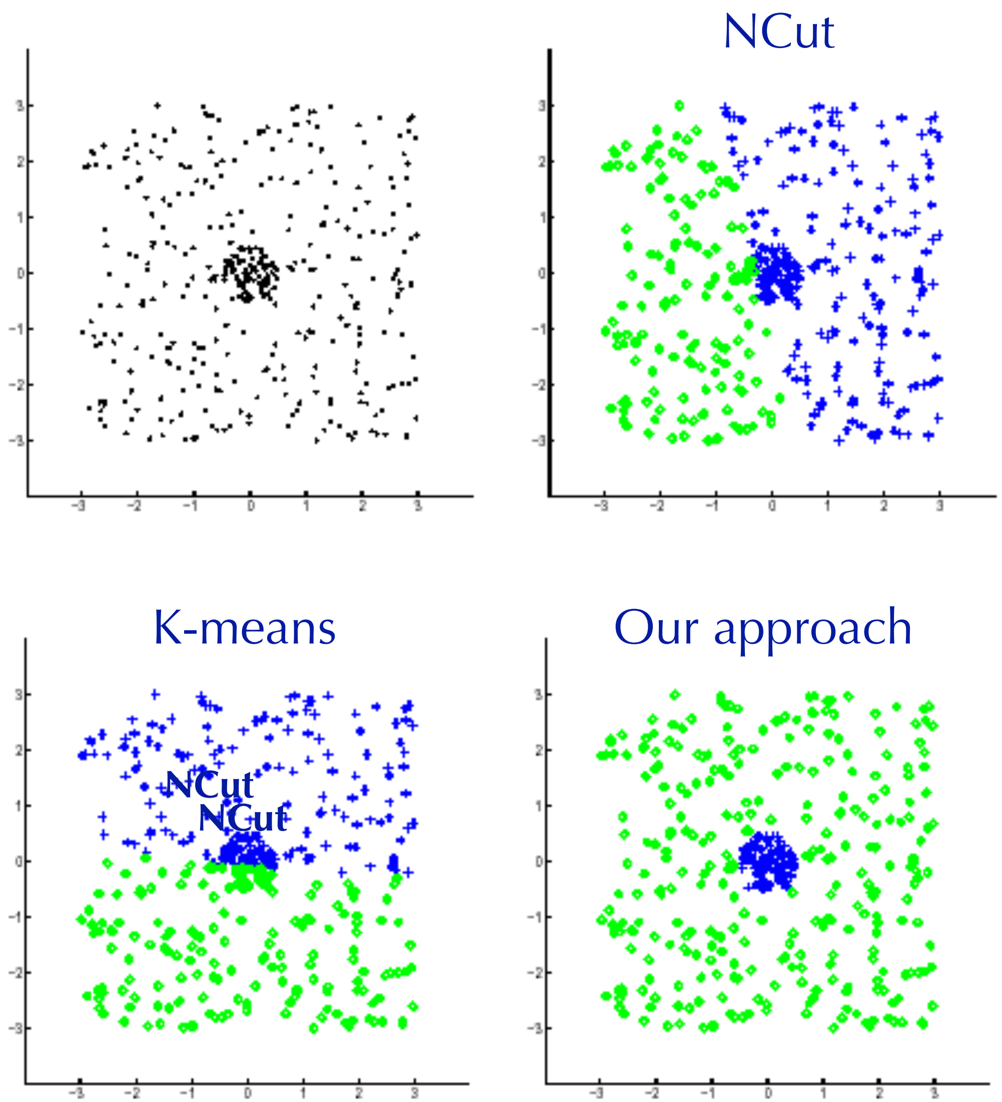
Figure 1b. Intersections were added to illustrate that the blobs are not distinguished by virtue of their intersections with the background curves.

Separating Structure from Clutter

Figure 2. A circle in a background of 200 randomly placed and oriented segments The circle is still perceived immediately although its contour is fragmented.

Figure 3. An edge image of a car in a cluttered background. Our attention is drawn immediately to the region of interest. It seems that the car need not be recognized to attract our attention. The car also remains salient when parallel lines and small blobs are removed, and when the less textured region surrounding parts of the car is filled in with more texture.

Separating Structure from Clutter



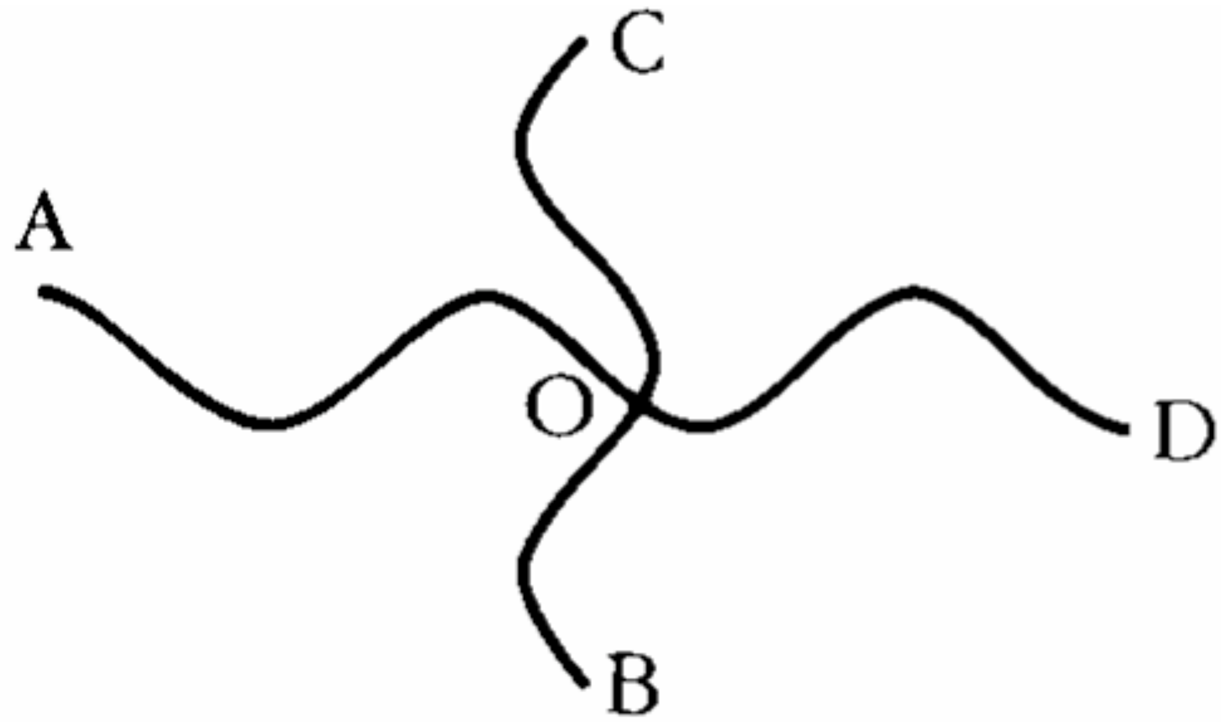
One-class Clustering

"[…] in certain real-world problems, natural groupings are found among only on a small subset of the data, while the rest of the data shows little or no clustering tendencies.

In such situations it is often more important to cluster a small subset of the data very well, rather than optimizing a clustering criterion over all the data points, particularly in application scenarios where a large amount of noisy data is encountered."

G. Gupta and J. Ghosh. Bregman bubble clustering: A robust framework for mining dense cluster. *ACM Trans. Knowl. Discov. Data* (2008).

When Groups Overlap



Does *O* belong to *AD* or to *BC* (or to none)?

The Need for Overlapping Clusters

Partitional approaches impose that each element cannot belong to more than one cluster. There are a variety of important applications, however, where this requirement is too restrictive.

Examples:

- ✓ clustering micro-array gene expression data
- ✓ clustering documents into topic categories
- ✓ perceptual grouping
- ✓ segmentation of images with transparent surfaces

References:

- ✓ N. Jardine and R. Sibson. The construction of hierarchic and non-hierarchic classifications. *Computer Journal*, 11:177–184, 1968
- ✓ A. Banerjee, C. Krumpelman, S. Basu, R. J. Mooney, and J. Ghosh. Model-based overlapping clustering. *KDD 2005*.
- ✓ K. A. Heller and Z. Ghahramani. A nonparametric Bayesian approach to modeling overlapping clusters. *AISTATS 2007*.

What is a Cluster?

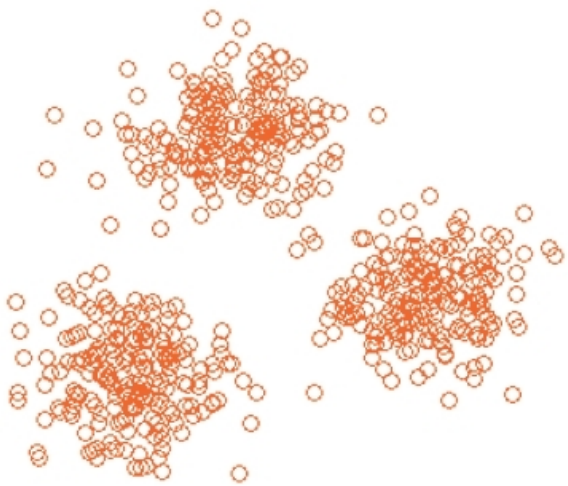
No universally accepted (formal) definition of a “cluster” but, informally, a cluster should satisfy two criteria:

Internal criterion

all “objects” *inside* a cluster should be highly similar to each other

External criterion

all “objects” *outside* a cluster should be highly dissimilar to the ones inside



The Notion of “Gestalt”

«In most visual fields the contents of particular areas “belong together” as circumscribed units from which their surrounding are excluded.»

W. Köhler, *Gestalt Psychology* (1947)

«In gestalt theory the word “Gestalt” means any segregated whole.»

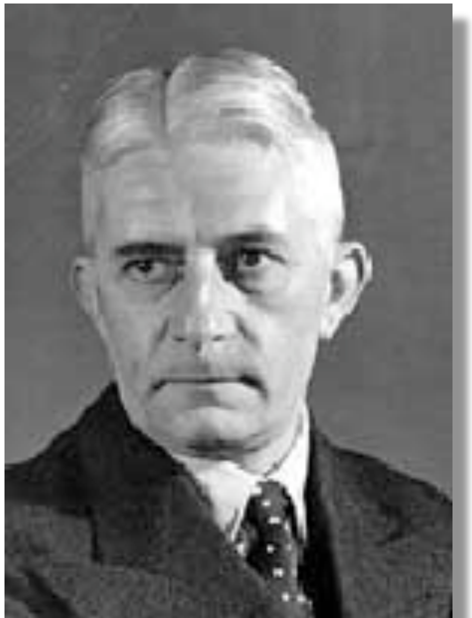
W. Köhler (1929)

Beyond Perception

«In fact, the concept "Gestalt" may be applied far beyond the limits of sensory experience.

According to the most general functional definition of the term, the processes of learning, of recall, of striving, of emotional attitude, of thinking, acting and so forth, may have to be included.»

W. Köhler, *Gestalt Psychology* (1947)

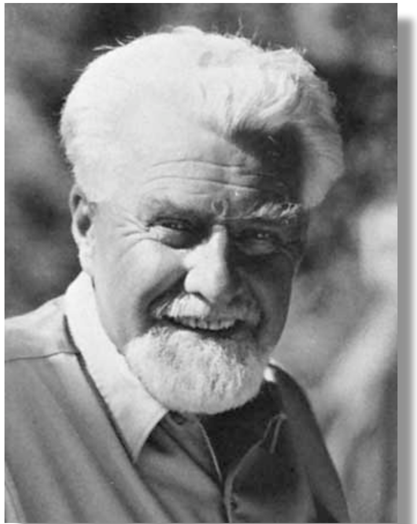


Beyond Perception

«The process of discovery is akin to the recognition of shapes as analysed by Gestalt psychology.»

Michael Polanyi

Science, Faith and Society (1946)



«In my opinion every discovery of a complex regularity comes into being through the function of gestalt perception.»

Konrad Lorenz (1959)

From Psychology to Sociology

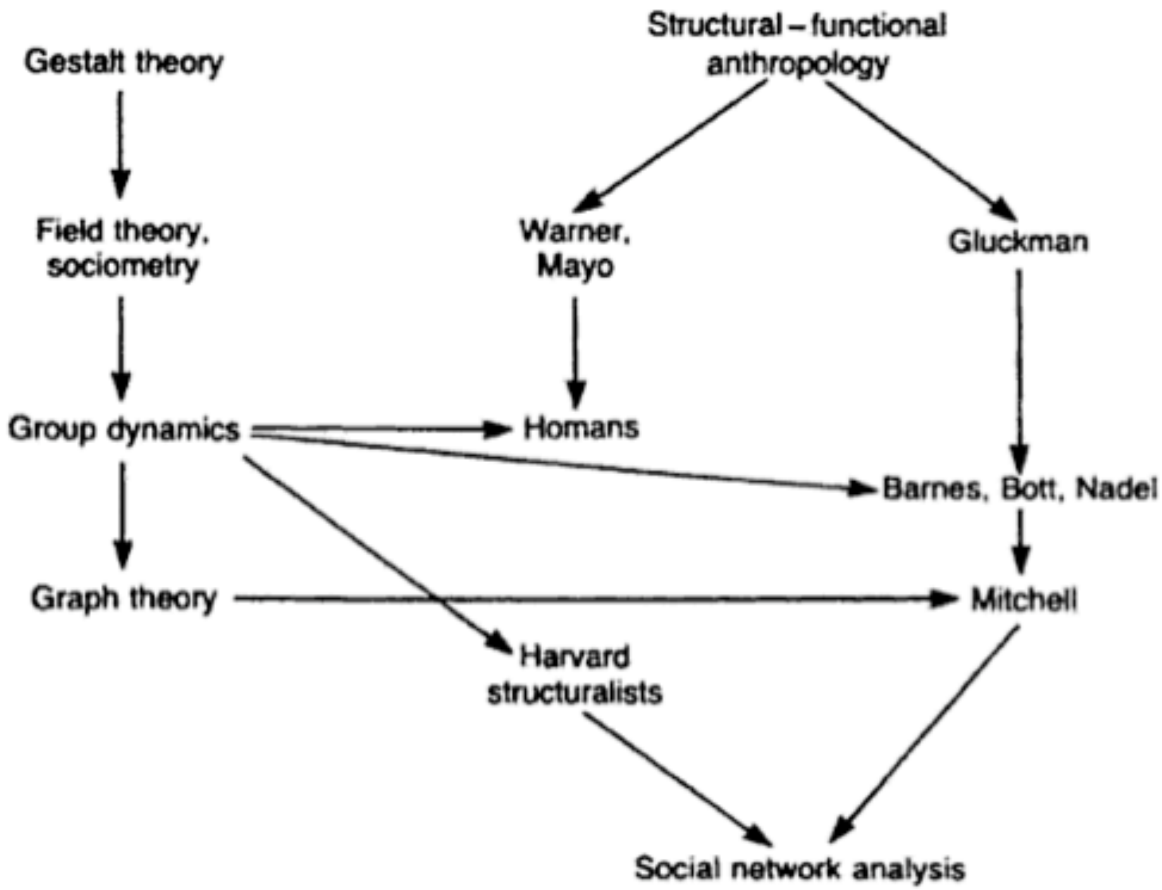


Figure 2.1 *The lineage of social network analysis*

From: J. Scott, *Social Network Analysis* (2000)

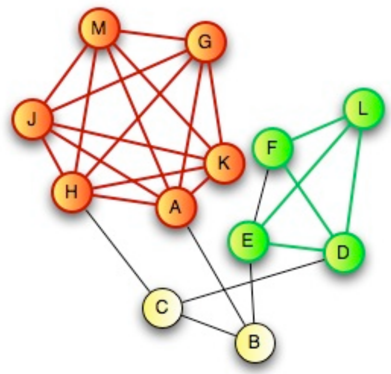
Data Clustering: Old *vs.* New

By answering the question "what is a cluster?" we get a novel way of looking at the clustering problem.

```
Clustering_old(V,A,k)
   V1,V2,...,Vk <- My_favorite_partitioning_algorithm(V,A,k)
   return V1,V2,...,Vk

------

Clustering_new(V,A)
   V1,V2,...,Vk <- Enumerate_all_clusters(V,A)
   return V1,V2,...,Vk

Enumerate_all_clusters(V,A)
   repeat
      Extract_a_cluster(V,A)
   until all clusters have been found
   return the clusters found
```

A Special Case: Binary Symmetric Affinities

Suppose the similarity matrix is a binary (0/1) matrix.

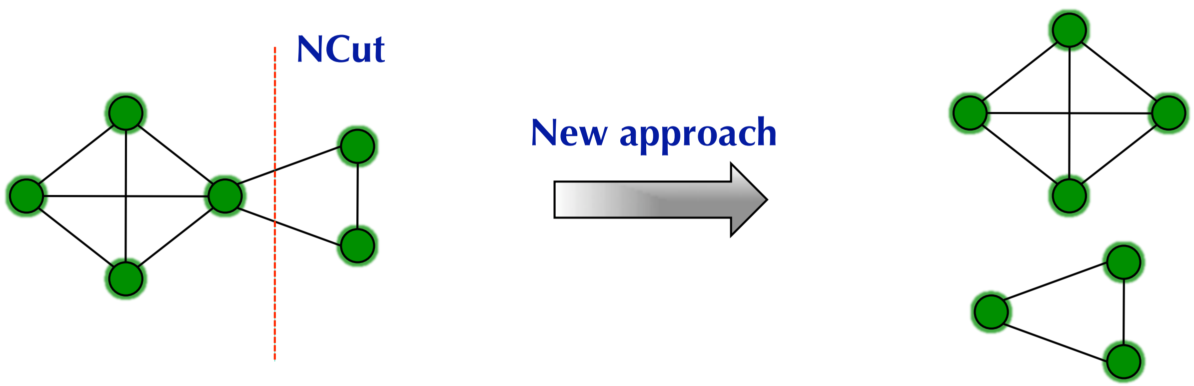


Given an unweighted undirected graph *G=(V,E)*:

A *clique* is a subset of mutually adjacent vertices

A *maximal clique* is a clique that is not contained in a larger one

In the 0/1 case, a meaningful notion of a cluster is that of a *maximal clique*.



Advantages of the New Approach

- ✓ No need to know the number of clusters in advance (since we extract them sequentially)
- ✓ Leaves clutter elements unassigned (useful, e.g., in figure/ground separation or one-class clustering problems)
- ✓ Allows extracting overlapping clusters

Need a partition?

```
Partition_into_clusters(V,A)
  repeat
     Extract_a_cluster
     remove it from V
  until all vertices have been clustered
```

The Symmetry Assumption

«Similarity has been viewed by both philosophers and psychologists as a prime example of a symmetric relation. Indeed, the assumption of symmetry underlies essentially all theoretical treatments of similarity.

Contrary to this tradition, the present paper provides empirical evidence for asymmetric similarities and argues that **similarity should not be treated as a symmetric relation**.»

Amos Tversky
"Features of similarities," *Psychol. Rev.* (1977)

Examples of asymmetric (dis)similarities

- ✓ Kullback-Leibler divergence
- ✓ Directed Hausdorff distance
- ✓ Tversky's contrast model

ESS's as Clusters

We claim that ESS's abstract well the main characteristics of a cluster:

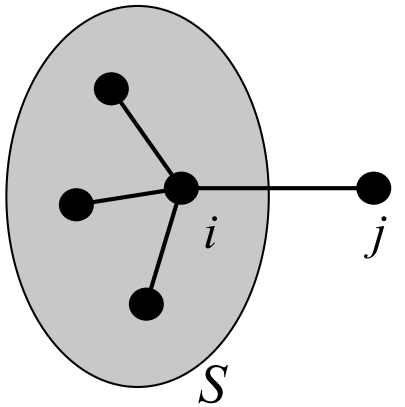
- ✓ **Internal coherency**: High mutual support of all elements within the group.
- ✓ **External incoherency**: Low support from elements of the group to elements outside the group.

Basic Definitions

Let $S \subseteq V$ be a non-empty subset of vertices, and $i \in S$.

The **(average) weighted degree** of i w.r.t. S is defined as:

$$\mathrm{awdeg}_S(i) = \frac{1}{|S|} \sum_{j \in S} a_{ij}$$



Moreover, if $j \notin S$, we define:

$$\phi_S(i,j) = a_{ij} - \mathrm{awdeg}_S(i)$$

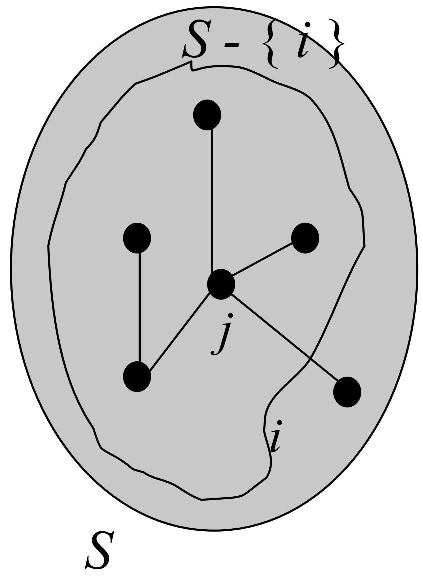
Intuitively, $\phi_S(i,j)$ measures the similarity between vertices j and i, with respect to the (average) similarity between vertex i and its neighbors in S.

Assigning Weights to Vertices

Let $S \subseteq V$ be a non-empty subset of vertices, and $i \in S$.

The **weight** of i w.r.t. S is defined as:

$$w_S(i) = \begin{cases} 1 & \text{if } |S| = 1 \\ \displaystyle\sum_{j \in S - \{i\}} \phi_{S-\{i\}}(j,i)\, w_{S-\{i\}}(j) & \text{otherwise} \end{cases}$$

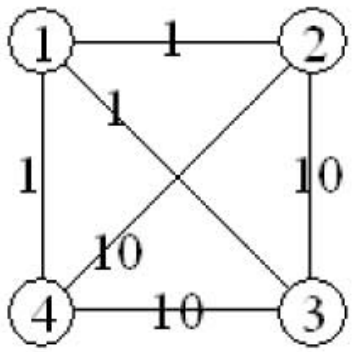


Further, the **total weight** of S is defined as:

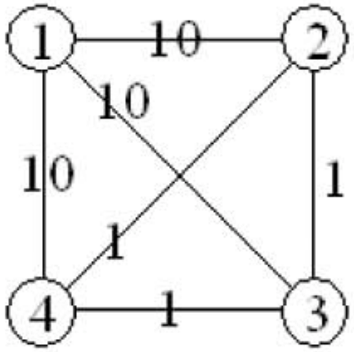
$$W(S) = \sum_{i \in S} w_S(i)$$

Interpretation

Intuitively, $w_S(i)$ gives us a measure of the overall (relative) similarity between vertex i and the vertices of S-{i} with respect to the overall similarity among the vertices in S-{i}.



$w_{\{1,2,3,4\}}(1) < 0$



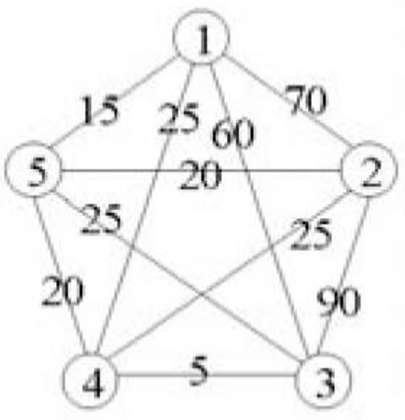
$w_{\{1,2,3,4\}}(1) > 0$

Dominant Sets

Definition (Pavan and Pelillo, 2003, 2007). A non-empty subset of vertices $S \subseteq V$ such that $W(T) > 0$ for any non-empty $T \subseteq S$, is said to be a **dominant set** if:

1. $w_S(i) > 0$, for all $i \in S$ (internal homogeneity)

2. $w_{S \cup \{i\}}(i) < 0$, for all $i \notin S$ (external homogeneity)



The set {1,2,3} is dominant.

Dominant sets ≡ clusters

The Clustering Game

Consider the following "clustering game."

- ✓ Assume a preexisting set of objects *O* and a (possibly asymmetric) matrix of affinities A between the elements of *O*.
- ✓ Two players with complete knowledge of the setup play by simultaneously selecting an element of *O*.
- ✓ After both have shown their choice, each player receives a payoff, monetary or otherwise, proportional to the affinity that the chosen element has with respect to the element chosen by the opponent.

Clearly, it is in each player's interest to pick an element that is strongly supported by the elements that the adversary is likely to choose.

Hence, in the (pairwise) clustering game:

- ✓ There are 2 players
- ✓ The objects to be clustered are the pure strategies
- ✓ The (null-diagonal) affinity matrix coincides with the similarity matrix

Dominant Sets are ESS's

Theorem (Torsello, Rota Bulò and Pelillo, 2006). Evolutionary stable strategies of the clustering game with affinity matrix A are in a one-to-one correspondence with dominant sets.

Note. Generalization of well-known Motzkin-Straus theorem from graph theory.

Dominant-set clustering

- ✓ To get a single dominant-set cluster use, e.g., replicator dynamics (but see Rota Bulò, Pelillo and Bomze, *CVIU* in press, for faster dynamics)
- ✓ To get a partition use a simple *peel-off* strategy: iteratively find a dominant set and remove it from the graph, until all vertices have been clustered
- ✓ To get overlapping clusters, enumerate dominant sets (see Bomze, 1992; Torsello, Rota Bulò and Pelillo, 2008)

Special Case: Symmetric Affinities

Given a symmetric real-valued matrix A (with null diagonal), consider the following Standard Quadratic Programming problem (StQP):

$$\begin{array}{ll} \text{maximize} & f(x) = x^T A x \\ \text{subject to} & x \in \Delta \end{array}$$

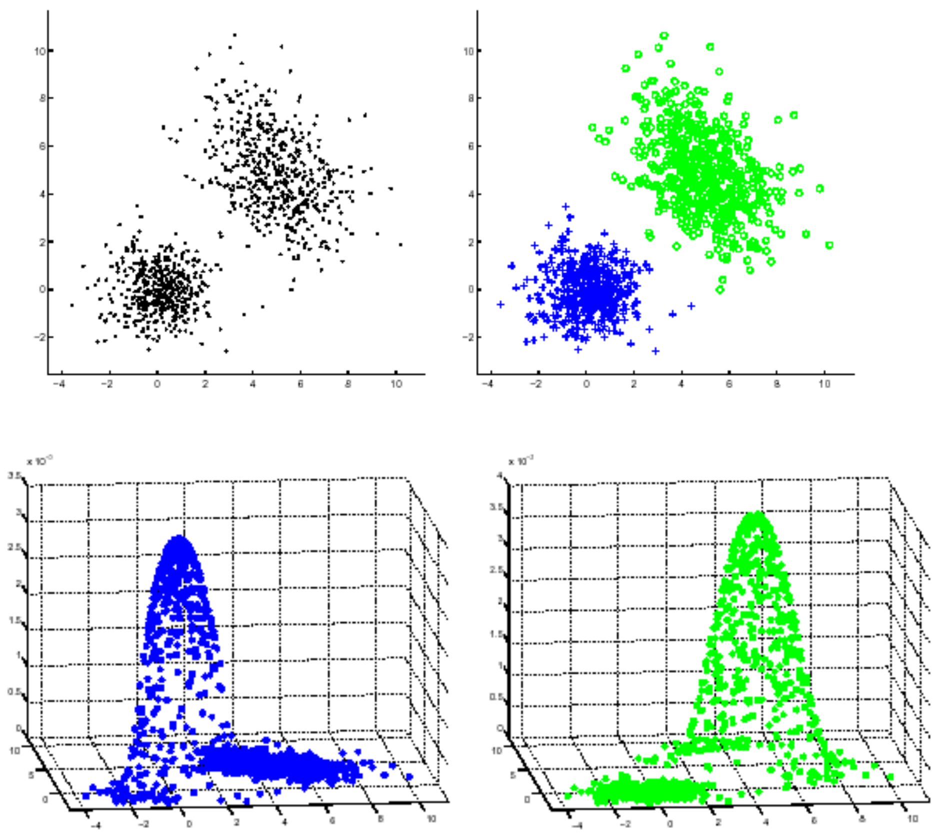
Note. The function $f(x)$ provides a measure of cohesiveness of a cluster (see Pavan and Pelillo, 2003, 2007; Sarkar and Boyer, 1998; Perona and Freeman, 1998).

ESS's are in one-to-one correspondence to (strict) local solutions of StQP

Note. In the 0/1 (symmetric) case, ESS's are in one-to-one correspondence to (strictly) maximal cliques (Motzkin-Straus theorem).

Measuring the Degree of Cluster Membership

The components of the converged vector give us a measure of the participation of the corresponding vertices in the cluster, while the value of the objective function provides of the cohesiveness of the cluster.



Applications of Dominant-Set Clustering

Image segmentation and grouping
Pavan and Pelillo (CVPR 2003, PAMI 2007); *Torsello and Pelillo* (EMMCVPR'09)

Bioinformatics
Identification of protein binding sites (*Zauhar and Bruist*, 2005)
Clustering gene expression profiles (*Li et al*, 2005)
Tag Single Nucleotide Polymorphism (SNPs) selection (*Frommlet*, 2010)

Security and video surveillance
Detection of anomalous activities in video streams (*Hamid et al.*, CVPR'05; AI'09)
Detection of malicious activities in the internet (*Pouget et al.*, J. Inf. Ass. Sec. 2006)

Content-based image retrieval
Wang et al. (Sig. Proc. 2008); *Giacinto and Roli* (2007)

Analysis of fMRI data
Neumann et al (NeuroImage 2006); *Muller et al* (J. Mag Res Imag. 2007)

Video analysis, object tracking, human action recognition
Torsello et al. (EMMCVPR'05); *Gualdi et al.* (IWVS'08); *Wei et al.* (ICIP'07)

Multiple instance learning
Erdem and Erdem (2011)

Feature selection
Hancock et al. (GbR'11; ICIAP'11)

Image matching and registration
Torsello et al. (IJCV 2011, ICCV'09, CVPR'10, ECCV'10)

In a nutshell…

The dominant-set (ESS) approach:

- ✓ makes no assumption on the underlying (individual) data representation
- ✓ makes no assumption on the structure of the affinity matrix, being it able to work with asymmetric and even negative similarity functions
- ✓ does not require *a priori* knowledge on the number of clusters (since it extracts them sequentially)
- ✓ leaves clutter elements unassigned (useful, e.g., in figure/ground separation or one-class clustering problems)
- ✓ allows principled ways of assigning out-of-sample items *(NIPS'04)*
- ✓ allows extracting overlapping clusters *(ICPR'08)*
- ✓ generalizes naturally to hypergraph clustering problems, i.e., in the presence of high-order affinities, in which case the clustering game is played by more than two players *(NIPS'09)*
- ✓ extends to hierarchical clustering *(ICCV'03: EMMCVPR'09)*

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