Tiered objects: a guided tour
(abstract)

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Tiered object is a generic name for species of structures A whose elements can be cumulatively arranged in layers according to some ranking that takes values in the set of extended natural numbers

$$\Omega =_{\text{def}} \{0 \sqsubset 1 \sqsubset 2 \sqsubset \cdots \sqsubset \infty\}.$$

Such structures arise often as (complete) ultrametric spaces or as final coalgebras; for example, the set $X^\infty =_{\text{def}} X^* \cup X^\omega$ of finite and infinite sequences over a non-empty set X, the set T_Σ^∞ of finite and infinite trees over a first-order signature Σ or, more generally, a wide class of final coalgebras, as shown by Adamek [1].

Tiering is given by a restriction operation

$$\cdot \restriction \cdot : A \times \Omega \to A$$

satisfying:

(i) For all $a \in A$ and $p, q \in \Omega$,

$$(a \restriction p) \restriction q = a \restriction \min\{p, q\} \tag{1}$$

(ii) If $\{a_i\}_{i \in \mathbb{N}}$ is a sequence of elements from A such that $a_i = a_{i+1} \restriction i$ for all $i \in \mathbb{N}$, there is a unique element $\lim_{i\to\infty} a_i \in A$ such that

$$a_n = \left(\lim_{i\to\infty} a_i\right) \restriction n \tag{2}$$

for all $n \in \mathbb{N}$.

Sometimes it is convenient to use the further property:

(iii) There is an element $\perp \in A$ such that, for all $a \in A$,

$$a \restriction 0 = \perp. \tag{3}$$

There is a wealth of examples of tiered objects in the recent literature, including e. g. *complete projection spaces* (see [4]), inverse limits of chains of embedding-projection pairs like the bifinite domains (see [6]), the *rank-ordered sets* introduced in [2]. Moreover, some notion of tiering is also exploited in proof techniques for lazy datatypes in functional programming, like the *generic approximation lemma* of [5], and is implicit in the use of families of equivalence relations in [8] and [7].

In the talk, I will discuss the basic properties of tiered objects together with some of their applications to coinductive datatypes: looking at the latter as tiered

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objects emphasizes the underlying approximation structure on which inductions can be carried out.

Then I will refine the basic framework observing that Ω is a complete Heyting algebra, and that in many instances there is a mapping ("extent") $\mathsf{E} : A \to \Omega$ with the properties:

$$a \restriction \mathsf{E}\, a = a, \tag{4}$$
$$\mathsf{E}(a \restriction p) = \mathsf{E}\, a \sqcap p, \tag{5}$$

thus obtaining *sheaves* over Ω, as described in [3]. The completeness property (ii) takes then the following form:

(iv) Let $a, b \in A$ be *compatible* if $a \restriction \mathsf{E}\, b = b \restriction \mathsf{E}\, a$. Then every $X \subseteq A$ whose elements are pairwise compatible has a *join* $\bigvee X$, the least upper bound of X w. r. t. the partial ordering where $a \leqslant b$ if and only of $a = b \restriction \mathsf{E}\, a$.

Finally, I will give examples of how the higher-order intuitionistic logic of sheaves [9] can be used in characterizing in a natural way constructions on tiered objects.

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