A new analysis technique for the Sprouts Game*

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Abstract

Sprouts is a two players game that was first introduced by M.S. Patterson and J.H. Conway in 1967. There are two players A and B that starting from a set of x_0 vertices draw a plane graph by alternatively connecting any pair of two vertices with degree less than three with an edge, and by inserting a new vertex in the new edge. The move is possible if and only if the new connection maintains the planarity of the graph. The player that executes the last possible move is the winner. We study some new topological properties of the Sprouts Game and we show their effectiveness by giving a complete analysis of the case $x_0 = 7$, for which no formal proof is, to the best of our knowledge, known.

1 Introduction

Sprouts is a simple pencil-and-paper game that was first introduced by M.S. Patterson and J.H. Conway in 1967, and was then exposed in the same year by Martin Gardner [5]. Its major diffusion started when Piers Anthony published the science-fiction novel "Macroscope" [1], where one of the characters presents this game. The game starts with a set of x_0 initial vertices drawn on the plane. Then, the two players alternatively connect any two vertices with a new edge and add a new vertex on it, with the rules that edges cannot cross each other (i.e., the graph has to remain planar), and no vertex can have degree (i.e., number of incident edges) more than three. The player that makes the last possible move wins.

As the space of possible moves grows exponentially with respect to x_0, the analysis of the game becomes nontrivial already for $x_0 = 4$, and topological properties are needed in order to avoid an infeasible (at least by hand) exhaustive

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search. An interesting analysis for the cases $x_0 \leq 6$ is given by Berlekamp and others in [3], where it is stated that the case $x_0 = 6$ "was first proved by D. Mollison, whose analysis of the game ran to 47 pages!".

The game has then been also studied by other authors [2, 4, 8]. Interesting results appear in [2] where the authors present a computer analysis of the game that solves the problem for $x_0 \leq 11$ through a program based on an optimized exhaustive search technique. No formal proof is however shown and the conjecture that B wins iff x_0 is 0,1 or 2 modulo 6 is presented. To the best of our knowledge, no "compact" formal analysis has been given for the cases $x_0 \geq 7$.

In this paper we present some new topological properties of the game, and we show how to apply them to solve the case of $x_0 = 7$ in a very compact and modular way. The properties we propose differ from the know ones [3, 4, 8] in two main aspects: (i) they can be applied to a region of the graph, thus allowing to prove lemmas for simple cases that can be then combined in order to analyze more complicated situations; (ii) they seem to be more efficient, as they usually allow (also exploiting the modularity described above) to prove that a strategy is winning after just a couple of moves.

The paper is organized as follows. In section 2 we introduce the definitions and the basic properties of planar graphs; in section 3 we formally introduce the Sprouts Game; in section 3.1 we present some known results about the game while in section 3.2 we give our new properties that allow to analyze the case $x_0 = 7$ in section 4. Finally, in section 5 we give some concluding remarks and we discuss open problems.

2 Definitions and basic properties

In this section we give some definitions and report some basic properties of planar graphs, mainly referring to [9]. We consider a graph $G = (V,E)$ where V is the set of *vertices* and E is the set of *edges*. Each edge $\{u,v\}$ is an unordered pair of distinct vertices (i.e., we consider undirected graphs), and is said to be *incident* to u and v. The *degree* of a vertex u is the number of edges incident to u. A *subgraph* $G' = (V',E')$ of G is a graph such that $V' \subseteq V$ and $E' \subseteq E$. We also say that a (*connected*) *component* of a graph G is a maximal connected subgraph G' of G, i.e., a subgraph with a maximal number of vertices and edges, such that any pair of vertices of V' is connected through a sequence of edges of E'. A *cut-vertex* is a vertex whose removal increases the number of components of G.

We say that a graph G is *planar* if it can be *embedded* in the plane, i.e., if we can draw it with points representing vertices and curves representing edges, so that no two edges intersect except at a common vertex. A *plane graph* is a particular embedding (i.e., a particular drawing) of a planar graph in the plane.

The parts of the plane delimited by the edges of a plane graph are called *faces*. More formally, faces are connected areas that can be obtained by deleting from the plane curves and points corresponding to the edges and vertices of a plane graph G. There is always an unbounded face which is not surrounded by edges and we call it *outer face*. Finally, the *boundary* of a face is defined as the set of edges in the closure of the face and the relative vertices are said to *cross* the face.

We also need the notion of *region*. A *region* R of G is a part of the plane covering exactly a set of faces of G with their corresponding boundaries. A *boundary vertex of* R is a vertex which is both in R and in the complementary region $\bar{R}$, i.e., the region corresponding to the faces not covered by R.

We now report the equation that relates the number of vertices, edges and faces of a plane graph:

Theorem 1 *(Euler 1750) Given a connected plane graph G with n vertices, m edges and f faces we have that $n - m + f = 2$.*

This theorem can be easily extended to the case where G is not connected:

Corollary 1 *If G has $c \geq 1$ components we have that $n - m + f - c = 1$.*

The idea is that we can compute the relation of theorem 1 in each component and we can combine it by counting the outer face only once therefore getting $n - m + f - (c - 1) = 2$.

3 The Sprouts Game

The Sprouts Game [3]: *We are given an initial (plane) graph $G_0 = (V_0, E_0)$ with $|V_0| = x_0$ and $|E_0| = \{\emptyset\}$ and two players A and B that alternatively execute a move. At a generic step i a player executes a move and obtains a new plane graph $G_i = (V_i, E_i)$ applying this rule:*

> *the player searches for two vertices x and y whose degree is < 3 and that can be connected together by drawing an edge e which does not intersect any other edge (also $y = x$ if the degree of x is < 2). After drawing edge e the player draws a new vertex v on e thus obtaining a new plane graph $G_i = (V_{i-1} \cup \{v\}, E_{i-1} \cup \{\{x, v\}, \{v, y\}\})$.*

If no such move is possible the current player stops and therefore looses.

Note that by construction all the vertices of the graphs G_i have degree ≤ 3. An example of a sequence of moves can be found in Figure 1.

In the following sections, we give some properties that are useful to the analysis of the Sprouts Game. In particular, in section 3.1 we present some known results about the maximum number of moves, while in section 3.2 we give our new properties that will allow to analyze the case $x_0 = 7$.

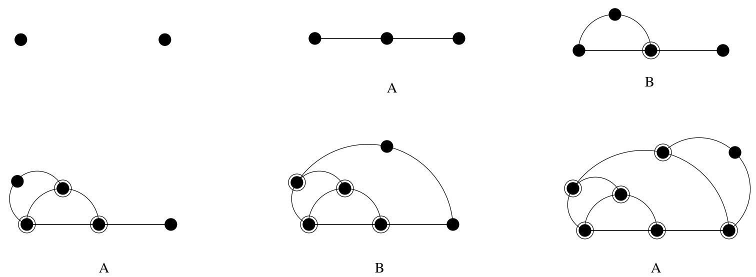

Figure 1: Maximal graph construction. Circles around vertices represent that such vertices have degree 3 and so cannot be used anymore.

3.1 Computing the maximal number of moves

It is quite simple to prove that that after at most $m_{max} = 3x_0 - 1$ moves, the game ends [3]. The idea is that every vertex has initially degree 0 and at every step can have degree at most 3, therefore the initial number of "free" degrees is $3x_0$. At every step this number decreases by 1 since by connecting two vertices x and y the total degree decreases by 2 but, at the same time, it is increased by 1 by the new added vertex. As, at the end, at least the one vertex will be still of degree 2 (it is the vertex added by the last move), we obtain that the number m of moves is such that $m \leq 3x_0 - 1$. Note that this also proves that the game will not last forever. Moreover, it is possible to prove [4] that a Sprouts Game with m_{max} moves is always possible [4] (see, e.g., Figure 1).

3.2 Isolating vertices

It is crucial to observe that the upper bound m_{max} is not always reached. Indeed, at the end of the game, it could be the case that some vertices are isolated and cannot be used anymore. Note that, such isolated vertices, may only be of degree 2, otherwise it could be possible to connect them through a self-loop. Let l be the number of isolated vertices at the end of a game. Then, the number of performed moves m is exactly $3x_0 - l$ as we miss one potential move for every isolated vertex. Since either A or B wins respectively depending on the oddness or evenness of the total number of moves m of the game, then a winning strategy for A (B) consists of properly isolating some vertices so to obtain an odd (even) number of moves (as also noted in [3]).

If, at a certain point of the game, l vertices are isolated then it is trivial to see that the final position will have at least l isolated vertices of degree 2. The main problem, for a player, is to be guaranteed that the other one will not be able to isolate an extra vertex (thus changing the parity of the number of moves). To this aim we develop some results that can be applied to find an upper bound to the number of isolated vertices at a certain point of the game. We first formalize the concept of isolated vertex.

Definition 1 *Consider a plane graph G. Two vertices x and y of G are* isolated one from the other in *G iff they cannot be connected with an edge without crossing another edge.*

We are interested in studying which is the maximal number of isolated vertices that can be obtained starting from a particular graph. In [3], this is done by estimating the number of vertices of degree three which are not neighbour of a "live" vertex (the so called *Pharisees*). [1] Here, we instead compute the minimum number of faces required to isolate them.

In the following we will denote by $f(G)$ the number of faces of a plane graph G and by $f(R)$ the number of faces of a region R.

Lemma 1 *Consider a plane graph G . If G has $p_0 + p_1 + p_2$ isolated vertices of degree respectively 0,1 and 2, and s is the number of isolated cut-vertices of degree 2 in G, then $f(G) \geq p_0 + p_1 + 2p_2 - s$.*

PROOF. The value $p_0 + p_1$ is trivially derived from the fact that every vertex of degree 0 or 1 can cross a unique face. Every vertex of degree 2 crosses two faces except if it is a cut-vertex. In this case it lies inside a single face. This leads to the value $2p_2 - s$. ■

Observe now that if in G no move is possible, then no vertex of degree 0 or 1 exists, therefore $p_0 = p_1 = 0$. We then have the following:

Corollary 2 *Consider a plane graph G where no move is possible, then $f(G) \geq 2p_2 - s$.*

Finally observe that if we want to compute the number of faces contained in a region R we also have to take into account the boundary vertices of R. Therefore we have the following:

Corollary 3 *Consider a region R of a plane graph G that has $B \geq 0$ boundary vertices of degree 2 and where no move is possible, then $f(R) \geq 2p_2 - s - B$.*

PROOF. It follows directly from corollary 2 and from the fact that boundary vertices of degree 2 cross one face in R and one outside it (i.e., in $\bar{R}$). ■

From lemma 1 we now know the minimum number of faces for $p_0 + p_1 + p_2$ isolated vertices of degree respectively 0,1 and 2. We now want to calculate which is the maximum number of faces we can generate starting from a certain graph with

[1] It is interesting to note that such a property is "global", in a sense that cannot be applied to a region of the graph, unless it is exactly an isolated Sprouts sub-game.

no isolated vertices and moving to a situation with $p_0+p_1+p_2$ isolated vertices. In this way we can check if this number of generated faces is sufficient to indeed isolate the $p_0+p_1+p_2$ vertices. This gives us a general relation on the maximal number of isolated vertices which depends on some observable parameters of the initial graph. We start with a definition:

Definition 2 *Let G be a plane graph and let v_0, v_1 and v_2 be the number of vertices of degree 0,1 and 2 respectively. We define $D(G) = 3v_0 + 2v_1 + v_2$.*

In other words $D(G)$ is the total number of edges that every vertex can still tolerate before reaching degree 3.

Let us now show a relation between $D(G)$ and the number of moves in a Sprouts Game, when moving from a graph to a new one.

Lemma 2 *Let G and G' be two plane graphs. If we execute n moves from G to G' then $D(G') = D(G) - n$.*

PROOF. Consider a single move from a generic H to H'. Then $D(H)$ is decreased by 2 because of the new edge but it is also increased by one because of the new vertex of degree 2. So $D(H') = D(H) - 1$. Iterating for n moves we obtain $D(G') = D(G) - n$. ■

We apply this result to calculate the number of faces generated when moving from a graph G to a graph G'. As expected, this number depends on the number of components of the two graphs.

Lemma 3 *Let G and G' be two plane graphs with c and c' components, respectively. If we execute some moves from G to G' then:*

$$f(G') - f(G) = D(G) - D(G') - c + c'$$

PROOF. Assume G' is obtained by executing a number x of moves starting from G, we then can obtain x by applying lemma 2: $x = D(G) - D(G')$. Let m, n, c be respectively the edges, vertices, and components of G and m', n', c' the ones of G'. By applying corollary 1, we have that:

$$f(G') - f(G) = m' - n' + c' + 1 - (m - n + c + 1) = m' - n' - (m - n) - c + c'$$

Note that every move adds two edges and one vertex and so the quantity $m-n$ increases by one after every move. Here we have $D(G) - D(G')$ moves and so: $f(G') - f(G) = D(G) - D(G') - c + c'$. ■

We can now present the main theorem which will be used in the next section to prove that a certain strategy for the Sprouts Game is winning. It gives the maximum number of isolated vertices which can be obtained by playing in a particular region of a certain initial graph. This maximum depends on the number of vertices of degree 0,1 and 2, of cut-vertices, of boundary vertices and of faces, that are contained in the selected region. We first need the following:

Lemma 4 *Let G and G' respectively be two plane graphs with c and c' components and s and s' cut vertices of degree 2, and let G' be obtained from G through some moves. Then $s' \leq s + c - c'$.*

PROOF. The cut vertices in G' can be at most the ones that were in G, i.e., s, plus the new ones that can only be created by connecting two separated components. During the moves from G to G' only $c - c'$ components were joined, therefore the bound. ■

Theorem 2 *Let G be a plane graph with c components. Let R be a region composed of F faces, with B boundary vertices, and s cut-vertices, and denote with v_0, v_1 and v_2 the number of vertices in R of degree 0,1 and 2 respectively. Suppose that after some moves inside R, the region R is transformed into a new region R' whose vertices are all isolated one from each other, and where no move is possible. Let v_2' be the number of (isolated) vertices of degree 2 in the new region R'. We then have:*

$$v_2' \leq \frac{3\,v_0 + 2\,v_1 + v_2 + s + B + F}{3}$$

PROOF. Let G' be the graph reached by G after the moves in R, and denote with c' its number of components and with s' its cut-vertices. By lemma 3 and by the hypothesis that moves are executed only inside the region R, we have that the number gf of new faces generated inside the new region R' moving from G to G' is: $gf = f(G') - f(G) = D(G) - D(G') - c + c'$. Therefore R' globally contains $gf + F$ faces, i.e., the new faces plus the F faces initially contained inside R. So by corollary 3 applied inside R' we have:

$$2v_2' - s' - B \leq gf + F = D(G) - D(G') - c + c' + F.$$

Observe now that $D(G) - D(G')$ depends only on the new moves inside R, therefore by this and by lemma 4 we have:

$$\begin{aligned} 2v_2' - s' - B &\leq gf + F \\ &= 3\,v_0 + 2\,v_1 + v_2 - v_2' - c + c' + F \\ &\leq 3\,v_0 + 2\,v_1 + v_2 - v_2' + s - s' + F \end{aligned}$$

and therefore $3v_2' \leq 3\,v_0 + 2\,v_1 + v_2 + s + F + B$, which proves the relation. ■

We can apply the theorem to the initial graph G_0 of the Sprouts Game. The above relation is very simple and can be used to calculate the maximum number of isolated vertices obtainable in a Sprouts Game.

Corollary 4 *In a Sprout Game with x_0 vertices, if a final configuration G' is reached with v_2' isolated vertices, then $v_2' \leq x_0$.*

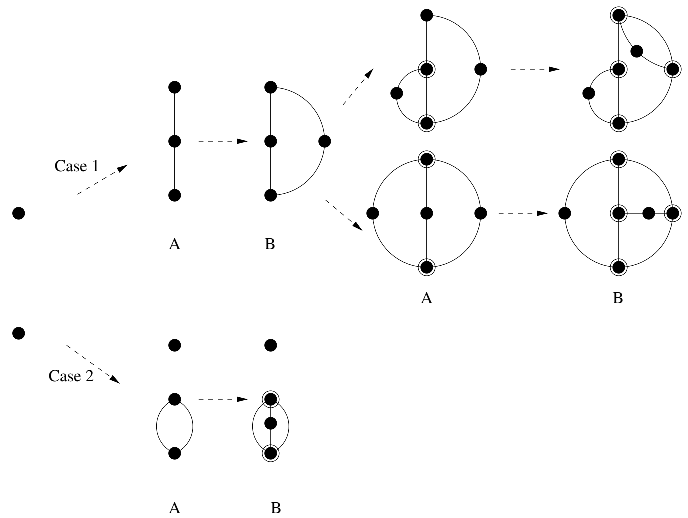

Figure 2: Strategy for B for the Sprouts Game on $x_0 = 2$. Only two embeddings are given at step 3 since all the others are symmetric.

PROOF. This is trivially derived by applying theorem 2 with $v_0 = x_0, v_1 = v_2 = s = B = 0$ and $F = 1$ and by observing that $v'_2 \leq \frac{3x_0+1}{3}$ implies $v'_2 \leq x_0$. ■

In [3, 4] it is also shown that the minimal number of moves of a Sprouts Game is $m_{min} = 2x_0$. It is also easy to see it from the corollary above. Indeed $m = 3x_0 - v'_2 \geq 3x_0 - x_0 = 2x_0$.

4 Winning strategies

In this section we first show some known winning strategies for the cases $x_0 = 0, 1, 2$ and we then show a new winning strategy and its analysis for the case $x_0 = 7$, for which only automated exhaustive search techniques are known [2]. The strategy for this last case can be a hint for the solution of the game with a generic value of x_0.

We will now show the detailed strategy for some values of x_0, note though that symmetric embeddings of the graph are not shown.

The cases $x_0 = 0$ and $x_0 = 1$ are trivially solved. Indeed if $x_0 = 0$ no moves are possible, therefore B wins. For the case $x_0 = 1$ we have that at least $2x_0 = 2$ moves and at most $3x_0 - 1 = 2$ moves may be done, therefore B will always win.

The case $x_0 = 2$ is more interesting as the number of moves m is such that $4 \leq m \leq 5$, so both A and B may win the game. Note that, in order for B to win, he must isolate 2 vertices, thus obtaining $m = 4$. Note also that, to isolate

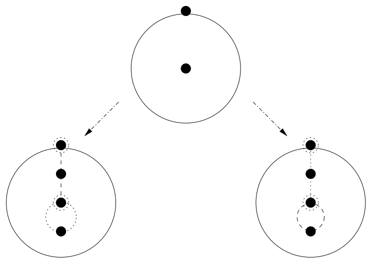

Figure 3: Strategy for lemma 5. B isolates a single vertex. Dashed lines correspond to A's move, dotted to B's.

$l > 2$ vertices we need $3x_0 - l < 6 - 2 = 4$ moves, but in our case this implies a contradiction. This means that, if at a certain point of the game B is sure that at least 2 vertices will be isolated, then he is guaranteed to win.

A winning strategy for B is described in the following and depicted in Figure 2 (we only describe the moves of B):

Case 1 B reconnects, at step 2, the two vertices connected by A at step 1. At step 4, B connects two vertices in the opposite face with respect to the one where A played at step 3, thus isolating 2 vertices and winning.

Case 2 B reconnects, at step 2, the two vertices connected by A at step 1 playing in the face where no degree 0 vertex is. B has now isolated 2 vertices so he wins.

The previous cases were only presented to give an intuition on how a player can force the isolation of some vertices. Let us now give the main result of the section: a formal strategy and correctness analysis for the case $x_0 = 7$. The complete strategy is presented step-wise using some sub-strategies for smaller problems, that we introduce in the following lemmas.

Lemma 5 *Player B can isolate a single vertex if he plays in a face with a vertex of degree 0 and a vertex of degree 2 as in Figure 3.*

PROOF. Consider Figure 3. A can execute only two possible moves. If he connects the two vertices, then B connects the degree 1 vertex to itself and succeeds, since by applying theorem 2 we get $v_2' \leq 5/3$, i.e., a single vertex of degree 2. Otherwise A connects the degree 0 vertex to itself and symmetrically to the previous case B connects the same vertex to the old degree 2 vertex and succeeds. ■

Lemma 6 *Player B can isolate exactly two vertices if he plays in a face with a single degree 2 vertex and two degree 0 vertices as in Figure 4, even if he does not use the degree 2 vertex. Moreover, the vertex on the boundary of the face is not one of the isolated vertices.*

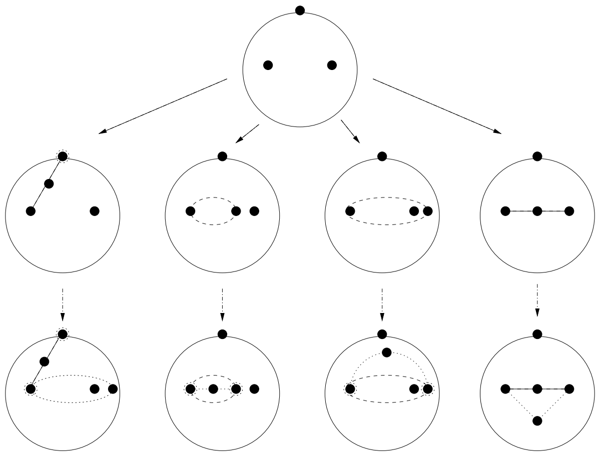

Figure 4: Strategy for lemma 6. B isolates two vertices.

PROOF. A can execute 4 possible moves (the others are symmetric) as in Figure 4 [2]. B's reply is shown in the figure. At this point note that in all the first three cases there are at least two vertices lying in different faces and by the properties that edges cannot cross each other we can deduct that at least two vertices will be isolated. Let us now prove that at most two vertices will be isolated.

In case A, a single vertex can be isolated inside the new face both if the degree 2 vertex is not used (since in the case $x_0 = 1$, B wins and there are exactly two moves) and if it is used (by lemma 5). Outside there is either a single degree 2 vertex (and we are done) or two degree 2 vertices and by applying theorem 2 we get $v'_2 \leq 5/3$, i.e., a single vertex of degree 2. In this case the initial degree 2 vertex was used by A.

In case B, a single degree 2 vertex has been isolated in a face and in the other face we can apply again either lemma 5 (if the vertex on the boundary can be used and therefore is not the isolated one) or the case $x_0 = 1$.

In case C, again a single degree 0 vertex has been isolated in a face and two degree 2 vertices in the other face. If a single one remains we are done, otherwise by theorem 2 we get $v'_2 \leq 5/3$, i.e., a single vertex of degree 2 which will not be on the boundary.

Finally in case D by theorem 2 we get $v'_2 \leq 8/3$, i.e., at most two vertices of degree 2. Moreover at least two vertices can be isolated if B replies in the face opposite to the one where A plays as in Figure 2. Therefore if the degree 2 vertex on the boundary has already been used we are done, otherwise by theorem 2 we get $v'_2 \leq 5/3$, i.e., a single vertex of degree 2 not on the boundary. ■

[2] In all the figures, dashed lines correspond to A's move, dotted to B's.

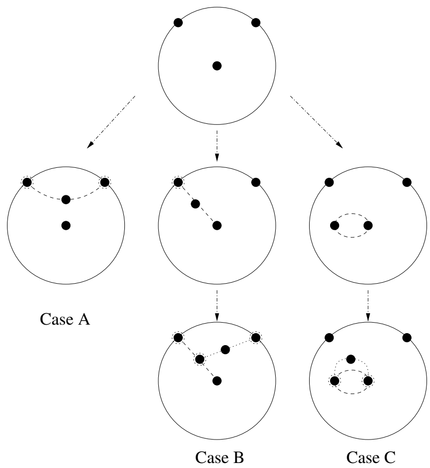

Figure 5: Strategy for lemma 7. *B* isolates a single vertex.

Lemma 7 *Player B can isolate exactly one vertex if he plays in a face with two degree 2 vertices and a single degree 0 vertex as in Figure 5, even if he does not use one or two degree 2 vertices. Moreover, the vertices on the boundary of the face are not one of the isolated vertices.*

PROOF. Consider Figure 5. If at the first step no vertex on the boundary can be used we are done since there is a single degree 0 vertex and *B* succeeds. If a single vertex can be used, we are back to lemma 5 and again we are done (and the boundary vertex will not be isolated). Finally let us assume both vertices are used, i.e., do not remain isolated. In this case there are three different situations as in Figure 5. Obviously in all situations at least a vertex is isolated. In case A, we can apply lemma 5 and we are done. In case B, if we apply theorem 2 we get $v_2' \leq 5/3$, i.e., a single vertex of degree 2. Finally, in case C there are only 3 vertices of degree 2 in the same face, thus by theorem 2 we would get $v_2' \leq 7/3$ which is not enough to prove that only one vertex will be isolated. However, note that in the next move, no matter which of the two vertices is connected, we get by theorem 2, $v_2' \leq 5/3$, i.e., a single vertex of degree 2. ■

Lemma 8 *Player B can isolate exactly two vertices if he plays in a face with a single degree 2 vertex and three degree 0 vertices as in Figure 6, even if he does not use the degree 2 vertex. Moreover, the vertex on the boundary of the face is not one of the isolated vertices.*

PROOF. Consider Figure 6. First observe that in all cases there are vertices in two distinct faces, therefore at least two vertices of degree 2 can be isolated. Moreover the fact that the vertex on the boundary of the face is not one of the isolated

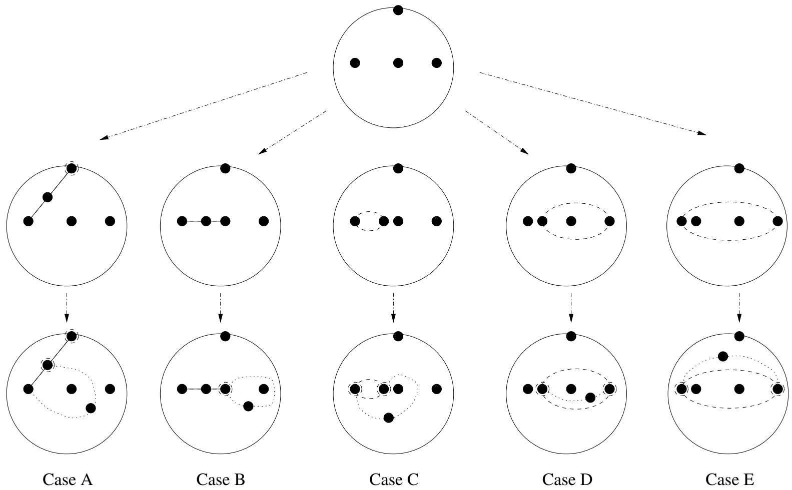

Figure 6: Strategy for lemma 8. B isolates two vertices.

vertices is induced by the analogous property in the used lemmas. Let us now show that we can isolate at most two vertices of degree 2. In case A, we can apply lemma 7 in both faces. The vertex on the boundary has been used by A. In case B, we can apply lemma 5 in the most internal face (or the case $x_0 = 1$), and a slight variant of lemma 7 on the other face, thus obtaining 2 isolated vertices. In case C, we can apply lemma 5 in one face (or the case $x_0 = 1$) and (a slight variant of) lemma 7 in the other face. In case D, we apply lemma 5 (therefore not isolating the vertex in the boundary) in both faces (or the case $x_0 = 1$ if the vertex in the boundary is used). Finally in case E, we trivially obtain a single vertex in the outer face (if the vertex in the boundary cannot be used, or eventually using, and therefore not isolating it) and another single vertex in the inner face since B can trivially force a lost and isolate a single vertex in the case $x_0 = 2$ (see also [3]). ■

Lemma 9 *Player B can isolate exactly three vertices if he plays in a face with a single degree 2 vertex and four degree 0 vertices as in Figure 7, even if he does not use the degree 2 vertex. Moreover, the vertex on the boundary of the face is not one of the isolated vertices.*

PROOF. The proof can be carried out similarly to the one of lemma 8. The winning moves of B are depicted in Figure 7. ■

We can now give the strategy for $x_0 = 7$. First observe that the number of possible moves ranges between $2 \times 7 = 14$ and $3 \times 7 - 1 = 20$. We want to show that B has a strategy for isolating exactly (i.e., at least and at most) 5 vertices of degree 2. This implies an even number of moves ($3 \times 7 - 5 = 16$), and therefore B wins.

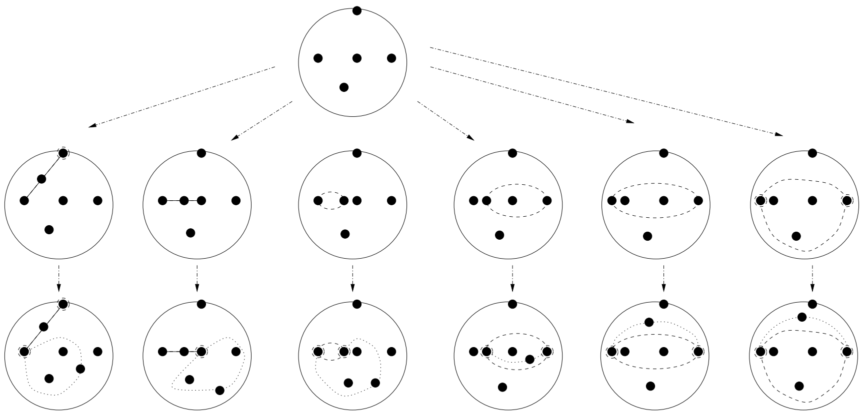

Figure 7: Strategy for lemma 9. *B* isolates three vertices.

The winning strategy for *B* for the case $x_0 = 7$ is illustrated in Figure 8, where again symmetric cases are not considered. Also note that a crucial property that we are using is the one stated in all the previous lemmas, i.e., the fact that in all different cases the vertex on the boundary of a considered face is not one of the isolated vertices (otherwise we would not be able to assure a certain number of isolated vertices). Formally the strategy for *B* consists of:

Strategy for $x_0 = 7$

We give the various winning moves for *B* corresponding to the five possible initial moves of *A*.

1. if at step 1 A has connected two initial vertices *a* and *c*, then *B* connects *c* with itself and includes one initial vertex plus *a* (and of course the vertex between *a* and *c*) inside the new face. Now, *B* responds to *A* moves following lemma 6 for the internal face (this is a slight variant but the strategy is exactly the same), and following lemma 9 for the external one. As a result, he is able to isolate exactly 5 vertices, thus winning.

In all the other cases *A* must have connected an initial vertex *a* to itself, and has therefore generated a new vertex *b*. Different cases arise depending on how many initial vertices were included in the new face:

2. if no vertex was included (Figure 8 case 2), then *B* connects *a* with *b* by playing inside the new face, and therefore isolating a single vertex. The game is back to the case $x_0 = 6$ for which a winning strategy for *B* has already been proved [3].

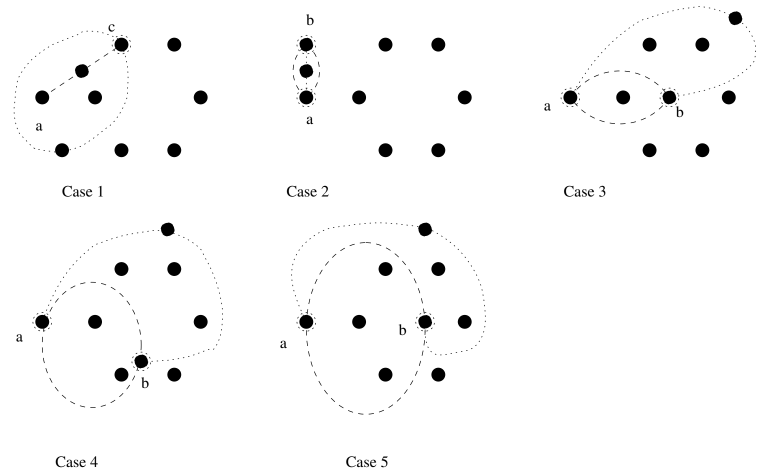

Figure 8: Strategy for B for $x_0 = 7$. Only some embeddings and initial moves are given.

3. if one vertex was included (Figure 8 case 3), then B plays in the external face by connecting a with b and including two initial vertices inside the new face. One vertex is completely isolated in the first face. Now B plays as in lemma 6 in the second face, isolating two vertices, and as in lemma 8 for the external face, isolating other two vertices, for a total of five isolated vertices.

4. if two vertices were included (Figure 8 case 4), then B plays in the external face by connecting a with b and including three initial vertices inside the new face. Now, in the first face B can isolate two vertices using lemma 6 (without using the vertex on the boundary); in the second face B can isolate two vertices using lemma 8; finally, in the third (external) face he can play following lemma 5 and isolate a single vertex (even if the boundary vertex is not used), for a total of five.

5. if three vertices were included (Figure 8 case 5), then B plays in the external face by connecting a with b and including two initial vertices inside the new face. Now, in the first face B can isolate two vertices using lemma 8 (without using the vertex on the boundary); in the second face B can isolate two vertices using lemma 6; finally, in the third (external) face he can play following lemma 5 and isolate a single vertex, for a total of five vertices.

Theorem 3 *The strategy above is a correct and winning for B.*

PROOF. The proof trivially follows by the the proofs of the lemmas. ■

5 Conclusions

In this paper we have presented some general topological properties of the Sprouts Game that relate, no matter what the value of x_0 is, the number of faces, and isolated vertices of degree 2 of a plane graph. We have also considered the case $x_0 = 7$ for which only exhaustive searches are known, we have shown a winning strategy and we have given a compact correctness analysis. Note that a slightly different strategy not based on lemma 8 could also be given. However, for such a strategy, it is necessary to consider the game $x_0 = 3$ where B wants to loose (*misère* game).

As an ongoing work we are considering how to use the results of theorem 2 and the main idea of the winning strategy (i.e., isolate proper sets of vertices so that only a certain subset of vertices of degree 2 remains isolated) in order to give a compact analysis and winning strategy to the Sprouts Game for values of $x_0 \geq 8$.

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